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MULTI-WINDOWED GRAPH FOURIER FRAMES

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Abstract:

Graph signal processing is a new research field in the signal processing community. Recently, windowed graph Fourier frames have been proposed to analyze graph signals. However, it is not easy to construct tight frames by using signal window function. In this paper, we extend the windowed graph Fourier transform to the multi-window case. The multi-windowed graph Fourier frames offer more freedom on constructing tight frames. We provide some results for constructing multi-windowed graph Fourier frames, dual frames and tight frames. When applying the multi-windowed graph Fourier transform on synthesis graph signals, a dual localization phenomenon can be found in the original and dual windowed graph Fourier transform coefficients.

Keywords:

Graph Fourier transform; Windowed graph Fourier frames; Tight frames; Dual frames

1. Introduction

In recent years, graph signal processing has become one of the hottest research fields in the signal processing community [1],[2]. Many types of data collected from or represented by networks, such as social networks and sensor networks, can be considered as signals defined on graphs. Due to the underlying irregular structures, classical signal processing techniques can not be applied for analyzing these graph signals. Thus, one of the research interests is to develop the classical techniques into the graph signal setting.

Time-frequency analysis is one of the basic theories in signal processing. Among the theoretical system of time-frequency analysis, windowed or short time Fourier transform (STFT) is a traditional tool for analyzing non-stationary signals, which are signals with frequency content changes in time, such as audio, acoustic and seismic signals [3]. STFT produces time-frequency representation for signals, from which we can obtain localized information about a signal, such as the instance where the frequency changed or the location of the frequencies of specific noises. Accordingly, researchers in the signal processing community are recently devoting themselves to establish a vertex-frequency analysis for graph signals. In [4], windowed graph Fourier transform is proposed under the framework of the spectral graph theory. The windowed graph Fourier atoms are capable to extract localized information from graph signals. In [5], the same authors gave a detailed theoretical discussion and real graph signal application analysis on windowed graph Fourier transform.

Unlike the classical windowed Fourier transform, the windowed graph Fourier atoms can not form a tight frame naturally. In addition, the construction of windowed graph Fourier tight frames requires a subtle design and analysis on the graph window functions. In this paper, we extend the windowed graph Fourier transform to the multi-window case. Multi-window functions offer more freedom on designing tight frames, and do not require complex techniques in the construction of window functions. We present some results for constructing multi-windowed graph Fourier frames and tight frames. We also apply the dual of a multi-windowed graph Fourier frame to establish reconstruction formulas for graph signals. When applying windowed graph Fourier transform on synthesis signals, a dual localization phenomenon can be found between the original and dual windowed graph Fourier transform coefficients. The rest of this paper is organized as follows. In Section 2, we briefly review the classical windowed Fourier transform, frames and dual frames. In Section 3, we summarize the framework of graph Fourier transform and graph windowed Fourier frames. In Section 4, we introduce the definition of multi-windowed graph Fourier frames, and provide some results on frames and tight frames. In Section 5, we present some results on dual frames. In Section 6, we provide experiment results for illustration of multi-windowed graph Fourier frames and duals. We conclude the paper in Section 7.

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2. Classical multi-windowed Fourier frames

In this section, we give a brief review of the multi-windowed Fourier frames. Other details can be found in [6]. For any window function $g \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$, the translation of g by u is defined by the operator T_u: $(T_u g)(t) := g(t-u)$.

For any $\xi \in \mathbb{R}$, the modulation of g by frequency ξ is defined by the operator M_ξ: $(M_\xi g)(t) := e^{2\pi i \xi t} g(t)$.

