Redundant Graph Fourier Transform

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***Abstract*—Signal processing on graphs is a new emerging field that processing high-dimensional data by spreading samples on networks or graphs. The new introduced definition of graph Fourier transform shows its importance in establishing the theory of frequency analysis or computational harmonic analysis on graph signal processing. We introduce the definition of redundant graph Fourier transform, which is defined via a Parseval frame transform generated from an extended Laplacian of a given graph. The flexibility and sparsity of the redundant graph Fourier transform are important properties that will be useful in signal processing. In certain applications and by selections of the extended Laplacian, redundant Fourier transform performs better than graph Fourier transform.**

***Index Terms*—Graph Fourier transform, redundant graph Fourier transform, graph Laplacian matrix, signal compression.**

I. Introduction

In traditional discrete signal processing, time-frequency approaches play an important role in signal processing in time or frequency domain, or even in time-frequency domain [1]. But for high dimensional datasets, such as transportation networks, social networks, traditional signal processing can not capture the underlying complex structure of the datasets. In order to solve this problem, an remarkable technique is to represent high dimensional datasets on weighted undirected graphs, then develop theories and processing methods in a similar manner as traditional discrete signal processing. Representing high dimensional data on a network or graph captures the underlying complex structure of the dataset. In applications such as social networks, electricity networks, transportation networks, and sensor networks, data naturally reside on the vertices of weighted graphs. Edges of these graphs are imposed weights that measuring the similarities, dependencies, or correlations between different pairs of vertices [2].

Recently, spectral graph signal processing become one of the hottest topics. The foundation of the theory is to apply graph Laplacian matrix and its eigenvectors to establish the graph Fourier transform, which captures the spectral property of the given datasets [2]. The motivation of graph Fourier transform is that the complex exponentials $e^{i\omega x}$ defining the classic Fourier transform are the eigenvectors of the $1-D$ Laplacian operator $\frac{d^2}{dx^2}$. Analogously, the graph Fourier transform is defined by the eigenvector matrix of the Laplacian matrix associated with the given graph. Based on graph Fourier transform, many new tools for signal processing on graphs have been developed and successfully applied in discrete time signals or high dimensional structured datasets [3], [4]. Applications of the graph Fourier transform leads to a computational harmonic analysis on the spectral domain of graphs, which involves the definitions and applications of operations on graphs such as convolution, translation, modulation, dilation [2]. Wavelet and windowed Fourier analysis on graphs are also established based on Graph Fourier transform [5], [6].

In classic Fourier analysis, the Fourier basis forms an orthonormal basis, correspondingly, Fourier frames are established by perturbation of the Fourier basis to an irregular Fourier frames, which is now widely studied and applied in applications [7], [8]. Our goal in this paper is to give a definition of Fourier frames on graphs. In fact, graph Fourier frames will be derived by using the graph Fourier transform. Then a redundant graph Fourier transform will be defined via a Parseval graph Fourier frame calculated from an extended Laplacian matrix of a given graph. Finally, we give an analysis on potential applications of this redundant transform to signals or high-dimensional signals processing.

II. Graph Fourier transform

In this section, we review the definition of graph Fourier transform. Consider a weighted graph $G=(V,E,W)$, where V denotes the set of vertices, E denotes the set of edges, of the graph, respectively. W is the weighted adjacency matrix. If there is an edge $e(i,j)$ connecting nodes i and j, $W_{i,j}$ represents the weight assign to the edge $e(i,j)$, otherwise, $W_{i,j}=0$. Define the degree matrix D associated to the graph G as a diagonal matrix whose i-th diagonal element d_i is equal to the sum of the weights of all the edges incident to vertex i, i.e. $D_{ii}=\sum_j W_{ij}$. Then the graph Laplacian, also called the combinatorial graph Laplacian of G is defined as $L=D-W$. Obviously, L is a real symmetric matrix, and therefore has a complete set of orthonormal basis. Denote these eigenvectors by χ_l for $l=0,1,\cdots,N-1$, with associated eigenvalues λ_l, i.e.

