# Extrinsic contribution to nonlinear current induced spin polarization

Ruda Guo ,[1,5,6] Yue-Xin Huang ,[2,3] Xiaoxin Yang,[4] Yi Liu ,[1,*] Cong Xiao ,[4,†] and Zhe Yuan [5,6,‡]

[1]*Center for Advanced Quantum Studies and Department of Physics, Beijing Normal University, Beijing 100875, China*
[2]*School of Sciences, Great Bay University, Dongguan, Guangdong 523000, China*
[3]*Great Bay Institute for Advanced Study, Dongguan 523000, China*
[4]*Institute of Applied Physics and Materials Engineering, University of Macau, Taipa, Macau, China*
[5]*Institute for Nanoelectronic Devices and Quantum Computing, Fudan University, Shanghai 200433, China*
[6]*Interdisciplinary Center for Theoretical Physics and Information Sciences (ICTPIS), Fudan University, Shanghai 200433, China*

(Received 7 March 2024; revised 23 May 2024; accepted 27 May 2024; published 12 June 2024)

Nonlinear spin polarization occurring in the second order of driving electric current is the dominant source of nonequilibrium magnetization in centrosymmetric or weakly noncentrosymmetric nonmagnetic materials, and induces nonlinear spin-orbit torque in magnets. Up to now, only the intrinsic mechanism based on anomalous spin polarizability dipole, which is the spin counterpart of Berry curvature dipole, has been studied, while disorder-induced mechanisms are still missing. Here, we derive these contributions, which include not only the anomalous distribution function due to skew scattering and coordinate shift, but also interband coherence effects given by disorder-induced spin shift and electric-field-induced anomalous scattering amplitude. We demonstrate these terms and show their importance in a minimal model. A scaling law for nonlinear current-induced spin polarization is constructed, which may help analyze experimental data in the future.

DOI: 10.1103/PhysRevB.109.235413

## I. INTRODUCTION

Current-induced spin polarization (CISP) is a central effect in spintronics towards spin-charge conversion and electrical control of spin [1,2]. In linear response to the driving electric current, the effect was originally proposed in nonmagnetic materials [3–5], can only appear in noncentrosymmetric crystals [3,6], and has been observed by magneto-optical means [7,8]. The physics of this effect falls into the standard Boltzmann response framework [9], and is parallel to the Drude conductivity of charge-current response. Recently, spin response to the square of driving current was proposed [10], which can be the leading effect in nonmagnetic crystals where the inversion symmetry is maintained or not severely broken. It stems from an *anomalous spin* carried by spin-orbit-coupled Bloch electrons under electric field, which is determined by the momentum-space dipole of anomalous spin polarizability (ASP), a geometric quantity intrinsic to the band structure. This is a Berry-phase effect and is exactly the spin counterpart of the widely studied nonlinear Hall effect induced by Berry curvature dipole [11,12].

As the nonlinear Hall effect receives significant disorder-induced contributions other than the Berry curvature dipole [13–15], one naturally asks about the role of disorder in nonlinear CISP. Moreover, it is anticipated that the interplay of the ASP-dipole intrinsic and disorder-induced extrinsic contributions can be manifested via tuning system parameters such as the temperature and gate, thus some scaling law [13–25] is highly desired for understanding experimental observations of nonlinear CISP. Despite the above importance, the extrinsic nonlinear CISP has not been studied.

In this work, we develop the semiclassical theory of extrinsic contributions to the nonlinear CISP, and derive systematic formulas for different terms. The focus is on the time-reversal ($\mathcal{T}$) even effect, which is allowed in both nonmagnetic and magnetic systems [10,26]. We find that skew scattering and coordinate shift, which play basic roles in anomalous Hall effect [27], also matter for nonlinear spin response. Besides, the electric field $E$ alters the scattering amplitude, which induces interband coherence during scattering. This effect is dubbed, following the terminology of anomalous velocity that arises from field-induced interband coherence in drift motion [28,29], as the anomalous scattering amplitude. These three mechanisms take action in the off-equilibrium electronic distribution function. In addition, the scattering potential dresses the Bloch state, leading to an interband coherence correction to the spin carried by a particular electron, i.e., a spin shift induced by scattering. We illustrate the extrinsic nonlinear CISP arising from these mechanisms in a minimal model, and find that they are in the same order of magnitude as the ASP dipole contribution [10]. We also construct the scaling law for the phenomenon of nonlinear CISP.

Our paper is organized as follows. We present the spin shift mechanism in Sec. II and emphasize the importance of field-induced anomalous scattering amplitude in Sec. III. We show the model calculation of extrinsic contributions to nonlinear CISP in Sec. IV and present the scaling law in Sec. V. In Sec. VI, we make some discussions and conclude

*Contact author: yiliu@bnu.edu.cn
†Contact author: xiaoziche@gmail.com
‡Contact author: yuanz@fudan.edu.cn

 

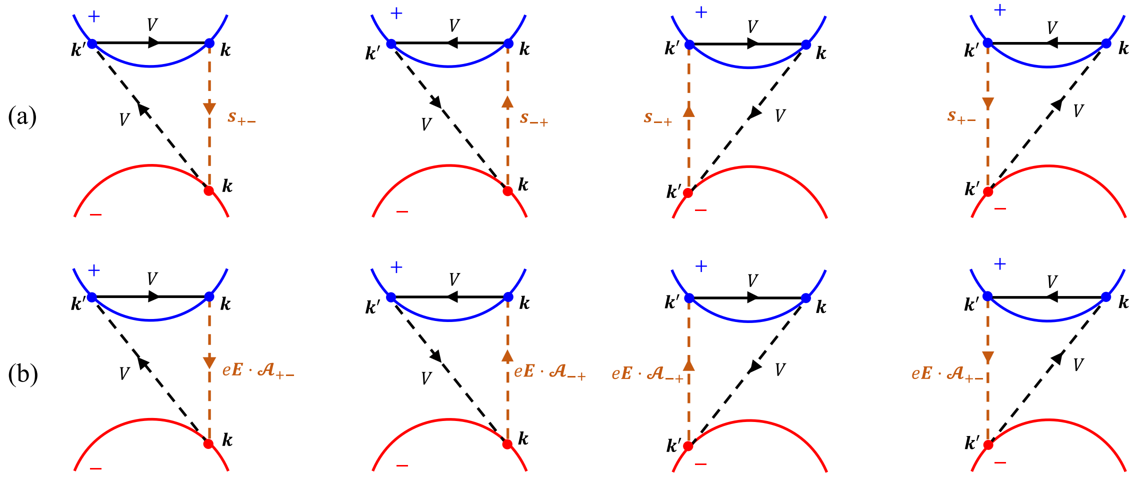


FIG. 1. Schematics of intraband and interband transition processes involved in (a) disorder-induced spin shift and (b) $E$-field-induced anomalous scattering amplitude.

our paper. The Appendixes contain the details of formulations and calculations.

## II. DISORDER-INDUCED SPIN SHIFT

The semiclassical response theory is a convenient tool for approaching $E$-field driven nonequilibrium phenomena. It works well when the disorder-induced band broadening $\hbar/\tau$ ($\tau$ is the carrier scattering time) is much less than the band-energy separation around Fermi surface [30]. It has been successfully applied to account for extrinsic contributions in anomalous and spin Hall effects [31–34], linear spin-orbit torque [35], as well as nonlinear Hall effects [13,36–39]. Within the semiclassical formalism, the spin density is given by the summation of the spin polarization $\boldsymbol{s}_l$ carried by each electron weighted by the distribution function $f_l$:

$$\boldsymbol{S} = \frac{1}{\mathcal{V}}\sum_l f_l \boldsymbol{s}_l. \tag{1}$$

Here, $\mathcal{V}$ represents the volume of the system, and $l = (\eta, \boldsymbol{k})$ corresponds to the band index $\eta$ and the wave vector $\boldsymbol{k}$, respectively.