Given a finite sequence of window functions $g^1, \cdots, g^L$, the family of discrete multi-windowed Fourier atoms is defined by:

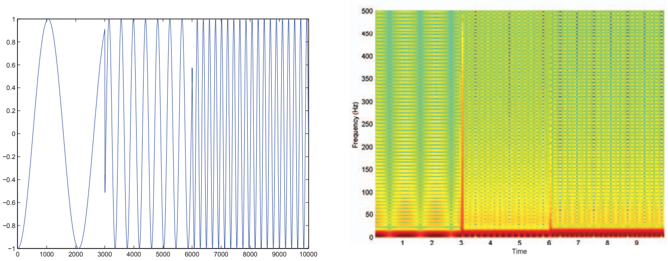
$$\mathcal{G}_c^w = \{g_{m,n}^l(t) := (M_m T_n) g^l(t) = e^{2\pi i m t} g^l(t-n)\}, \quad (1)$$

where $l = 1, \cdots, L$; $m \in \mathbb{Z}$; $n \in \mathbb{Z}$.

$\mathcal{G}_c^w$ is called a multi-windowed Fourier frame if there exist two positive constant $A, B > 0$, such that for any $f \in L^2(\mathbb{R})$,

$$A\|f\|^2 \le \sum_{l=1}^{L} \sum_{m,n \in \mathbb{Z}} |\langle f, (M_m T_n) g^l \rangle|^2 \le B\|f\|^2. \quad (2)$$

The constants A, B are called frame bounds. $\mathcal{G}_c^w$ is called a tight frame if $A = B$. Here $\langle f, (M_m T_n) g^l \rangle = \int f(t) \overline{g^l(t-n)} e^{-2\pi m t} dt$ are the windowed Fourier transform coefficients. When $L = 1$, the coefficients $\{|\langle f, (M_m T_n) g \rangle|^2\}_{m,n \in \mathbb{Z}}$ is generally called the "spectrogram" of signal f. Figure 1 presents an example of the spectrogram generated by a signal windowed Fourier transform of a non-stationary signal by a rectangle window function.



(a) (b)

FIGURE 1. (a) A non-stationary signal; (b) Spectrogram.

Tight frames provide stable reconstructions of the signal functions, whereas the window functions must be subtly designed to generate a tight frame. Nevertheless, any function $f \in L^2(\mathbb{R})$ can be reconstructed by using the windowed Fourier transform coefficients and a dual multi-windowed Fourier frame.

A multi-windowed Fourier frame $\tilde{\mathcal{G}}_c^w = \{\tilde{g}_{m,n}^l(t)\}$ generated by window functions $\tilde{g}^1, \cdots, \tilde{g}^L$, is called a dual to $\mathcal{G}_c^w = \{g_{m,n}^l(t)\}$ if for any $f \in L^2(\mathbb{R})$,

$$f = \sum_{l=1}^{L} \sum_{m,n \in \mathbb{Z}} \langle f, (M_m T_n) g^l \rangle (M_m T_n) \tilde{g}^l. \quad (3)$$

3. Windowed graph Fourier transform

In this section, we shortly summarize the basic concepts and results for graph windowed Fourier frames. Under the framework in [4], we consider a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{V}$ denotes the set of vertices, $\mathcal{E}$ denotes the set of edges, of the graph, respectively. W is the weighted adjacency matrix. If there is an edge $e(i,j)$ connecting nodes i and j, $w_{i,j}$ represents the weight assigned to the edge $e(i,j)$. The degree matrix D associated to the graph $\mathcal{G}$ is defined as a diagonal matrix whose i-th diagonal element d_i is the degree of vertex i, i.e. the sum of the weights of all the edges connected to vertex i: $d_i = \sum_j w_{ij}$. Then the graph Laplacian matrix of $\mathcal{G}$ is defined by $L = D - W$. Obviously, L is a real symmetric matrix, and therefore has a complete set of orthonormal basis. These eigenvectors are denoted by u_l for $l = 0, 1, \cdots, N-1$, with associated eigenvalues λ_l, i.e. $Lu_l = \lambda_l u_l$. We denote the eigen-matrix by $U = (u_0, u_1, \cdots, u_{N-1})$.