$$L\chi_l=\lambda_l\chi_l.$$

Then for any vector $f\in\mathbb{R}^N$ defined on the vertices of G, its graph Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(l)=\langle\chi_l,f\rangle=\sum_{n=1}^{N}\chi_l^*(n)f(n).$$

The inverse transform can be derived by:

$$f(n)=\sum_{l=0}^{N-1}\widehat{f}(l)\chi_l(n).$$

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Fig. 1. Circle graph

The Parseval relation holds for the graph Fourier transform, that is, for any $f, g \in \mathbb{R}^N$,

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle.$$

Note that graph Fourier transform is consistent with the traditional Fourier transform if a finite discrete periodic signal is indexed by a circle graph in Figure 1, since the eigenvector matrix of its Laplacian is the discrete Fourier transform matrix.

III. Redundant Graph Fourier Transform

In this section, we introduce the definition of redundant graph Fourier transform. In order to establish the transform, we have to construct an extended Laplacian matrix, which will involve two graph structures on a given graph signal, and a bipartite graph that presents the connections between the two graphs. By using the first N rows of the eigenvector matrix of the extended Laplacian matrix, we give a definition of the redundant graph Fourier transform. The reason why it called "redundant" is that it is defined by using a Parseval frame rather than an orthonormal basis in the setting of graph Fourier transform. A detailed introduction of frame theory and its applications in signal processing can be found in [1].

Definition III.1. *A sequence of vectors $\{v_n\}_{n\in\Lambda}$ is a frame of a Hilbert space H if there exist two constants $B \geq A > 0$ such that*

$$A\|u\|^2 \leq \sum_{n\in\Lambda} |\langle u, v_n\rangle|^2 \leq B\|u\|^2, \forall u \in H. \tag{1}$$

where Λ is a countable index set. When $A = B = 1$ the frame is said to be Parseval.

A system of complex exponentials $\{e^{i\lambda_k x}\}_{\lambda_k\in\Lambda}$, where $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers, is called a Fourier frame if $\{e^{i\lambda_k x}\}_{\lambda_k\in\Lambda}$ satisfy the frame inequalities (1) for all $u \in L^2(-\pi, \pi)$ [8].

Finite Fourier frames, or harmonic frames for $\mathbb{R}^N$ or $\mathbb{C}^N$ are obtained by keeping arbitrary N rows from an $M \times M$ discrete Fourier transform matrix [10]. Harmonic frames have been proved to be useful in applications [11]. In this paper, we will extend this approach to establish the redundant Fourier transform, but by choosing the first N rows of the eigenvector matrix.

For that, we need the following well-known lemma in frame theory.

Lemma III.2. *Let Q be an $m \times n$ matrix in which the rows form an orthonormal set of vectors in $\mathbb{R}^n$, where $n \geq m$. Let $F = \{v_1, v_2, \cdots, v_n\}$ be the columns of Q, then F is a Parseval frame for $\mathbb{R}^m$.*

Besides, we also need the following result from matrix theory.

Lemma III.3. ([12]) *Let A, B, C, D be $n \times n$ matrix on $\mathbb{R}$. If matrix A is invertible and $AC = CA$, then*

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(AD - CB). \tag{2}$$

Redundant graph Fourier transform Now we establish the redundant graph Fourier transform by the steps as follows:

- The motivation of redundant graph Fourier transform is, in some cases, one graph is not enough to capture the spectral information in a dataset, then an extra graph would be useful to improve the representation. Given a graph $G = \{V, E_1, W_1\}$, let $G' = \{V', E_2, W_2\}$ be another graph imposed on subset of the original dataset with $V' \subseteq V$. Establish a bipartite graph $\widetilde{G} = \{(V, V'), E_3, W_3\}$, then we obtain a connected extended graph of G with respect to G' and $\widetilde{G}$, denoted by $\overline{G}$, with vertex set $V(\overline{G}) = V \cup V'$, edge set $E(\overline{G}) = E_1 \cup E_2 \cup E_3$. Suppose that graph G has N vertices, their indices in $\overline{G}$ remain the same as in G. Suppose that G' has M vertices, and the i-th node of the graph G' will be indexed by $N + i$. Select a weight $\alpha > 0$ measuring the impact of the bipartite graphs $\widetilde{G}$ on $\overline{G}$.
- Let the Laplacian of the graph G be L, Laplacian of the graph G' be L'. Then the Laplacian of the extended graph $\overline{G}$ will be written as