From the study of anomalous Hall effect, we know that both electric field and scattering potential can polarize the Bloch state, hence alter the expectation value of an observable on that state [32,33]. Here, in the presence of scattering and $E$ field, the spin polarization carried by a particular electron is given by

$$\boldsymbol{s}_l = \boldsymbol{s}_l^0 + \boldsymbol{s}_l^{\mathrm{a}} + \boldsymbol{s}_l^{\mathrm{ss}}, \tag{2}$$

where $\boldsymbol{s}_l^0$ denotes the expectation value of spin operator on the unperturbed Bloch state $|\eta\boldsymbol{k}\rangle$, $\boldsymbol{s}_l^{\mathrm{a}}$ and $\boldsymbol{s}_l^{\mathrm{ss}}$ arise from field- and scattering-perturbed electronic state, respectively. Their expressions can be acquired by the recently developed semiclassical approach for evaluating observables other than electric current [10,35,40]. In particular,

$$\left(\boldsymbol{s}_l^{\mathrm{a}}\right)_\alpha = -\frac{e}{\hbar}(\Upsilon_l)_{\alpha\beta}E_\beta \tag{3}$$

shares the same origin as the $E$-field-induced anomalous velocity [29], hence is dubbed as the anomalous spin [10]. Here and hereafter, the summation over repeated Cartesian indices $\alpha, \beta$.. is implied. The rank-2 tensor

$$\Upsilon_{\eta\boldsymbol{k}} = -2\hbar^2\,\mathrm{Im}\sum_{\eta'\neq\eta}\frac{\boldsymbol{s}_{\eta\eta'}(\boldsymbol{k})\boldsymbol{v}_{\eta'\eta}(\boldsymbol{k})}{(\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta'\boldsymbol{k}})^2} \tag{4}$$

is the anomalous spin polarizability (ASP). In Eq. (4), $\varepsilon_{\eta\boldsymbol{k}}$ is the band energy, and the numerator involves the interband matrix elements of spin and velocity operators. On the other hand,

$$\begin{aligned}\boldsymbol{s}_l^{\mathrm{ss}} =& -2\pi\sum_{\eta'\boldsymbol{k}'}W^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'}\delta(\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta'\boldsymbol{k}'})\\ &\times\mathrm{Im}\left[\sum_{\eta''\neq\eta'}\frac{\langle u_{\eta\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\boldsymbol{s}_{\eta'\eta''}(\boldsymbol{k}')\langle u_{\eta''\boldsymbol{k}'}|u_{\eta\boldsymbol{k}}\rangle}{\varepsilon_{\eta'\boldsymbol{k}'}-\varepsilon_{\eta''\boldsymbol{k}'}}\right.\\ &\left.-\sum_{\eta''\neq\eta}\frac{\langle u_{\eta''\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\langle u_{\eta'\boldsymbol{k}'}|u_{\eta\boldsymbol{k}}\rangle\boldsymbol{s}_{\eta\eta''}(\boldsymbol{k})}{\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta''\boldsymbol{k}}}\right]\end{aligned} \tag{5}$$

characterizes an effective spin shift due to scattering-induced interband coherence processes [40] [see schematics in Fig. 1(a)]. Here, $W^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'} = W^{\mathrm{o}}_{\boldsymbol{k}',\boldsymbol{k}}$ is the plane-wave part of the Born scattering amplitude, and we assume scalar disorder for concreteness. In the case of static impurity, one has $W^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'} = \langle|V^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'}|^2\rangle_c$, where $\langle\ldots\rangle_c$ indicates average over random impurity configuration, and $V^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'}$ is the plane-wave part of the scattering matrix element.

It is interesting to note that the appearance of disorder-induced spin shift is also connected to the band geometric quantity ASP. This connection can be made explicit by considering scattering in the long-range limit. In this limit, in Eq. (5), $\boldsymbol{k}$ is very close to $\boldsymbol{k}'$, hence $\eta'$ is forced to be equal to $\eta$. Then, expanding the integrand of Eq. (5) up to the first order of $\boldsymbol{k}'-\boldsymbol{k}$, we get

$$\left(\boldsymbol{s}_l^{\mathrm{ss}}\right)_\alpha = (\Upsilon_l)_{\alpha\beta}\sum_{\boldsymbol{k}'}\tilde{\omega}^{(2)}_{\boldsymbol{k}\boldsymbol{k}'}(k_\beta - k'_\beta) \simeq (\Upsilon_l)_{\alpha\beta}k_\beta/\tau, \tag{6}$$

where $\tilde{\omega}^{(2)}_{\boldsymbol{k}\boldsymbol{k}'} = \frac{2\pi}{\hbar}\langle|V^{\mathrm{o}}_{\boldsymbol{k},\boldsymbol{k}'}|^2\rangle_c\delta(\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta\boldsymbol{k}'})$ is the scattering rate for long-ranged disorder in the lowest Born order. The integration in Eq. (6) indicates the momentum relaxation $k_\beta/\tau$ within the momentum relaxation time ($\tau$) approximation.

The link between $\boldsymbol{s}_l^{\rm ss}$ and ASP makes it convenient to compare the relative importance of field-induced anomalous spin and disorder-induced spin shift. For electrons around the Fermi surface, the relative ratio of the two takes the form of

$$\frac{(\boldsymbol{s}_l^{\rm a})_\alpha}{(\boldsymbol{s}_l^{\rm ss})_\alpha} \sim \frac{-eE\tau}{\hbar k_F}. \tag{7}$$

Here $-eE\tau$ measures the shift of Fermi surface in momentum space, which is usually much less than the Fermi momentum $\hbar k_F$ [41].

Although $\boldsymbol{s}^{\rm ss}$ is much larger than $\boldsymbol{s}^{\rm a}$ on the Fermi surface, their contributions to macroscopic nonlinear spin response are anticipated to be generally in the same order of magnitude. To see this, we inspect the semiclassical Boltzmann equation that describes the distribution function $f_l$ of electrons:

$$\frac{e}{\hbar}\boldsymbol{E}\cdot\partial_{\boldsymbol{k}} f_l = -\sum_{l'}(\omega_{l'l} f_l - \omega_{ll'} f_{l'}). \tag{8}$$

The right-hand side is the collision integral, where $\omega_{l'l}$ is the scattering rate from state $l$ to $l'$. $f_l$ can be solved in ascending powers of $E$ field:

$$f_l = f_{0,l} + f_{1,l} + f_{2,l}, \tag{9}$$

where $f_{n,l}$ is the distribution function in the $E^n$ order. Under the relaxation-time approximation, one has $f_{2,l}/f_{1,l} \sim -eE\tau/\hbar k_F$. Then, according to Eq. (1), the ASP and spin-shift contributions to spin response in the $E^2$ order are given by $\sum_l f_{1,l}\boldsymbol{s}_l^{\rm a}$ and $\sum_l f_{2,l}\boldsymbol{s}_l^{\rm ss}$, respectively, and are of the same order of magnitude.

The above qualitative analysis shows that the extrinsic contribution to nonlinear spin from the spin-shift mechanism is comparable to the intrinsic ASP term in the case of smooth disorder potential. In the later quantitative calculation on a minimal model, we show the same conclusion for short-ranged disorder (see the blue and red curves in Fig. 2). One can thus expect that for nonlinear spin, the extrinsic and intrinsic contributions are in general both important.

## III. FIELD-INDUCED ANOMALOUS SCATTERING AMPLITUDE

From the study of nonlinear Hall effect [15], it is known that the skew scattering and the field effect during scattering, which are beyond the above simple relaxation-time approximation, can also contribute to nonlinear response by altering the distribution function. The skew scattering stems from higher-order Born expansions of the scattering rate, including the non-Gaussian conventional skew scattering and the Gaussian skew scattering [32,42–45]. The field effect during scattering consists of not only the electric field working upon the coordinate shift process, which has been known for a long time as a part of the side-jump mechanism for linear anomalous Hall effect [27,32], but also the field-corrected scattering amplitude, which is unique to nonlinear response and starts to contribute from the $E^2$ order [38].

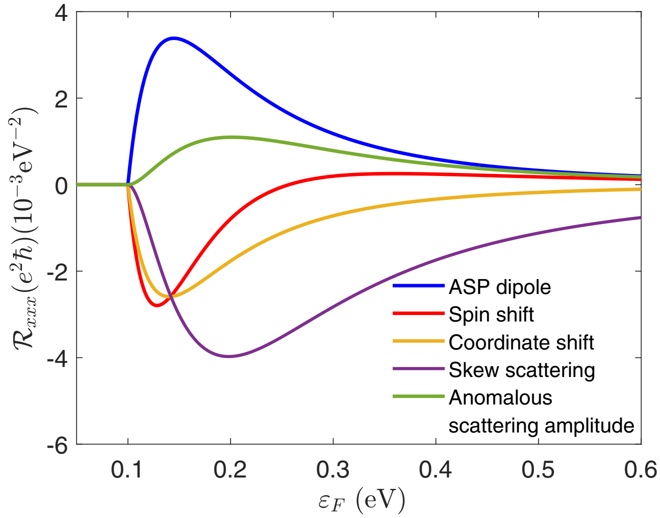


FIG. 2. The second-order nonlinear CISP of model (12), plotted according to the expressions in Table. I. The total skew scattering comes from the sum of Gaussian and non-Gaussian terms. Parameters are chosen as $w = 0.1$ eV Å, $v = 1$ eV Å, $\Delta = 0.1$ eV, $n_iV_0^2 = 10^2$ (eV Å)$^2$ and $n_iV_1^3 = 10^4$ eV$^3$ Å$^4$.