For a graph signal $f \in \mathbb{R}^n$ defined on a N-vertex graph $\mathcal{G}$, its graph Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\lambda_l) = \langle f, u_l \rangle = \sum_{n=1}^{N} u_l^*(n) f(n). \quad (4)$$

The inverse transform can be derived by:

$$f(n) = \sum_{l=0}^{N-1} \widehat{f}(\lambda_l) u_l(n). \quad (5)$$

Thus, the Parseval equation holds for the graph Fourier transform, that is, for any $f, g \in \mathbb{R}^N$, $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$.

The generalized translation of f by i-vertex is defined by the operator T_i:

$$(T_i f)(n) := \sqrt{N} \sum_{p=0}^{N-1} \hat{f}(p) u_p^*(i) u_p(n). \quad (6)$$

The generalized modulation of f by frequency k is defined by the operator M_k:

$$(M_k f)(n) := \sqrt{N} f(n) u_k(n). \quad (7)$$

Using the generalized translation and modulation operators, the windowed graph Fourier atoms generated by a window function $g \in \mathbb{R}^N$ on $\mathcal{G}$ are defined by

$$g_{i,k}(n) := (M_k T_i g)(n) = N u_k^*(n) \sum_{p=0}^{N-1} \hat{g}(p) u_p^*(i) u_p(n), \quad (8)$$

with $i = 1, 2, \cdots, N$, $k = 0, 1, \cdots, N-1$. We denote the set of these windowed graph Fourier atoms by

$$\mathcal{G}^w = \{g_{i,k}\}_{i=1,2,\cdots,N; k=0,1,\cdots,N-1}. \quad (9)$$

In [4] and [5], the authors showed that the condition for a window $g \in \mathbb{R}^N$ to generate a windowed graph Fourier frame is $\hat{g}(0) \neq 0$.

Lemma 3.1. [4] For a window function $g \in \mathbb{R}^N$ defined on graph $\mathcal{G}$, if $\hat{g}(0) \neq 0$, then the windowed graph Fourier atoms defined in (9) is a frame with lower frame bound

$$A := \min_{n \in \{1,2,\cdots,N\}} \{N \|T_n g\|_2^2\}, \quad (10)$$

and upper lower frame bound

$$B := \max_{n \in \{1,2,\cdots,N\}} \{N \|T_n g\|_2^2\}, \quad (11)$$

There are two types of windowed graph Fourier tight frames can be easily constructed:

a) If $|u_k(n)| = \frac{1}{\sqrt{N}}$, for all $k = 0, 1, \cdots, N$, and $n = 1, 2, \cdots, N$, then for any window function g, $\mathcal{G}^w$ is a tight frame with frame bounds $A = B = N\|g\|_2^2$;

b) If $\hat{g}(\lambda_l) = \delta_0(\lambda_l)$, for all $l = 0, 1, \cdots, N-1$, then $\mathcal{G}^w$ is a tight frame with frame bounds $A = B = N$.

The construction of tight windowed graph Fourier frames is constrained by the properties of the graph Fourier transform matrix. Technical design of tight window functions will involved with complicated computation and analysis on the graph Fourier transform matrix. Compared with the single-windowed case, multi-window functions offer more freedom on constructing tight frames, since it is not related to the properties of the graph Fourier transform matrix.

4. Multi-windowed graph Fourier frames

Correponding to (1) and (9), for a finite sequence of window functions $g^1, g^2, \cdots, g^L \in \mathbb{R}^N$, we can define the set of multi-windowed graph Fourier atoms by

$$\mathcal{G}_L^w = \{g_{i,k}^l\}_{i=1,2,\cdots,N; k=0,1,\cdots,N-1; l=1,2,\cdots,L}. \quad (12)$$

Then we have the following result for multi-windowed graph Fourier frames:

Theorem 4.1. Let $\mathcal{G}_L^w$ be the set of multi-windowed graph Fourier atoms defined in (12). If $\sum_{l=1}^L |\hat{g}^l(0)|^2 \neq 0$, then $\mathcal{G}_L^w$ is a frame with lower frame bound

$$A := \min_{n \in \{1,2,\cdots,N\}} \{N \sum_{l=1}^L \|T_n g^l\|_2^2\}, \quad (13)$$

and upper lower frame bound

$$B := \max_{n \in \{1,2,\cdots,N\}} \{N \sum_{l=1}^L \|T_n g^l\|_2^2\}, \quad (14)$$

Proof.

$$\begin{aligned}
&\sum_{l=1}^L \sum_{i=1}^N \sum_{k=0}^{N-1} |\langle f, g_{i,k}^l \rangle|^2 = \sum_{l=1}^L \sum_{i=1}^N \sum_{k=0}^{N-1} |\langle f, M_k T_i g^l \rangle|^2 \\
&= N \sum_{l=1}^L \sum_{i=1}^N \sum_{k=0}^{N-1} |\langle f (T_i g^l)^*, u_k \rangle|^2 \\
&= N \sum_{l=1}^L \sum_{i=1}^N \|f (T_i g^l)\|^2 = N \sum_{l=1}^L \sum_{i=1}^N \sum_{n=1}^N |f(n)|^2 |(T_i g^l)(n)|^2 \\
&= N \sum_{l=1}^L \sum_{i=1}^N \sum_{n=1}^N |f(n)|^2 |(T_n g^l)(i)|^2 \quad (15) \\
&= N \sum_{n=1}^N |f(n)|^2 \sum_{l=1}^L \|(T_n g^l)\|_2^2,
\end{aligned}$$

where (15) follows from the symmetry of L and the definition of operator T_i in (6). In addition, if $\sum_{l=1}^L |\hat{g}^l(0)|^2 \neq 0$, we have

$$\sum_{l=1}^L \|T_n g^l\|_2^2 = N \sum_{p=0}^{N-1} \sum_{l=1}^L |\widehat{g^l}(p)|^2 |u_p(n)|^2 > \sum_{l=1}^L |\widehat{g^l}(0)|^2 > 0 \quad (16)$$

By (16), taking the minimum and maximum of $\sum_{l=1}^L \|T_n g^l\|_2^2$, we have the lower frame bound $A > 0$. Thus, $\mathcal{G}_L^w$ is a frame with lower and upper frame bounds defined in (13) and (14). □

Corollary 4.1. $\mathcal{G}_L^w$ is a tight frame if and only if there exists a constant $C > 0$, such that $\sum_{l=1}^L \|T_n g^l\|_2^2 = C$ for $n = 1, 2, \cdots, N$.

Proof. In the proof of Theorem 4, for any $f \in \mathbb{R}^N$, we have

$$\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}|\langle f, g_{i,k}^l\rangle|^2 = N\sum_{n=1}^{N}|f(n)|^2\sum_{l=1}^{L}\|(T_n g^l)\|_2^2. \tag{17}$$

Thus, if $\sum_{l=1}^{L}\|T_n g^l\|_2^2 = C$ for $n = 1, 2, \cdots, N$, then

$$\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}|\langle f, g_{i,k}^l\rangle|^2 = NC\|f\|_2^2, \tag{18}$$

which implies that $\mathcal{G}_L^w$ is a tight frame with frame bounds $A = B = NC$.