$$L_{II} = \begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix} + \alpha \begin{pmatrix} S & -W_3 \\ -W_3^T & \widetilde{S} \end{pmatrix} \tag{3}$$

  where S is a diagonal matrix, with its diagonal entries equal to the l_1 norm of rows of W_3, that is, $s_{ii} = \sum_{k=1}^{M} w_{3,ik}$. $\widetilde{S}$ is also a diagonal matrix, whose diagonal entries equal to the l_1 norm of columns of W_3, i.e. $\widetilde{s}_{ii} = \sum_{j=1}^{N} w_{3,ji}$. Then L_{II} is still a positive semidefinite matrix with rank less than $2N$, we can sort its eigenvalues as

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N+M-1}.$$

- Denote the corresponding orthonormal eigenvectors by $\phi_0, \phi_1, \cdots, \phi_{N+M-1}$. Putting together these eigenvectors together to have an orthonormal matrix Φ, which is an $(N+M) \times (N+M)$ matrix. Taking the first N rows of Φ, and denote them with respect to the column indices by $\mathcal{W} = (w_0, w_1, \cdots, w_{N+M-1})$, where $w_k \in \mathbb{R}^N$, $k = 0, 1, \cdots, N+M-1$. By Lemma III.2, W is a Parseval frame and thus is redundant.
- Given a structured signal f, and impose it with an extended graph $\overline{G}$ as defined above. Suppose that the Parseval frame derived from the Laplacian L_{II} of $\overline{G}$ is $\mathcal{W} = (w_0, w_1, \cdots, w_{N+M-1})$. The redundant graph

Fourier transform of f is defined as

$$\hat{\hat{f}}(k) = \langle f, w_k \rangle = \sum_{l=1}^{N} w_k^*(l) f(l)$$

for $k = 0, 1, \cdots, N + M - 1$.

And the inverse redundant graph Fourier transform is then given by

$$f(n) = \sum_{k=0}^{N+M-1} \hat{\hat{f}}(k) w_k(n).$$

The structure of redundant graph Fourier transform considers the extra impact of the graphs $G^{'}$ and $\widetilde{G}$, which leads to the redundancy of the transform. A redundant transform generate a freedom on stable representations of the given signals [1]. In cases where redundancy is important, our redundant graph Fourier transform would be useful.

In the following theorem, we will discuss a special choice of α and $L^{'} = L$, $W_3 = I_N$ for the redundant graph Fourier transform. And the reason why select the first N rows of Φ will be explained based on this result.

Theorem III.4. *Given a graph G and its extended graph with respect to $G^{'}$ and $\widetilde{G}$ as defined above, $\overline{G}$, denote their Laplacian matrix by L and L_{II} defined in (3), respectively. Let $L^{'} = L$, $W_3 = I_N$. α be the weight assigned to the bipartite graph. Suppose that the eigenvalues of L are ordered as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$, and the eigenvalues of L_{II} are ordered as $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2N-1}$. Then the spectrum of L_{II} contains that of L. If α satisfies the condition $2\alpha < min_{k=1}^{N-1}\{\lambda_k - \lambda_{k-1}\}$. Then we have $\mu_{2l} = \lambda_l$, $l = 0, 1, \cdots, N - 1$, and $\mu_{2l+1} = \lambda_l + 2\alpha$. That is, we insert new eigenvalues between the eigenvalues of the Laplacian L.*

Proof. Let $P = \alpha W_3$. First we have $\lambda I - (L + P)$ is invertible since $det(\lambda I - (L + P))$ is a polynomial. By the assumptions, $L^{'} = L$ and $W_3 = I_N$, $L + P$ is symmetric, then by Lemma III.3, the characteristic function of L_{II} is

$$\begin{aligned} & det \begin{pmatrix} \lambda I - (L+P) & P \\ P & \lambda I - (L+P) \end{pmatrix} \\ & = det((\lambda I - (L+P))^2 - P^2) \\ & = det((\lambda I - (L+2P))(\lambda I - L)) \\ & = det(\lambda I - (L+2P)) det(\lambda I - L) \end{aligned} \tag{4}$$

Equation (4) shows that the eigenvalues of L is also the eigenvalues of L_{II}.