As the skew scattering and coordinate shift are well known, the pertaining formulation is relegated to the Appendixes. As for the relatively new nonlinear contribution from field-corrected scattering amplitude, it has thus far often been regarded as another kind of side-jump contribution, although its physical origin and picture are not related to any side jump of electron. Here, considering its origin in the field-polarized Bloch state, which also underlies the anomalous velocity and anomalous spin of Bloch electrons [10], we dub this contribution as the anomalous scattering amplitude. In the case of scalar disorder, the corresponding change of scattering rate in the lowest Born order is given by $\omega_{l'l}^{(2),\rm asa} = \omega_{ll'}^{(2),\rm asa} = \frac{2\pi}{\hbar}W_{l'l}^{\rm asa}\delta(\varepsilon_l - \varepsilon_{l'})$, where the anomalous scattering amplitude reads

$$W_{l'l}^{\rm asa} = W_{\boldsymbol{k},\boldsymbol{k}'}^{\rm o}(-e\boldsymbol{E})\cdot 2\,{\rm Re}\left[\sum_{\eta''\neq\eta'}\frac{\langle u_{\eta\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\boldsymbol{\mathcal{A}}_{\eta'\eta''}(\boldsymbol{k}')\langle u_{\eta''\boldsymbol{k}'}|u_{\eta\boldsymbol{k}}\rangle}{\varepsilon_{\eta'\boldsymbol{k}'}-\varepsilon_{\eta''\boldsymbol{k}'}} + \sum_{\eta''\neq\eta}\frac{\langle u_{\eta\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\langle u_{\eta'\boldsymbol{k}'}|u_{\eta''\boldsymbol{k}}\rangle\boldsymbol{\mathcal{A}}_{\eta''\eta}(\boldsymbol{k})}{\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta''\boldsymbol{k}}}\right], \tag{10}$$

with $\boldsymbol{\mathcal{A}}_{\eta'\eta}(\boldsymbol{k}) = \langle u_{\eta'\boldsymbol{k}}|i\partial_{\boldsymbol{k}}|u_{\eta\boldsymbol{k}}\rangle$ being the interband Berry connection. The schematics of intraband and interband transition processes involved in $W_{l'l}^{\rm asa}$ are shown in Fig. 1(b). Comparing the $E$-field-induced anomalous scattering amplitude Eq. (10) [Fig. 1(b)] with the disorder induced spin shift Eq. (5) [Fig. 1(a)], one observes interesting structural similarity.

One may immediately ask why this anomalous scattering amplitude has not been found in any linear response of current

TABLE I. Mechanisms and expressions for extrinsic contributions to the nonlinear CISP response tensor of model (12). Short-range random scalar impurity potential $V(\boldsymbol{r}) = \sum_i V_i \delta(\boldsymbol{r} - \boldsymbol{R}_i)$ is considered, with $\langle V_i \rangle_c = 0$, $\langle V_i^2 \rangle_c = V_0^2$, and $\langle V_i^3 \rangle_c = V_1^3$, and $n_i$ is the density of impurities.

| Mechanism | Nonlinear response coefficient |
|---|---|
| Anomalous spin polarizability dipole | $\mathcal{R}_{xxx}^{\text{asp}} = -\frac{3e^2\hbar v w \Delta(\Delta^2-\varepsilon_F^2)}{2\pi n_i V_0^2 \varepsilon_F^3 (3\Delta^2+\varepsilon_F^2)}$ |
| Spin shift | $\mathcal{R}_{xxx}^{\text{ss}} = \frac{e^2\hbar v w \Delta(\Delta^2-\varepsilon_F^2)(33\Delta^2-5\varepsilon_F^2)}{4\pi n_i V_0^2 \varepsilon_F^3 (3\Delta^2+\varepsilon_F^2)^2}$ |
| Coordinate shift | $\mathcal{R}_{xxx}^{\text{cs}} = \frac{e^2\hbar v w \Delta(\Delta^2-\varepsilon_F^2)(17\Delta^2+3\varepsilon_F^2)}{4\pi n_i V_0^2 \varepsilon_F^3 (3\Delta^2+\varepsilon_F^2)^2}$ |
| Anomalous scattering amplitude | $\mathcal{R}_{xxx}^{\text{asa}} = \frac{3e^2\hbar v w \Delta(\Delta^2-\varepsilon_F^2)^2}{2\pi n_i V_0^2 \varepsilon_F^3 (3\Delta^2+\varepsilon_F^2)^2}$ |
| Conventional skew scattering | $\mathcal{R}_{xxx}^{\text{csk}} = -\frac{e^2\hbar v w V_1^3 \Delta(\Delta^2-\varepsilon_F^2)^2(9\Delta^2+5\varepsilon_F^2)}{\pi \varepsilon_F^2 n_i^2 V_0^6 (3\Delta^2+\varepsilon_F^2)^3}$ |
| Gaussian skew scattering | $\mathcal{R}_{xxx}^{\text{Gsk}} = -\frac{e^2\hbar v w \Delta(\Delta^2-\varepsilon_F^2)^2(77\Delta^2+13\varepsilon_F^2)}{4\pi n_i V_0^2 \varepsilon_F^3 (3\Delta^2+\varepsilon_F^2)^3}$ |

and spin. In fact, substituting $\omega_{l'l}^{(2),\text{asa}}$ into the collision integral of Boltzmann equation (8), one finds that the pertaining term vanishes at the first order of $E$ field. This means that the contribution from anomalous scattering amplitude is a purely nonlinear response phenomenon. In particular, in the $E^2$ order, the anomalous scattering amplitude gives rise to an effective driving term in the Boltzmann equation, which reads

$$\sum_{l'} \omega_{l'l}^{(2),\text{asa}} (f_{1,l} - f_{1,l'}) = -\sum_{l'} \omega_{l'l}^{(2)} \big(f_{2,l}^{\text{asa}} - f_{2,l'}^{\text{asa}}\big). \tag{11}$$

The solution of this equation, $f_{2,l}^{\text{asa}}$, yields an additional distribution function in the $E^2$ order.

## IV. MODEL CALCULATION

Gathering the aforementioned ingredients, one can get several extrinsic contributions to the $\mathcal{T}$-even nonlinear CISP by Eq. (1), with the detailed formulation presented in the Appendixes. To illustrate these contributions, we apply the theory to a four-band $\boldsymbol{k} \cdot \boldsymbol{p}$ model with inversion symmetry

$$H(\boldsymbol{k}) = w k_x \sigma_z + v(k_x s_y - k_y s_x)\sigma_z + \Delta s_z, \tag{12}$$

where $s_i$'s and $\sigma_i$'s are the Pauli matrices representing the spin and orbital degrees of freedom, respectively; $\boldsymbol{k} = (k_x, k_y)$ is the wave vector; $w$, $v$, and $\Delta$ are the model parameters. $w$ tilts the Dirac cone along the $x$ direction. This model consists of two copies of tilted Weyl models connected by the inversion operation. It is defined around one valley in the Brillouin zone of nonmagnetic systems, whereas its time-reversed counterpart can be written down for the other valley. As we are considering a $\mathcal{T}$-even response, the existence of $\mathcal{T}$-connected two valleys simply doubles the result. Furthermore, regarding symmetry, the presence of inversion forbids the linear CISP, and the $\mathcal{M}_x\mathcal{T}$ symmetry of $H(\boldsymbol{k})$ ensures that for the second-order nonlinear response defined by

$$S_\alpha = \mathcal{R}_{\alpha\beta\gamma} E_\beta E_\gamma, \tag{13}$$

only $\mathcal{R}_{xxx}$, $\mathcal{R}_{xyy}$, $\mathcal{R}_{y(xy)}$, and $\mathcal{R}_{z(xy)}$ are allowed. Here $\mathcal{R}_{z(xy)} \equiv (\mathcal{R}_{zxy} + \mathcal{R}_{zyx})/2$. For illustrative purpose, we calculate $\mathcal{R}_{xxx}$ in the following.

We consider short-range random impurity potential $V(\boldsymbol{r}) = \sum_i V_i \delta(\boldsymbol{r} - \boldsymbol{R}_i)$, with $\langle V_i \rangle_c = 0$, $\langle V_i^2 \rangle_c = V_0^2$ and $\langle V_i^3 \rangle_c = V_1^3$, and solve the Boltzmann equation with skew scattering, coordinate shift, and anomalous scattering amplitude, following the method and approximations adopted in Refs. [13,36–38]. In particular, to obtain analytic result, we assume $w \ll v$. The noncrossing approximation is employed to simplify calculations of the Gaussian skew scattering. Although the Gaussian skew scattering from crossed disorder lines is expected to be quantitatively important in general, as in the anomalous Hall effect [42–45], we can take the noncrossing approximation because the main purpose here is to illustrate the existence and importance of extrinsic contributions to nonlinear CISP. Moreover, in qualitative aspect, the crossed Gaussian skew scattering has the same scaling behavior as its noncrossed companion.

The expressions of $\mathcal{R}_{xxx}$ induced by different semiclassical mechanisms are shown in Table I. The calculation details are provided in the Appendixes. One sees that the common factor $(\Delta^2 - \varepsilon_F^2)$ in the numerator of all contributions ensures the vanishing of each term at the band edge. The tilt term $w$ is crucial, because it breaks out-of-plane rotational axis that would otherwise forbid the in-plane response. In Fig. 2, we plot different contributions as a function of the Fermi energy, and find them to be comparable in magnitude.