On the other hand, if $\mathcal{G}_L^w$ is a tight frame with frame bounds $A = B = NC$, we can select N vectors $f_1, f_2, \cdots, f_N \in \mathbb{R}^N$ such that $\|f_k\|_2 = 1$, for $k = 1, 2, \cdots, N$ and the vectors $\{f_k^2 := (|f_k(1)|^2, |f_k(2)|^2, \cdots, |f_k(N)|^2)^T\}_{k=1}^N$ are linearly independent. Then $F = \left(f_1^2, f_2^2, \cdots, f_N^2\right)$ is non-singular. Let $X = (\sum_{l=1}^{L}\|(T_1 g^l)\|_2^2, \sum_{l=1}^{L}\|(T_2 g^l)\|_2^2, \cdots, \sum_{l=1}^{L}\|(T_N g^l)\|_2^2)^T$, by applying $f_1, f_2, \cdots, f_N$ in (17) and (18), we have

$$NF^T X = NC \cdot \mathit{1}, \tag{19}$$

where $\mathit{1} = (1, 1, \cdots, 1)^T$, $(\cdot)^T$ denotes the transpose of a vector or a matrix.

Since *F* is non-singular, Equation (19) has a unique solution. It is easy to verify that $X = C \cdot \mathit{1}$ is the solution. That is, $\sum_{l=1}^{L}\|T_n g^l\|_2^2 = C$ for $n = 1, 2, \cdots, N$. □

Corollary 4.2. Let $\mathcal{G}_L^w$ be the set of multi-windowed graph Fourier atoms defined in (12). If there exists a constant C, such that $\sum_{l=1}^{L}|\hat{g}^l(\lambda_p)|^2 = C$, for $p = 0, 1, \cdots, N-1$, then $\mathcal{G}_L^w$ is a tight frame with frame bounds $A = B = N^2 C$.

Proof. By Equation (16) in the proof of Theorem 4, for any $f \in \mathbb{R}^N$, we have,

$$\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}|\langle f, g_{i,k}^l\rangle|^2 = N\sum_{n=1}^{N}|f(n)|^2\sum_{l=1}^{L}\|(T_n g^l)\|_2^2$$
$$= N^2\sum_{n=1}^{N}|f(n)|^2\sum_{p=0}^{N-1}\sum_{l=1}^{L}|\widehat{g^l}(\lambda_p)|^2|u_p(n)|^2. \tag{20}$$

Since the eigen-matrix U of L is orthogonal, we have $\sum_{p=0}^{N-1}|u_p(n)|^2 = 1$, for $n = 1, 2, \cdots, N$. If $\sum_{l=1}^{L}|\widehat{g^l}(\lambda_p)|^2 = C$, for $p = 0, 1, \cdots, N-1$, from Equation (20), we have

$$\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}|\langle f, g_{i,k}^l\rangle|^2 = N^2 C\|f\|_2^2. \tag{21}$$

Thus, $\mathcal{G}_L^w$ is a tight frame with frame bounds $A = B = N^2 C$.□

Note that Corollary 4 presents a method for constructing multi-windowed graph Fourier tight frames. For example, select a window function $g^1 \in \mathbb{R}^N$, we can calculate its graph Fourier transform coefficients such that $|\widehat{g^1}|^2 := \left(|\widehat{g^1}(1)|^2, |\widehat{g^1}(2)|^2, \cdots, |\widehat{g^1}(N)|^2\right)^T = a \in \mathbb{R}^N$. We can then select a constant C large enough, such that vector $b = C \cdot \mathit{1} - a > 0$. Then let $|\widehat{g^2}|^2 = b$ for some proper window function g^2, by Corollary 4, $\{g^1, g^2\}$ generate a 2-windowed graph Fourier tight frame.

5. Dual of multi-windowed graph Fourier frames

In [5], the authors presented a reconstruction formula for single windowed graph Fourier frames. In this section, we provide another reconstruction formula by introducing the definition of dual of multi-windowed graph Fourier frames. Similar to the definition in (3), the multi-windowed graph Fourier atoms generated by a finite sequence of window functions $\tilde{g}^1, \tilde{g}^2, \cdots, \tilde{g}^N$ in the sense of (12): $\tilde{\mathcal{G}}_L^w$ is called a dual to $\mathcal{G}_L^w$ if for any $f \in \mathbb{R}^N$, if there exists a constant C, such that

$$f = C\sum_{l=1}^{L}\sum_{k=0}^{N-1}\sum_{i=1}^{N}\langle f, M_k T_i g^l\rangle M_k T_i \tilde{g}^l. \tag{22}$$