The remaining eigenvalues of L_{II} is the eigenvalues of $L + 2P$. It is easy to see that $\lambda_i(L + 2P) = \lambda_i(L) + 2\alpha$. Then if $2\alpha < min_{k=1}^{N-1}\{\lambda_k - \lambda_{k-1}\}$, we have $\mu_{2l} = \lambda_l$, $\mu_{2l+1} = \lambda_l + 2\alpha$ for $l = 0, 1, \cdots, N - 1$. □

Discussion By Theorem III.4, selection of $L^{'} = L$, $W_3 = I_N$ in L_{II} lead to new eigenvalues inserted between that of L. Note that, the eigenvectors of Laplacian L_{II} has a special structure. In fact, suppose that the k-th eigenvector of L_{II} is $(v_k^T, u_k^T)^T$, $v_k, u_k \in \mathbb{R}^N$. Then we have,

$$\begin{pmatrix} L+P & -P \\ -P & L+P \end{pmatrix} \begin{pmatrix} v_k \\ u_k \end{pmatrix} = \mu_k \begin{pmatrix} v_k \\ u_k \end{pmatrix} \tag{5}$$

then we have,

$$\left\{ \begin{array}{c} (L+P)v_k - Pu_k = \mu_k v_k \\ -Pv_l + (L+P)u_k = \mu_k u_k \end{array} \right. \tag{6}$$

for $l = 0, 1, \cdots, N - 1$, and thus

$$\left\{ \begin{array}{c} L(v_k + u_k) = \mu_k(v_k + u_k) \\ (L+2P)(v_k - u_k) = \mu_k(v_k - u_k) \end{array} \right. \tag{7}$$

Then $v_k + u_k$ eigenvectors of L with eigenvalue μ_k, $l = 0, 1, \cdots, 2N - 1$. Since L only has eigenvalues $\mu_{2l} = \lambda_l$, $l = 0, 1, \cdots, 2N - 1$, we have $v_k + u_k = 0$ if $k = 2l + 1$, $l = 0, 1, \cdots, 2N - 1$. Then $v_k = -u_k$, and substitute it into the second equation in (7), and by Theorem III.4, we have $Lv_k = Lv_{2l+1} = \lambda_l v_k$, if the eigenvalues λ_k are distinct, that is, the eigen-subspace are one dimensional, then we have $v_k = c\chi_l$ for some constant c.

If $k = 2l$, then $\mu_k = \lambda_l$, and by the second equation in (7), we have $v_k = u_k$, and therefore, v_k is also a eigenvector of L by the first equation. Therefore, the first N rows and the last N rows of the matrix Φ of L_{II} are almost the same with respect to a switch of negative symbol in the odd columns. This is the reason why we choose the first N rows of Φ. And recall that a harmonic frame in frame theory is also generated by taking the first N rows of a $K \times K$ discrete Fourier transform matrix, with $N \leq K$.

Note that other selection of $L^{'}$, W_3 and α can not generate a structured spectral similar to the previous selection.

IV. Properties of RGFT

Flexibility. The properties of redundant graph Fourier transform depends on the selections $L^{'}$ and W_3 and the weight parameter α, with a good selection of them, RGFT improves the performance of GFT in signal compression or denoising. And the selection of $L^{'}$, W_3 and α can be adaptive to different considerations.

Sparsity. The coefficients of the RGFT can be more sparser than GFT, or we can select proper $L^{'}$, W_3 and α to generate a sparser RGFT. More researches have to be done on the problem of improving the sparsity. Note that a bad selection of E and α will produce noise-like impact on RGFT and therefore lead to poor application performance.

V. Experiments on Signal Compression

Traditional signal compression algorithms are based on expanding signals into suitable bases with the expectation that the representation is sparse, that is, most information of the signal is captured by a few basis functions. By selecting the basis components with largest magnitudes, one can always reconstruct the signal with small approximation error in the least-squares sense. Frames are generally considered as generalized bases, simply because frames give more sparse and more stable representations than bases or orthogonal bases [1].