## V. SCALING LAW OF THE NONLINEAR CISP

Because of the coexistence of nonlinear CISP from multiple origins, it is helpful to have some guidelines for understanding experimental data. In this regard, the scaling law between the detected spin signal and longitudinal resistivity (conductivity) may render useful information, which has been shown in anomalous and spin Hall effects [17,18], spin-orbit torque [2], as well as nonlinear Hall effect [13,14]. Given the parallel formulation of nonlinear CISP and nonlinear Hall effect, they should possess the same scaling law (one can readily check this). The key reason for this coincidence is that the scaling law does not rely on details of individual contributions to spin or current but only on their scaling forms with respect to the disorder concentration and strength. More formally, both these two can be obtained from $A = \frac{1}{\mathcal{V}} \sum_l f_l a_l$ where $a_l = e v_l$ for electric current and $a_l = s_l$ for spin polarization. The skew scattering, coordinate shift, and anomalous scattering amplitude mechanisms contribute to response through the distribution function $f_l$, which are the same for both responses. Hence these mechanisms have the same scaling law. Furthermore, both the anomalous velocity (corresponding to Berry curvature dipole) and anomalous spin (corresponding to ASP dipole) come from the $E$-field-induced modification of Bloch state, and hence they also obey the same scaling law. Besides, the side-jump velocity and spin shift are both from disorder-induced dressing of Bloch state, thus have the same scaling dependence on disorder. For instance, in the presence of two types of static disorder, in which one is impurity ($i = 0$) and the other is phonon ($i = 1$) [13,17], one has

$$\mathcal{R}\rho = C + A_0\frac{\rho_0}{\rho^2} + \sum_{i=0,1} C_i\frac{\rho_i}{\rho} + \sum_{i,j=0,1} C_{ij}\frac{\rho_i\rho_j}{\rho^2}, \tag{14}$$

where $\rho = \rho_0 + \rho_1$ is the longitudinal resistivity, and $\rho_0$ is the residual resistivity. The scaling parameter $C$ stands for the ASP dipole contribution, $A_0$ and $C_{ij}$ are from conventional and Gaussian skew scattering, respectively, and $C_i = C_i^{\rm ss} + C_i^{\rm cs} + C_i^{\rm asa}$ from the spin shift, coordinate shift, as well as the anomalous scattering amplitude. The scaling law can also be expressed in terms of longitudinal conductivity ($\sigma \simeq 1/\rho, \sigma_0 \simeq 1/\rho_0$):

$$\mathcal{R}/\sigma - A_0\sigma^2/\sigma_0 = B + B'\sigma/\sigma_0 + B''(\sigma/\sigma_0)^2, \tag{15}$$

where $B = C + C_1 + C_{11}$, $B' = C_0 - C_1 + C_{01} + C_{10} - 2C_{11}$, and $B'' = C_{00} + C_{11} - C_{01} - C_{10}$. Note that the static approximation of electron-phonon scattering is practically valid as long as $\rho$ has a nearly linear temperature dependence [19], and this behavior usually extends to quite low temperatures in moderately disordered samples fabricated in most spintronics experiments [2].

At low temperatures where $\sigma \simeq \sigma_0$, scaling Eq. (15) becomes $\mathcal{R}/\sigma = A_0\sigma_0 + C + C_0 + C_{00}$, by which $A_0$ can be determined in experiments. Then, at finite temperatures, $\mathcal{R}/\sigma - A_0\sigma^2/\sigma_0$ can be fitted as a parabolic function of $\sigma/\sigma_0$. Noticeably, the linear and quadratic terms in this fitting can only arise from extrinsic mechanisms.

## VI. DISCUSSION

We have shown the importance of extrinsic contributions to nonlinear CISP by semiquantitative analysis and quantitative model calculations. We highlight the nonlinear responses from disorder-induced spin shift and field-induced anomalous scattering amplitude, which have received little attention in previous studies of spintronics and nonlinear electronics. The proposed scaling law is expected to serve as a first step to understand the interplay of intrinsic and extrinsic contributions in experimental data.

Besides nonlinear CISP, the $\mathcal{T}$-even nonlinear current-induced orbital magnetization has also received recent interest [46]. Despite the complexity of accurately formulating the nonequilibrium orbital magnetization in metals introduced by the nonlocality of orbital magnetic dipole operator [47], the scaling law as a qualitative result should still be the same as that for nonlinear CISP.

In this work we focused on $\mathcal{T}$-even nonlinear CISP, thus the scaling law obtained is the same as that for $\mathcal{T}$-even nonlinear charge current response [13]. In magnetic systems, $\mathcal{T}$-odd nonlinear CISP can also occur [48,49], and the pertinent scaling law is more involved, which is the same as that for $\mathcal{T}$-odd nonlinear charge transport [39].

When employing the scaling law, temperature varying measurements have been usually taken to tune the relative proportions of electron-phonon and electron-impurity scattering [13–25]. However, Joule heating will alter the practical temperature of electron and phonon systems, and hence influences the scaling analysis. The subtraction of Joule heating effect in analyzing nonlinear responses is thus a question that deserves thorough studies in the future.

## ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grants No. 12174028 and No. 12374101), and UM Start-up Grant (Grant No. SRG2023-00033-IAPME).

## APPENDIX A: NONLINEAR SPIN RESPONSE FORMULATION

According to the Fermi golden rule, the scattering rate in Eq. (8) in the main text is

$$\omega_{l'l} = \frac{2\pi}{\hbar}|T_{l'l}|^2\delta(\epsilon_l - \epsilon_{l'}). \tag{A1}$$

Here, $T_{l'l}$ which is known as the T-matrix is defined as

$$T_{l'l} = \langle l'|\hat{V}_{\rm imp}|\Psi_l\rangle, \tag{A2}$$

where $\hat{V}$ is the impurity potential operator and $|\Psi_l\rangle$ is the eigenstate of the full Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$. $|\Psi_l\rangle$ satisfies the Lippman-Schwinger equation

$$|\Psi_l\rangle = |l\rangle + \frac{\hat{V}_{\rm imp}}{\varepsilon_l - \hat{H}_0 + i\delta}|\Psi_l\rangle. \tag{A3}$$

We consider weak disorder here and expand the scattering rate up to the fourth order of $V$

$$\omega_{l'l} \approx \omega_{l'l}^{(2)} + \omega_{l'l}^{(3a)} + \omega_{l'l}^{(4a)}, \tag{A4}$$

where $\omega_{l'l}^{(3a)}$ and $\omega_{l'l}^{(4a)}$ are the asymmetric parts of the third- and fourth-order scattering rate respectively. Here we omit the symmetric parts of both of them because they only renormalize the second-order result. In the nonlinear response of spin to the electric field, we need to consider the effect of the $\boldsymbol{E}$ field during the scattering of semiclassical electron. We only consider the modification to the second-order of $V$ part and ignore the mixed contributions from different mechanisms. Hence, $\omega_{l'l}^{(2)}$ can be written as [38]

$$\omega_{l'l}^{(2)} = \omega_{l'l}^{(2s)} + \omega_{l'l}^{(2),\rm cs} + \omega_{l'l}^{(2),\rm asa}. \tag{A5}$$

The first term on the right side of Eq. (A5)

$$\omega_{l'l}^{(2s)} = \frac{2\pi}{\hbar}\langle|V_{l'l}|^2\rangle_c\delta(\varepsilon_l - \varepsilon_{l'}) \tag{A6}$$

is $\boldsymbol{E}$-field independent term, where $\langle\cdots\rangle_c$ stands for disorder average. And the second term

$$\omega_{l'l}^{(2),\rm cs} = \frac{2\pi}{\hbar}\langle|V_{l'l}|^2\rangle_c\frac{\partial\delta(\varepsilon_l - \varepsilon_{l'})}{\partial\varepsilon_l}e\boldsymbol{E}\cdot\delta\boldsymbol{r}_{l'l} \tag{A7}$$

arises from the coordinate shift during scattering, where

$$\delta \boldsymbol{r}_{l'l} = \boldsymbol{\mathcal{A}}_{l'} - \boldsymbol{\mathcal{A}}_{l} - (\partial_{\boldsymbol{k}} + \partial_{\boldsymbol{k}'})\arg(V_{l'l}) \tag{A8}$$

and $\boldsymbol{\mathcal{A}}_l = \langle u_{\eta\boldsymbol{k}}|i\partial_{\boldsymbol{k}}|u_{\eta\boldsymbol{k}}\rangle$ is the intraband Berry connection. The third term originating from interband virtual transition only contributes to nonlinear responses, that is

$$\omega_{l'l}^{(2),\mathrm{asa}} = \frac{2\pi}{\hbar} W_{l'l}^{\mathrm{asa}} \delta(\varepsilon_l - \varepsilon_{l'}), \tag{A9}$$

$$W_{l'l}^{\mathrm{asa}} = -e\boldsymbol{E} \cdot \sum_{l''}{}' 2\,\mathrm{Re}\left\langle \frac{V_{ll'}V_{l'l''}\boldsymbol{\mathcal{A}}_{l''l}}{\varepsilon_l - \varepsilon_{l''}} + \frac{V_{ll'}\boldsymbol{\mathcal{A}}_{l'l''}V_{l''l}}{\varepsilon_{l'} - \varepsilon_{l''}} \right\rangle, \tag{A10}$$