Theorem 5.1. Suppose that $\mathcal{G}_L^w$ is a multi-windowed graph Fourier frame as defined in (12). If there exists a finite sequence of window functions $\tilde{g}^1, \tilde{g}^2, \cdots, \tilde{g}^N$ and a constant $\mu > 0$, such that $\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu$ for $p = 0, 1, \cdots, N-1$, then $\tilde{\mathcal{G}}_L^w$ is a dual of $\mathcal{G}_L^w$.

Proof. Suppose that $\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu$ for $p =$

$0, 1, \cdots, N-1$. Then we have

$$\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}\langle f, g_{i,k}^{l}\rangle \tilde{g}_{i,k}^{l}(n)$$
$$=\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{k=0}^{N-1}\Big(N\sum_{m=1}^{N} f(m)u_k^*(m)\sum_{p=0}^{N-1}\hat{g}^l(\lambda_p)u_p(i)u_p^*(m)\Big)\cdot$$
$$\Big(Nu_k(n)\sum_{p'=0}^{N-1}\hat{g}^l(\lambda_{p'})u_{p'}^*(i)u_{p'}(n)\Big)$$
$$=N^2\sum_{m=1}^{N} f(m)\sum_{l=1}^{L}\sum_{p=0}^{N-1}\sum_{p'=0}^{N-1}\hat{g}^l(\lambda_p)\hat{\tilde{g}}^l(\lambda_{p'})u_p^*(m)u_{p'}(n)\cdot$$
$$\sum_{i=1}^{N}u_p(i)u_{p'}^*(i)\sum_{k=0}^{N-1}u_k^*(m)u_k(n)$$
$$=N^2\sum_{m=1}^{N} f(m)\sum_{l=1}^{L}\sum_{p=0}^{N-1}\sum_{p'=0}^{N-1}\hat{g}^l(\lambda_p)\hat{\tilde{g}}^l(\lambda_{p'})u_p^*(m)u_{p'}(n)\cdot$$
$$\delta_{pp'}\delta_{mn}=(*)$$

$$(*)=N^2 f(n)\sum_{p=0}^{N-1}\sum_{l=1}^{L}\hat{g}^l(\lambda_p)\hat{\tilde{g}}^l(\lambda_p)|u_p(n)|^2$$
$$=N^2\mu f(n).$$

Thus, $\tilde{\mathcal{G}}_L^w$ is a dual of $\mathcal{G}_L^w$ with $C = N^2\mu$ in (22). □

Theorem 5 implies that there is a freedom on constructing dual multi-windowed graph Fourier frames. For a single windowed graph Fourier frame $\mathcal{G}^w$ generated by $g \in \mathbb{R}^N$, the dual $\tilde{\mathcal{G}}^w$ can be constructed by selecting a constant μ then the dual window can be constructed by letting $\hat{\tilde{g}}(\lambda_p) = \frac{\mu}{\hat{g}(\lambda_p)}$, such that $\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu$, i.e. for $p = 0, 1, \cdots, N-1$. But in case that $\hat{g}(\lambda_l) = 0$ for some l, the dual can not be constructed in this scheme. In addition, the proof of Theorem 5 implies that the dual of a multi-windowed graph Fourier frame is possible to be a frame.

Corollary 5.1. Suppose that $\mathcal{G}_L^w$ is a multi-windowed graph Fourier frame as defined in (12). If there exists a finite sequence of window functions $\tilde{g}^1, \tilde{g}^2, \cdots, \tilde{g}^N$ and a constant $\mu > 0$, such that $\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu$ for $p = 0, 1, \cdots, N-1$, then $\tilde{\mathcal{G}}_L^w$ is also a multi-windowed graph Fourier frame.