The redundant graph Fourier transform is based on the construction of a Parseval frame that captures the spectrum information of a given dataset. In the following, we would apply this transform on signal compression.

Signal compression algorithm Given a signal s, suppose that the extended Laplacian associated with s is L_{II}, and the Parseval transform matrix we obtained is $\mathcal{W}$, then we denote the redundant graph Fourier transform frame coefficients by $(\mathcal{W}^T s)(n) = \hat{\hat{s}}_n$. We compress s by keeping only K coefficients with largest magnitudes in $\hat{\hat{s}}_n$. Without loss of generality, assume that $|\hat{\hat{s}}_0| \geq |\hat{\hat{s}}_n| \geq \cdots \geq |\hat{\hat{s}}_{N+M-1}|$. Then the signal reconstruction after compression is $s_{II} = \mathcal{W}(\hat{\hat{s}}_0, \hat{\hat{s}}_1, \cdots, \hat{\hat{s}}_{K-1}, 0, \cdots, 0)^T$. The approximation error is calculated by

$$err(s_{II}, s) = \frac{\|s_{II} - s\|_2}{\|s\|_2}.$$

In the following, we give the performance of data compression of RGFT by letting $L^{'} = L, W_3 = W_1$, and $\alpha = 0.1$, on a simple signal $f(t) = t, t = 1, 2, \cdots, 200,\ t \neq 50$, $f(50) = 100$, and compare that with GFT. Since every element in the signal f is strongly related with its neighbor elements, the graph structure imposed to signal $f(t)$ is a path, and the weights all equal to 1. When keeping 15 to 90 percents of the largest coefficients, Figure 2 shows that RGFT have better approximation errors than GFT.

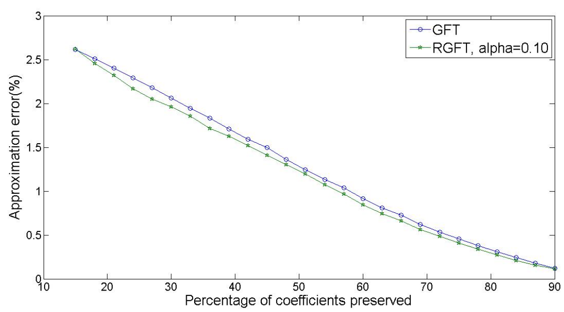

Fig. 2. Approximation errors of GFT and RGFT with different percentages of the largest coefficients preserved for signal f, with $L^{'} = L, W_3 = W_1$ and the weight alpha=0.1.

When selecting the $L^{'} = L, W_3 = I_{200}$ and $\alpha = 0.1$, RGFT performs almost the same as GFT in approximation error for signal $f(t)$, but still slightly better than GFT at some points, see Figure 3.

Figure 3 shows that the redundant graph Fourier transform almost reduce to graph Fourier transform by a proper selection on the parameters.

VI. Conclusion

In this paper, a redundant graph Fourier transform is proposed based on the framework of graph Fourier transform. The redundant graph Fourier transform captures the impact of two graphs and their connections on a structured signal. Flexibility of this transform ensures that it can performs better than the graph Fourier transform by selecting suitable graphs, weights and the connection parameter.

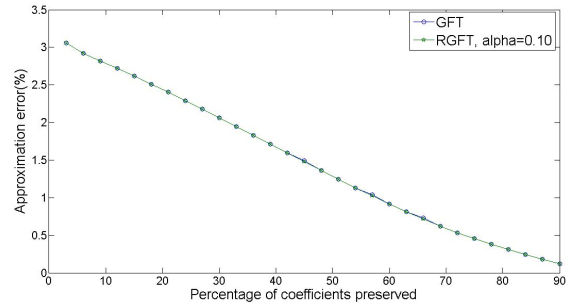

Fig. 3. Approximation errors of GFT and RGFT with different percentages of the largest coefficients preserved for signal f, with $L^{'} = L, W_3 = I_{200}$ and the weight alpha=0.1.

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