where $\boldsymbol{\mathcal{A}}_{l'l} = \langle u_{\eta'\boldsymbol{k}}|i\partial_{\boldsymbol{k}}|u_{\eta\boldsymbol{k}}\rangle$ is the interband Berry connection. Substituting Eqs. (A4) and (A5) into Eq. (8) in the main text, we can decompose the Boltzmann equation into five equations based on the different scattering mechanisms. The distribution function can be decomposed as

$$f_{n,l} = f_{n,l}^{\mathrm{L}} + f_{n,l}^{\mathrm{cs}} + f_{n,l}^{\mathrm{asa}} + f_{n,l}^{\mathrm{csk}} + f_{n,l}^{\mathrm{Gsk}} \tag{A11}$$

corresponding to the following five equations:

$$-\frac{e}{\hbar}\boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{L}} = \sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{\mathrm{L}} - f_{n,l'}^{\mathrm{L}}\big), \tag{A12}$$

$$-\frac{e}{\hbar}\boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{cs}} - \sum_{l'} \omega_{l'l}^{(2),\mathrm{cs}}\big(f_{n-1,l}^{\mathrm{L}} - f_{n-1,l'}^{\mathrm{L}}\big) = \sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{\mathrm{cs}} - f_{n,l'}^{\mathrm{cs}}\big), \tag{A13}$$

$$-\frac{e}{\hbar}\boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{asa}} - \sum_{l'} \omega_{l'l}^{(2),\mathrm{asa}}\big(f_{n-1,l}^{\mathrm{L}} - f_{n-1,l'}^{\mathrm{L}}\big) = \sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{\mathrm{asa}} - f_{n,l'}^{\mathrm{asa}}\big), \tag{A14}$$

$$-\frac{e}{\hbar}\boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{csk}} = \sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{\mathrm{csk}} - f_{n,l'}^{\mathrm{csk}}\big) + \sum_{l'} \big(\omega_{l'l}^{(3a)} f_{n,l}^{\mathrm{L}} - \omega_{ll'}^{(3a)} f_{n,l'}^{\mathrm{L}}\big), \tag{A15}$$

and

$$-\frac{e}{\hbar}\boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{Gsk}} = \sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{\mathrm{Gsk}} - f_{n,l'}^{\mathrm{Gsk}}\big) + \sum_{l'} \big(\omega_{l'l}^{(4a)} f_{n,l}^{\mathrm{L}} - \omega_{ll'}^{(4a)} f_{n,l'}^{\mathrm{L}}\big). \tag{A16}$$

$f_{n,l}^{\mathrm{L}}$ arises from conventional symmetric scattering, and $f_{0,l}^{\mathrm{L}}$ is just Fermi distribution. $f_{n,l}^{\mathrm{cs}}$ and $f_{n,l}^{\mathrm{asa}}$ arise from the modification of the scattering rate by the electric field. $f_{n,l}^{\mathrm{csk}}$ and $f_{n,l}^{\mathrm{Gsk}}$ correspond to the conventional non-Gaussian disorder contribution and Gaussian disorder contribution, respectively. All contributions except $f_{n,l}^{\mathrm{L}}$ are zero when $n = 0$. In particular, $f_{1,l}^{\mathrm{asa}} = 0$, hence it does not contribute to the linear response. To solve the decomposed Boltzmann equations Eqs. (A12)–(A16), we take the relaxation-time approximation [13] that for $n > 0$

$$\sum_{l'} \omega_{l'l}^{(2s)}\big(f_{n,l}^{i} - f_{n,l'}^{i}\big) = \frac{f_{n,l}^{i}}{\tau_l}, \tag{A17}$$

where the superscript $i$ represents any of the contributions mentioned above. Then we can obtain the iterative form for the distribution functions:

$$f_{n,l}^{\mathrm{L}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l,}^{\mathrm{L}} \tag{A18}$$

$$f_{n,l}^{\mathrm{cs}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{cs}} - \tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{cs}}\big(f_{n-1,l}^{\mathrm{L}} - f_{n-1,l'}^{\mathrm{L}}\big), \tag{A19}$$

$$f_{n,l}^{\mathrm{asa}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{asa}} - \tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{asa}}\big(f_{n-1,l}^{\mathrm{L}} - f_{n-1,l'}^{\mathrm{L}}\big), \tag{A20}$$

$$f_{n,l}^{\mathrm{csk}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{csk}} - \tau_l \sum_{l'} \big(\omega_{l'l}^{(3a)} f_{n,l}^{\mathrm{L}} - \omega_{ll'}^{(3a)} f_{n,l'}^{\mathrm{L}}\big), \tag{A21}$$

and

$$f_{n,l}^{\mathrm{Gsk}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{n-1,l}^{\mathrm{Gsk}} - \tau_l \sum_{l'} \big(\omega_{l'l}^{(4a)} f_{n,l}^{\mathrm{L}} - \omega_{ll'}^{(4a)} f_{n,l'}^{\mathrm{L}}\big). \tag{A22}$$

Theoretically, we can obtain distribution functions of any order. To obtain the second-order spin responses, we firstly consider the first-order distribution functions. We can easily obtain

$$f_{1,l}^{\mathrm{L}} = -\frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}}, \tag{A23}$$

$$f_{1,l}^{\mathrm{cs}} = -\tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{cs}}\big(f_{0,l}^{\mathrm{L}} - f_{0,l'}^{\mathrm{L}}\big), \tag{A24}$$

$$f_{1,l}^{\mathrm{asa}} = 0, \tag{A25}$$

$$f_{1,l}^{\mathrm{csk}} = \frac{e}{\hbar}\tau_l \sum_{l'} \omega_{l'l}^{(3a)}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}} + \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big), \tag{A26}$$

and

$$f_{1,l}^{\mathrm{Gsk}} = \frac{e}{\hbar}\tau_l \sum_{l'} \omega_{l'l}^{(4a)}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}} + \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big). \tag{A27}$$

One should note that $f_{1,l}^{\mathrm{asa}} = 0$ because $\delta(\varepsilon_l - \varepsilon_{l'})$ in $\omega_{l'l}^{(2),\mathrm{asa}}$ times $(f_{0,l}^{\mathrm{L}} - f_{0,l'}^{\mathrm{L}})$ is zero. Then, we substitute the first-order results back into Eqs. (A18)–(A22), obtaining

$$f_{2,l}^{\mathrm{L}} = \frac{e^2}{\hbar^2}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}}\big), \tag{A28}$$

$$f_{2,l}^{\mathrm{cs}} = \frac{e}{\hbar}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}}\left[\tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{cs}}\big(f_{0,l}^{\mathrm{L}} - f_{0,l'}^{\mathrm{L}}\big)\right] + \frac{e}{\hbar}\tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{cs}}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}} - \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big), \tag{A29}$$

$$f_{2,l}^{\mathrm{asa}} = \frac{e}{\hbar}\tau_l \sum_{l'} \omega_{l'l}^{(2),\mathrm{asa}}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}} - \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big), \tag{A30}$$

$$f_{2,l}^{\mathrm{csk}} = -\frac{e^2}{\hbar^2}\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}}\left[\tau_l \sum_{l'} \omega_{l'l}^{(3a)}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}} + \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big)\right] - \frac{e^2}{\hbar^2}\tau_l \sum_{l'} \omega_{l'l}^{(3a)}\big[\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}}\big(\tau_l \boldsymbol{E} \cdot \partial_{\boldsymbol{k}} f_{0,l}^{\mathrm{L}}\big) + \tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'}\big(\tau_{l'} \boldsymbol{E} \cdot \partial_{\boldsymbol{k}'} f_{0,l'}^{\mathrm{L}}\big)\big], \tag{A31}$$

and

$$f_{2,l}^{\rm Gsk} = -\frac{e^2}{\hbar^2}\tau_l \boldsymbol{E}\cdot\partial_{\boldsymbol{k}}\left[\tau_l\sum_{l'}\omega_{l'l}^{(4a)}\left(\tau_l\boldsymbol{E}\cdot\partial_{\boldsymbol{k}}f_{0,l}^{\rm L}+\tau_{l'}\boldsymbol{E}\cdot\partial_{\boldsymbol{k}'}f_{0,l'}^{\rm L}\right)\right] - \frac{e^2}{\hbar^2}\tau_l\sum_{l'}\omega_{l'l}^{(4a)}\left[\tau_l\boldsymbol{E}\cdot\partial_{\boldsymbol{k}}\left(\tau_l\boldsymbol{E}\cdot\partial_{\boldsymbol{k}}f_{0,l}^{\rm L}\right) + \tau_{l'}\boldsymbol{E}\cdot\partial_{\boldsymbol{k}'}\left(\tau_{l'}\boldsymbol{E}\cdot\partial_{\boldsymbol{k}'}f_{0,l'}^{\rm L}\right)\right]. \quad (A32)$$