Proof. If $\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu > 0$ for $p = 0, 1, \cdots, N-1$, by the Cauchy-Schwartz inequality, we have

$$\mu^2 = |\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p)|^2 \le \sum_{l=1}^{L}|\hat{\tilde{g}}(\lambda_p)|^2\sum_{l=1}^{L}|\hat{g}(\lambda_p)|^2, \quad (23)$$

for $p = 0, 1, \cdots, N-1$. Then we have $\sum_{l=1}^{L}|\hat{g}^l(\lambda_p)|^2 \neq 0$, and $\sum_{l=1}^{L}|\hat{\tilde{g}}^l(\lambda_p)|^2 \neq 0$, for $p = 0, 1, \cdots, N-1$. By Theorem 4, we have $\tilde{\mathcal{G}}_L^w$ is also a multi-windowed graph Fourier frame. □

Note that a pair sequence of generators of dual multi-windowed graph Fourier frame is unnecessary to satisfy the conditions $\sum_{l=1}^{L}\hat{g}(\lambda_p)\hat{\tilde{g}}(\lambda_p) = \mu$ for $p = 0, 1, \cdots, N-1$. For example, in the case that *L* is a circulant matrix, the eigenmatrix is the discrete Fourier transform matrix. Thus, we have $|u_p(n)| = \frac{1}{\sqrt{N}}$. In this case, any pair of multi-windowed graph Fourier frames can be dual to each other.

6. Examples

In this section, we present two examples to provide further intuition on multi-windowed graph Fourier frames and their duals. In the first example, we present the spectrogram generated by a pair of dual frames for a signal defined on a unweighted path graph of 180 vertices. It has been proved in [7] that the graph Laplacian eigenvectors for the path graphs are the basis vectors in the DCT-II transform. In our experiments, we design a signal in Figure 2(a) on the path graph by composing three Laplacian eigenvectors: u_5 restricted to the first 60 vertices, u_{15} restricted to the next 60 vertices, and u_{25} restricted to the final 60 vertices. These Laplacian eigenvectors can be generated by using the function dctmtx(·) in Matlab. In Figure 2, we present the windowed graph Fourier transform coefficients of this signal generated by a window g by setting $\hat{g}(\lambda_l) = e^{-\tau\lambda_l}$ with $\tau = 20$ and its dual window $\tilde{g}$ by $\hat{\tilde{g}}(\lambda_l) = \frac{\mu}{e^{-\tau\lambda_l}}$, $\mu > 0$. Both windows are normalized such that $\|g\| = \|\tilde{g}\| = 1$. Figure 2(b) gives the curves of the window and dual window functions in the graph spectral domain, where the original window is localized in the low frequencies, while the dual window localized in high frequencies. Figure 2(c) and (d) present the spectrograms of f, i.e. $|Sf(i,k)|^2$ for all $i \in \{1, 2, \cdots, 180\}$ and $k \in \{0, 1, \cdots, 180\}$ generated by $\mathcal{G}^w$ and $\tilde{\mathcal{G}}^w$. Corresponding to the window spectral curves, the spectrogram generated by g is localized in the low frequencies with vertex localization corresponding to f, while the spectrogram generated by $\tilde{g}$ is localized in the high frequencies with the same vertex localization.

In the second example, we follow the same structure in the first example and present the spectrogram generated by a 2-