For electron wave packet, the spin is not equal to the expectation of the spin operator when spin-orbit coupling exists. Both $\boldsymbol{E}$-field and disorder scattering can correct the spin carried by wave packet. The wave-packet spin induced by electric field called anomalous spin [10] is

$$\boldsymbol{s}_l^{\rm a} = -2\,{\rm Re}\left[\sum_{l'\neq l}\boldsymbol{s}_{\eta\eta'}(\boldsymbol{k})\frac{e\boldsymbol{E}\cdot\boldsymbol{\mathcal{A}}_{\eta'\eta}(\boldsymbol{k})}{\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta'\boldsymbol{k}}}\right]. \quad (A33)$$

And disorder scattering can also bring spin correction, called side-jump spin, is

$$\boldsymbol{s}_l^{\rm ss} = -2\pi\sum_{\eta'\boldsymbol{k}'}\left\langle\left|V_{\boldsymbol{k},\boldsymbol{k}'}^{\rm o}\right|^2\right\rangle_c\delta(\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta'\boldsymbol{k}'}) \times {\rm Im}\left[\sum_{\eta''\neq\eta'}\frac{\langle u_{\eta\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\langle u_{\eta''\boldsymbol{k}'}|u_{\eta\boldsymbol{k}}\rangle\boldsymbol{s}_{\eta'\eta''}(\boldsymbol{k}')}{\varepsilon_{\eta'\boldsymbol{k}'}-\varepsilon_{\eta''\boldsymbol{k}'}} - \sum_{\eta''\neq\eta}\frac{\langle u_{\eta''\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle\langle u_{\eta'\boldsymbol{k}'}|u_{\eta\boldsymbol{k}}\rangle\boldsymbol{s}_{\eta\eta''}(\boldsymbol{k})}{\varepsilon_{\eta\boldsymbol{k}}-\varepsilon_{\eta''\boldsymbol{k}}}\right], \quad (A34)$$

where $V_{k,k'}^0$ is the plane-wave part of $V_{l'l}$. Substituting Eq. (2) in the main text and decomposed distribution function Eq. (A11) into Eq. (1) in the main text, the spin density can be written as

$$\boldsymbol{s}_n = \frac{1}{\mathcal{V}}\sum_l\left(f_{n,l}^{\rm L}\boldsymbol{s}_l^0 + f_{n,l}^{\rm L}\boldsymbol{s}_l^{\rm ss} + f_{n-1,l}^{\rm L}\boldsymbol{s}_l^{\rm a} + f_{n,l}^{\rm csk}\boldsymbol{s}_l^0 + f_{n,l}^{\rm Gsk}\boldsymbol{s}_l^0 + f_{n,l}^{\rm cs}\boldsymbol{s}_l^0 + f_{n,l}^{\rm asa}\boldsymbol{s}_l^0\right). \quad (A35)$$

## APPENDIX B: TILTED 2D MODEL

### 1. Electronic Structure

Now consider the 2D model Eq. (12) in the main text. The band structure reads

$$\varepsilon_{\boldsymbol{k}}^{\pm} = wk_x \pm\sqrt{v^2k^2+m^2} \quad (B1)$$

with $k=\sqrt{k_x^2+k_y^2}$. And the corresponding eigenstates are

$$|\pm,\boldsymbol{k}\rangle = \frac{1}{\sqrt{\mathcal{V}}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|u_{\pm,\boldsymbol{k}}\rangle, \quad (B2)$$

where $\mathcal{V}$ is the volume of the system and the spinor part is

$$|u_{+,\boldsymbol{k}}\rangle = \begin{pmatrix} i\cos\dfrac{\theta}{2} \\ -\sin\dfrac{\theta}{2}e^{i\phi}\end{pmatrix}, \quad |u_{-,\boldsymbol{k}}\rangle = \begin{pmatrix} i\sin\dfrac{\theta}{2} \\ \cos\dfrac{\theta}{2}e^{i\phi}\end{pmatrix}. \quad (B3)$$

Here, we adopt

$$\cos\theta = \frac{\Delta}{\sqrt{v^2k^2+\Delta^2}}, \quad \tan\phi = \frac{k_y}{k_x} \quad (B4)$$

for simplicity. The group velocity is

$$v_{\pm,\boldsymbol{k}}^{0,x} = \frac{1}{\hbar}(w\pm v\sin\theta\cos\phi), \quad (B5)$$

$$v_{\pm,\boldsymbol{k}}^{0,y} = \frac{1}{\hbar}(\pm v\sin\theta\sin\phi). \quad (B6)$$

And the spin expectation of the spin operator is

$$\boldsymbol{s}_{\pm,\boldsymbol{k}}^0 = \pm\frac{\hbar}{2}(-\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta). \quad (B7)$$

The intraband Berry connections are

$$\mathcal{A}_{\boldsymbol{k}}^{x,+} = \frac{v\sin\phi\cos\theta\tan\left(\frac{\theta}{2}\right)}{2\Delta}, \quad (B8)$$

$$\mathcal{A}_{\boldsymbol{k}}^{y,+} = -\frac{v\cos\phi\cos\theta\tan\left(\frac{\theta}{2}\right)}{2\Delta}, \quad (B9)$$

$$\mathcal{A}_{\boldsymbol{k}}^{x,-} = \frac{v\sin\phi\cos\theta\cot\left(\frac{\theta}{2}\right)}{2\Delta}, \quad (B10)$$

$$\mathcal{A}_{\boldsymbol{k}}^{y,-} = -\frac{v\cos\phi\cos\theta\cot\left(\frac{\theta}{2}\right)}{2\Delta}. \quad (B11)$$

And interband Berry connections are

$$\mathcal{A}_{\boldsymbol{k}}^{x,+-} = -\frac{v\cos\theta(\sin\phi - i\cos\phi\cos\theta)}{2\Delta}, \quad (B12)$$

$$\mathcal{A}_{\boldsymbol{k}}^{y,+-} = \frac{v\cos\theta(\cos\phi + i\sin\phi\cos\theta)}{2\Delta}, \quad (B13)$$

$$\mathcal{A}_{\boldsymbol{k}}^{x,-+} = -\frac{v\cos\theta(\sin\phi + i\cos\phi\cos\theta)}{2\Delta}, \quad (B14)$$

$$\mathcal{A}_{\boldsymbol{k}}^{y,-+} = \frac{v\cos\theta(\cos\phi - i\sin\phi\cos\theta)}{2\Delta}. \quad (B15)$$

Then the anomalous spin for upper band is

$$\boldsymbol{s}_{+,\boldsymbol{k}}^{\rm a} = \frac{ev\hbar\cos^2\theta}{4\Delta^2}(E_x\cos\theta, E_y\cos\theta, E_x\sin\phi\sin\theta - E_y\cos\phi\sin\theta) \quad (B16)$$

and the Berry curvature for each band is

$$\Omega_{\boldsymbol{k}}^{\pm} = \mp\frac{v^2\cos^3\theta}{2\Delta^2}. \quad (B17)$$

### 2. Disorder Scattering

Assuming $\hat{V}_{\rm imp} = \sum_j V_j\delta(\boldsymbol{r}-\boldsymbol{P_j})$, the element of impurity potential is

$$V_{l,l'} = V_{\boldsymbol{k},\boldsymbol{k}'}^{\rm o}\langle u_{\eta\boldsymbol{k}}|u_{\eta'\boldsymbol{k}'}\rangle, \quad (B18)$$

where $V_{k,k'}^0 = \sum_j V_j e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{P}_j}/\mathcal{V}$. We assume that the Fermi level lies in the upper bands, then the second-order scattering rate is

$$\omega_{\boldsymbol{k}'\boldsymbol{k}}^{(2s)} = \frac{2\pi}{\hbar}\langle|V_{\boldsymbol{k}'\boldsymbol{k}}^{++}|^2\rangle_c\delta(\epsilon_l-\epsilon_{l'}) = \frac{\pi n_i V_0^2}{\hbar}[1+\cos\theta\cos\theta' + \cos(\phi-\phi')\sin\theta\sin\theta'] \times\delta(\varepsilon_{\boldsymbol{k}}^+ - \varepsilon_{\boldsymbol{k}'}^+). \quad (B19)$$

The asymmetric part of third-order scattering is

$$\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(3a)} = \frac{\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(3)} - \omega_{\boldsymbol{k}'\boldsymbol{k}}^{(3)}}{2} = \frac{\pi}{\hbar}\sum_{l''}\left\langle \frac{V_{\boldsymbol{k}\boldsymbol{k}'}^{++}V_{\boldsymbol{k}'\boldsymbol{k}''}^{+\eta''}V_{\boldsymbol{k}''\boldsymbol{k}}^{\eta''+} - V_{\boldsymbol{k}'\boldsymbol{k}}^{++}V_{\boldsymbol{k}\boldsymbol{k}''}^{+\eta''}V_{\boldsymbol{k}''\boldsymbol{k}'}^{\eta''+}}{\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}''}^{\eta''} - i\delta} + c.c.\right\rangle_c \delta(\varepsilon_{\boldsymbol{k}}^{+} - \varepsilon_{\boldsymbol{k}'}^{+})$$

$$\approx \frac{\pi\,\Delta n_i V_1{}^3}{4v^3\hbar}[2v\sin\theta\sin\theta'\sin(\phi-\phi') + w(\cos\theta+\cos\theta')(\sin\phi\tan\theta - \sin\phi'\tan\theta')]\delta(\varepsilon_{\boldsymbol{k}}^{+} - \varepsilon_{\boldsymbol{k}'}^{+}). \tag{B20}$$

For fourth-order scattering, we only take into account the Gaussian part in the noncrossing approximation. Hence the fourth-order scattering rate for the upper band is

$$\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(4)} = \frac{2\pi}{\hbar}\left[\sum_{\eta''\boldsymbol{k}''}\sum_{\eta'''\boldsymbol{k}'''}\frac{\langle V_{\boldsymbol{k}''\boldsymbol{k}}^{\eta''+}V_{\boldsymbol{k}\boldsymbol{k}'''}^{+\eta'''}\rangle_c\langle V_{\boldsymbol{k}'''\boldsymbol{k}'}^{\eta'''+}V_{\boldsymbol{k}'\boldsymbol{k}''}^{+\eta''}\rangle_c}{(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}''}^{\eta''} - i\delta)(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}'''}^{\eta'''} + i\delta)}\right]\delta(\varepsilon_{\boldsymbol{k}}^{+} - \varepsilon_{\boldsymbol{k}'}^{+})$$

$$+ \frac{2\pi}{\hbar}\left[\sum_{\eta''\boldsymbol{k}''}\sum_{\eta'''\boldsymbol{k}'''}\frac{\langle V_{\boldsymbol{k}'\boldsymbol{k}}^{++}V_{\boldsymbol{k}\boldsymbol{k}''}^{+\eta''}\rangle_c\langle V_{\boldsymbol{k}''\boldsymbol{k}'''}^{\eta''\eta'''}V_{\boldsymbol{k}'''\boldsymbol{k}'}^{\eta'''+}\rangle_c}{(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}''}^{\eta''} + i\delta)(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}'''}^{\eta'''} + i\delta)} + c.c.\right]\delta(\varepsilon_{\boldsymbol{k}}^{+} - \varepsilon_{\boldsymbol{k}'}^{+})$$

$$+ \frac{2\pi}{\hbar}\left[\sum_{\eta''\boldsymbol{k}''}\sum_{\eta'''\boldsymbol{k}'''}\frac{\langle V_{\boldsymbol{k}'\boldsymbol{k}}^{++}V_{\boldsymbol{k}'''\boldsymbol{k}}^{\eta'''+}\rangle_c\langle V_{\boldsymbol{k}\boldsymbol{k}''}^{+\eta''}V_{\boldsymbol{k}''\boldsymbol{k}'''}^{\eta''\eta'''}\rangle_c}{(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}''}^{\eta''} + i\delta)(\varepsilon_{\boldsymbol{k}'}^{+} - \varepsilon_{\boldsymbol{k}'''}^{\eta'''} + i\delta)} + c.c.\right]\delta(\varepsilon_{\boldsymbol{k}}^{+} - \varepsilon_{\boldsymbol{k}'}^{+}). \tag{B21}$$

After some algebraic manipulations, the asymmetric part of fourth-order scattering is

$$\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(4a)} = \frac{\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(4)} - \omega_{\boldsymbol{k}'\boldsymbol{k}}^{(4)}}{2} \approx \frac{3\pi n_i^2 V_0^4}{8v^2\hbar}\sin\theta\sin\theta'\sin(\phi-\phi')(\cos\theta+\cos\theta')$$

$$+ \frac{\pi w n_i^2 V_0^4}{8v^3\hbar}[-\sin\theta\sin\theta'\sin(\phi-\phi')(\cos\phi\sin\theta\cos\theta + \cos\phi'\sin\theta'\cos\theta')]$$

$$+ \frac{\pi w n_i^2 V_0^4}{32v^3\hbar}(\cos 2\theta + \cos 2\theta' + 14)(\sin\phi\sin\theta\cos\theta' - \sin\phi'\cos\theta\sin\theta')$$

$$+ \frac{\pi w n_i^2 V_0^4}{8v^3\hbar}(\sin\phi\tan\theta - \sin\phi'\tan\theta')(\cos^2\theta + \cos^2\theta'). \tag{B22}$$

To obtain an analytic result, we take the isotropic constant relaxation time ($t = 0$), that is

$$\frac{1}{\tau} = \sum_{\boldsymbol{k}'}\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(2s)}[1 - \cos(\phi-\phi')]\delta(\varepsilon_F - \varepsilon_{\boldsymbol{k}'}^{+}) = \frac{n_i V_0^2(\varepsilon_F^2 + 3\Delta^2)}{4\hbar v^2\varepsilon_F}. \tag{B23}$$

We can similarly define the relaxation time for non-Gaussian scattering

$$\frac{1}{\tau_{\rm sk}} = \sum_{\boldsymbol{k}'}\omega_{\boldsymbol{k}\boldsymbol{k}'}^{(3a)}\sin(\phi-\phi')\delta(\varepsilon_F - \varepsilon_{\boldsymbol{k}'}^{+}) = \frac{n_i V_1{}^3\Delta(\varepsilon_F^2 - \Delta^2)}{8\hbar v^4\varepsilon_F}. \tag{B24}$$

The coordinate shift of semiclassical electron during scattering for upper band is

$$\delta\boldsymbol{r}_{\boldsymbol{k}'\boldsymbol{k}}^{++} = \frac{v(\cos\theta+\cos\theta')(\sin\phi'\cos\theta\sin\theta' - \sin\phi\sin\theta\cos\theta')}{2\Delta(\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta' + 1)}. \tag{B25}$$

Then the side-jump velocity $\boldsymbol{v}_l^{sj} = \sum_{l'}\omega_{l'l}^{(2s)}\delta\boldsymbol{r}_{\boldsymbol{k}'\boldsymbol{k}}$ is

$$v_{+,\boldsymbol{k}}^{x,\rm sj} = \frac{n_{\rm i}V_0{}^2\sin\theta\cos\theta[w\sin(2\phi)\sin\theta - 4v\sin\phi]}{8v^2\hbar}, \tag{B26}$$

$$v_{+,\boldsymbol{k}}^{y,\rm sj} = \frac{n_{\rm i}V_0{}^2\cos\theta[-2\cos\phi\sin\theta(w\cos\phi\sin\theta - 2v) + w(\cos^2\theta + 3)]}{8v^2\hbar}. \tag{B27}$$

And the spin shift is

$$s_{+,\boldsymbol{k}}^{x,\rm sj} = -\frac{n_{\rm i}V_0{}^2\hbar\cos\theta[-2\cos\phi\sin\theta(w\cos\phi\sin\theta - 2v) + w\cos^2\theta + 3w]}{16v^3}, \tag{B28}$$

$$s_{+,\boldsymbol{k}}^{y,\rm sj} = \frac{n_{\rm i}V_0{}^2\hbar\sin\phi\sin\theta\cos\theta(w\cos\phi\sin\theta - 2v)}{8v^3}, \tag{B29}$$

$$s_{+,\boldsymbol{k}}^{z,\rm sj} = -\frac{n_{\rm i}V_0{}^2 w\hbar\sin\phi\sin\theta[\cos(2\theta) + 7]}{32v^3}. \tag{B30}$$

[1] D. Awschalom and N. Samarth, Spintronics without magnetism, Physics **2**, 50 (2009).

[2] A. Manchon, J. Železný, I. M. Miron, T. Jungwirth, J. Sinova, A. Thiaville, K. Garello, and P. Gambardella, Current-induced spin-orbit torques in ferromagnetic and antiferromagnetic systems, Rev. Mod. Phys. **91**, 035004 (2019).

[3] E. L. Ivchenko and G. E. Pikus, New photogalvanic effect in gyrotropic crystals, JETP Lett. **27**, 604 (1978).

[4] A. G. Aronov and Y. B. Lyanda-Geller, Nuclear electric resonance and orientation of carrier spins by an electric field, JETP Lett. **50**, 431 (1989).

[5] Y. M. Edelstein, Spin polarization of conduction electrons induced by electric current in two-dimensional asymmetric electron systems, Solid State Commun. **73**, 233 (1990).

[6] D. Culcer and R. Winkler, Generation of spin currents and spin densities in systems with reduced symmetry, Phys. Rev. Lett. **99**, 226601 (2007).

[7] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Current-induced spin polarization in strained semiconductors, Phys. Rev. Lett. **93**, 176601 (2004).

[8] N. P. Stern, S. Ghosh, G. Xiang, M. Zhu, N. Samarth, and D. D. Awschalom, Current-induced polarization and the spin Hall effect at room temperature, Phys. Rev. Lett. **97**, 126603 (2006).

[9] A. Manchon and S. Zhang, Theory of nonequilibrium intrinsic spin torque in a single nanomagnet, Phys. Rev. B **78**, 212405 (2008).