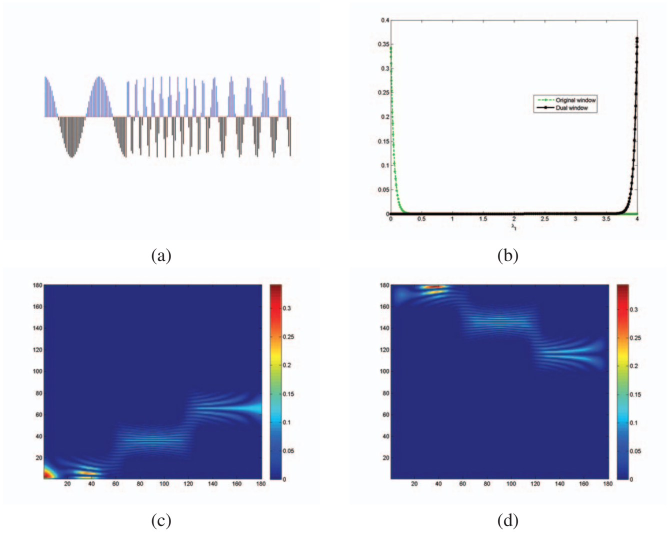


FIGURE 2. (a) A signal on path graph by composing Laplacian eigenvectors; (b) The spectral curves of a window function and its dual; (c) Spectrogram generated by $\mathcal{G}^w$; (d) Spectrogram generated by $\tilde{\mathcal{G}}^w$.

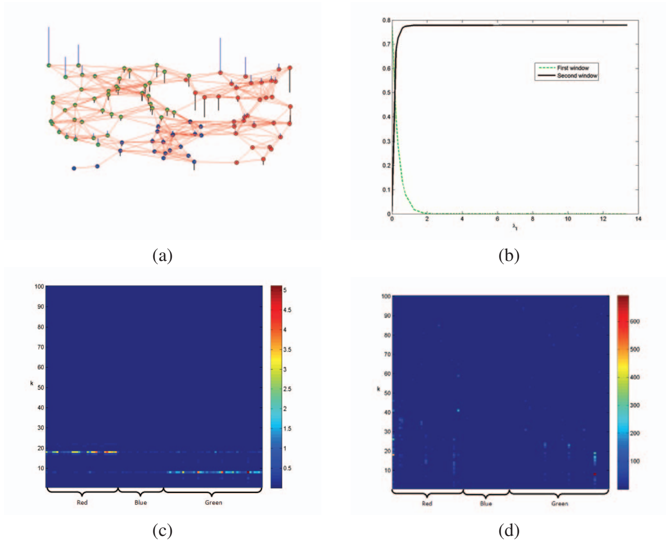


FIGURE 3. (a) A signal on random graph by composing Laplacian eigenvectors; (b) The spectral curves of window functions g_1 and g_2; (c) Spectrogram generated by $\mathcal{G}_1^w$; (d) Spectrogram generated by $\mathcal{G}_2^w$.

window graph Fourier tight frame for a signal defined on a random graph with 100 vertices. As presented in Figure 3, the vertices of the graph is partitioned into 3 classes: red, blue and green, by using spectral clustering. Similar to the example above, a signal on this random graph is taken by restricting u_7 to the red vertices, u_{17} to the blue vertices, and u_{27} to the green vertices. Figure 3(c) and (d) are the spectrograms of this signal generated by the same window used in the first example with $\tau = 3$: $\hat{g}^1(\lambda_l) = e^{-\tau\lambda_l}$ and $\hat{g}^2(\lambda_l) = \sqrt{\mu - |\hat{g}^1(\lambda_l)|^2}$, with $\mu = \max\{\hat{g}^1(\lambda_l)\}$. However, as shown in Figure 3(b), if one of the windows in a 2-windowed graph Fourier frame is localized in the low frequency, the other window do not localized in any frequency. Additionally, as presented in Figure 3(c), spectrogram generated by g^1 is well localized in the vertex clusters, whereas spectrogram generated by g^2 is not localized on both vertex or frequency domain.

7. Conclusion

We extended the theory of windowed graph Fourier transform to the multi-windowed case. The construction of frames in the multi-window case is more flexible than the case of single window. We presented results for constructing multi-windowed graph Fourier frames, tight frames and dual frames, respectively. When applying the transform to synthesis graph signals, a dual localization phenomenon can be found in the windowed and dual windowed graph Fourier transform coefficients.

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