[10] C. Xiao, W. Wu, H. Wang, Y.-X. Huang, X. Feng, H. Liu, G.-Y. Guo, Q. Niu, and S. A. Yang, Time-reversal-even nonlinear current induced spin polarization, Phys. Rev. Lett. **130**, 166302 (2023).

[11] I. Sodemann and L. Fu, Quantum nonlinear Hall effect induced by Berry curvature dipole in time-reversal invariant materials, Phys. Rev. Lett. **115**, 216806 (2015).

[12] Q. Ma, S.-Y. Xu, H. Shen, D. MacNeill, V. Fatemi, T.-R. Chang, A. M. Mier Valdivia, S. Wu, Z. Du, C.-H. Hsu, S. Fang, Q. D. Gibson, K. Watanabe, T. Taniguchi, R. J. Cava, E. Kaxiras, H.-Z. Lu, H. Lin, L. Fu, N. Gedik, and P. Jarillo-Herrero, Observation of the nonlinear Hall effect under time-reversal-symmetric conditions, Nature (London) **565**, 337 (2019).

[13] Z. Z. Du, C. M. Wang, S. Li, H.-Z. Lu, and X. C. Xie, Disorder-induced nonlinear Hall effect with time-reversal symmetry, Nat. Commun. **10**, 3047 (2019).

[14] K. Kang, T. Li, E. Sohn, J. Shan, and K. F. Mak, Nonlinear anomalous Hall effect in few-layer $WTe_2$, Nat. Mater. **18**, 324 (2019).

[15] Z. Z. Du, H.-Z. Lu, and X. Xie, Nonlinear Hall effects, Nat. Rev. Phys. **3**, 744 (2021).

[16] Y. Tian, L. Ye, and X. Jin, Proper scaling of the anomalous Hall effect, Phys. Rev. Lett. **103**, 087206 (2009).

[17] D. Hou, G. Su, Y. Tian, X. Jin, S. A. Yang, and Q. Niu, Multivariable scaling for the anomalous Hall effect, Phys. Rev. Lett. **114**, 217203 (2015).

[18] Y. Omori, E. Sagasta, Y. Niimi, M. Gradhand, L. E. Hueso, F. Casanova, and Y. C. Otani, Relation between spin Hall effect and anomalous Hall effect in 3*d* ferromagnetic metals, Phys. Rev. B **99**, 014403 (2019).

[19] C. Xiao, H. Zhou, and Q. Niu, Scaling parameters in anomalous and nonlinear Hall effects depend on temperature, Phys. Rev. B **100**, 161403(R) (2019).

[20] J. Duan, Y. Jian, Y. Gao, H. Peng, J. Zhong, Q. Feng, J. Mao, and Y. Yao, Giant second-order nonlinear Hall effect in twisted bilayer graphene, Phys. Rev. Lett. **129**, 186801 (2022).

[21] P. He, G. K. W. Koon, H. Isobe, J. Y. Tan, J. Hu, A. H. C. Neto, L. Fu, and H. Yang, Graphene moiré superlattices with giant quantum nonlinearity of chiral Bloch electrons, Nat. Nanotechnol. **17**, 378 (2022).

[22] T. Ma, H. Chen, K. Yananose, X. Zhou, L. Wang, R. Li, Z. Zhu, Z. Wu, Q.-H. Xu, J. Yu *et al.*, Growth of bilayer $MoTe_2$ single crystals with strong non-linear Hall effect, Nat. Commun. **13**, 5465 (2022).

[23] X.-G. Ye, H. Liu, P.-F. Zhu, W.-Z. Xu, S. A. Yang, N. Shang, K. Liu, and Z.-M. Liao, Control over Berry curvature dipole with electric field in $WTe_2$, Phys. Rev. Lett. **130**, 016301 (2023).

[24] H. Li, M. Li, R.-C. Xiao, W. Liu, L. Wu, W. Gan, H. Han, X. Tang, C. Zhang, and J. Wang, Current induced second-order nonlinear Hall effect in bulk $WTe_2$, Appl. Phys. Lett. **123**, 163102 (2023).

[25] M. Suárez-Rodríguez, B. Martín-García, W. Skowroński, F. Calavalle, S. S. Tsirkin, I. Souza, F. De Juan, A. Chuvilin, A. Fert, M. Gobbi, F. Casanova, and L. E. Hueso, Odd nonlinear conductivity under spatial inversion in chiral tellurium, Phys. Rev. Lett. **132**, 046303 (2024).

[26] J. Železný, H. Gao, A. Manchon, F. Freimuth, Y. Mokrousov, J. Zemen, J. Mašek, J. Sinova, and T. Jungwirth, Spin-orbit torques in locally and globally noncentrosymmetric crystals: Antiferromagnets and ferromagnets, Phys. Rev. B **95**, 014403 (2017).

[27] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Anomalous Hall effect, Rev. Mod. Phys. **82**, 1539 (2010).

[28] M.-C. Chang and Q. Niu, Berry phase, hyperorbits, and the Hofstadter spectrum, Phys. Rev. Lett. **75**, 1348 (1995).

[29] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. **82**, 1959 (2010).

[30] H. Li, H. Gao, L. P. Zârbo, K. Výborný, X. Wang, I. Garate, F. Doğan, A. Čejchan, J. Sinova, T. Jungwirth, and A. Manchon, Intraband and interband spin-orbit torques in noncentrosymmetric ferromagnets, Phys. Rev. B **91**, 134402 (2015).

[31] N. A. Sinitsyn, A. H. MacDonald, T. Jungwirth, V. K. Dugaev, and J. Sinova, Anomalous Hall effect in a two-dimensional Dirac band: The link between the Kubo-Streda formula and the semiclassical Boltzmann equation approach, Phys. Rev. B **75**, 045315 (2007).

[32] N. A. Sinitsyn, Semiclassical theories of the anomalous Hall effect, J. Phys.: Condens. Matter **20**, 023201 (2008).

[33] C. Xiao, B. Xiong, and F. Xue, Boltzmann approach to spin–orbit-induced transport in effective quantum theories, J. Phys.: Condens. Matter **30**, 415002 (2018).

[34] C. Xiao, Semiclassical Boltzmann theory of spin Hall effects in giant Rashba systems, Front. Phys. **13**, 137202 (2018).

[35] C. Xiao and Q. Niu, Semiclassical theory of spin-orbit torques in disordered multiband electron systems, Phys. Rev. B **96**, 045428 (2017).

[36] E. J. König, M. Dzero, A. Levchenko, and D. A. Pesin, Gyrotropic Hall effect in Berry-curved materials, Phys. Rev. B **99**, 155404 (2019).

[37] H. Isobe, S.-Y. Xu, and L. Fu, High-frequency rectification via chiral Bloch electrons, Sci. Adv. **6**, eaay2497 (2020).

[38] C. Xiao, Z. Z. Du, and Q. Niu, Theory of nonlinear Hall effects: Modified semiclassics from quantum kinetics, Phys. Rev. B **100**, 165422 (2019).
[39] Y.-X. Huang, C. Xiao, S. A. Yang, and X. Li, Scaling law for time-reversal-odd nonlinear transport, arXiv:2311.01219 [cond-mat.mes-hall].
[40] C. Xiao, Y. Liu, Z. Yuan, S. A. Yang, and Q. Niu, Temperature dependence of the side-jump spin Hall conductivity, Phys. Rev. B **100**, 085425 (2019).
[41] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).
[42] J. M. Luttinger, Theory of the Hall effect in ferromagnetic substances, Phys. Rev. **112**, 739 (1958).
[43] I. Ado, I. Dmitriev, P. Ostrovsky, and M. Titov, Anomalous Hall effect with massive Dirac fermions, Europhys. Lett. **111**, 37004 (2015).
[44] E. J. König, P. M. Ostrovsky, M. Dzero, and A. Levchenko, Anomalous Hall effect on the surface of topological Kondo insulators, Phys. Rev. B **94**, 041403(R) (2016).
[45] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, Anomalous Hall effect in a 2D Rashba ferromagnet, Phys. Rev. Lett. **117**, 046601 (2016).
[46] I. Baek, S. Han, S. Cheon, and H.-W. Lee, Nonlinear orbital and spin Edelstein effect in centrosymmetric metals, arXiv:2310.05113.
[47] C. Xiao, H. Liu, J. Zhao, S. A. Yang, and Q. Niu, Thermoelectric generation of orbital magnetization in metals, Phys. Rev. B **103**, 045401 (2021).
[48] Y.-L. Zhou, H.-J. Duan, Y.-J. Wu, M.-X. Deng, L. Wang, D. Culcer, and R.-Q. Wang, Nonlinear antidamping spin-orbit torque originating from intraband transport on the warped surface of a topological insulator, Phys. Rev. B **105**, 075415 (2022).
[49] C. Xiao, H. Liu, W. Wu, H. Wang, Q. Niu, and S. A. Yang, Intrinsic nonlinear electric spin generation in centrosymmetric magnets, Phys. Rev. Lett. **129**, 086602 (2022).