Chapter 5
Design Issues Related to Text-Based Tasks

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5.1 Introduction

In this chapter, we focus on design issues related to written tasks and prompts for mathematical action and the sequencing and norms of collections of tasks, such as textbooks, that shape the expected action. We take a *task* to be the written presentation of a planned mathematical experience for a learner, which could be one action or a sequence of actions that form an overall experience. Thus, a task could consist of anything from a single problem, or a textbook exercise, to a complex interdisciplinary exploration. The design process for such tasks is not necessarily long or cyclic, but we are interested in particular issues that might or should be considered when designing tasks to be presented in text.

A *text*-based task is intended to create mathematical action through prepared and published inert written and visual images, in worksheets, textbooks, screen images, video, assessment instruments, digital interactive textbooks, and other digital technologies. We emphasize that text-based tasks are any such inert, static tasks with which learners interact and do not refer just to tasks in textbooks. In collections of text-based materials, such as textbooks, tasks do not exist on their own, but as components of the whole collection. Hence, in this chapter we also consider how the overall principles and aims of a collection (e.g., a textbook, a set of worksheets) can be embodied in different kinds of tasks. When we talk about prepared, published

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collections of tasks, we focus solely on the school-level curriculum and learners in educational environments. We also consider freestanding tasks that might be used by learners of any age and in varied phases of learning as relevant. Chapters 3 and 4 look at how teachers and learners, respectively, work with tasks, including digital tasks. Further insights into dynamic digital technologies are found in Chap. 6.

Throughout this chapter, we highlight design principles and research perspectives that are salient whether the tasks are freestanding or part of textbook collections. Although the design of text-based tasks and the design of textbooks share many common features, we believe that text-based task design is worthy of study in its own right, separate from but related to design and research on textbooks.

5.1.1 Research on **Text-Based Tasks** *Within Textbooks*

Some work on task design has delineated how different interpretations of curriculum aims and standards influence the design of collections of tasks embodied in textbooks. For instance, in the USA a number of large-scale curriculum projects were developed in the 1990s in response to the publication of the *Curriculum and Evaluation Standards* by their National Council of Teachers of Mathematics (1989). As Hirsch (2007) indicated, this Standards document provided a "basic design framework" that influenced authors regarding the nature and scope of content, integration of technology, embedded assessments, professional development for teachers, and active engagement of learners through explorations within cooperative groups. Thus, a design issue facing the curriculum developers was to provide text-based tasks or materials that embodied both intention and implementation (see Chap. 2). Other research on textbooks has focused on how the principles of design of the textbook tasks might have influenced teachers' enactment of those tasks and how such enactment influenced learner achievement (see Chap. 3; also see multiple perspectives on enactment in Remillard, Herbel-Eisenmann, and Lloyd, 2009). Still others have focused on design principles through comparisons of the tasks within textbooks, sometimes from a neutral stance of simply highlighting differences (e.g., Pepin and Haggarty, 2001) and other times from a more critical stance that highlights differences in tasks and their affordances for student learning (Huntley and Terrell, 2014). A summary of textbook research presented by Fan, Zhu, and Miao (2013) points to the shortage of research about the design process itself and a shortage of research into relationships between textbooks, teaching, and learning.

Our focus is on text-based tasks, and there is a growing body of research that approaches textbook analysis through the opportunities afforded by task content. In these approaches, it is presumed that some affordances are desirable according to some theoretical frame or curriculum aims, and research generally reveals the presence or lack of a particular feature. For example, research into some textbooks has shown a lack of explanations (Dole and Shield, 2008; Stacey and Vincent, 2009), of higher-order thinking (Nicely, 1985; Nicely, Fiber, and Bobango, 1986), of worked-out examples (Mayer, Sims, and Tajika, 1995), of opportunities for reasoning (Stylianides, 2009, 2014; Thompson, Senk, and Johnson, 2012), of problem variation (Stigler, Fuson, Ham, and Kim, 1986), of word problems (Xin, 2008), of conceptual connections (Sun, 2011), or of conceptual robustness (Harel, 2009).

Detailed examination of such textbook analysis research is beyond the scope of our chapter as we are interested in individual and sequential task design and how design embodies curriculum principles and theories of learning, either implicitly or explicitly. However, the assumptions behind many of these studies are that (a) the textbook defines the learners' mathematical experience because of the prevalence of textbooks in schools around the world and (b) textbooks could or should provide all these experiences. With both of these assumptions, learners' experiences with the tasks are intertwined with the associated teaching and how the teacher enacts the tasks as part of instruction (see Thompson and Usiskin (2014) for more insights on the enactment of curriculum). Thus, while we consider presence or lack of particular textual features, we cannot claim that presence ensures experience or absence implies teaching deficits. Rather, we consider design features that underlie the development of tasks, generally irrespective of the implementation of those tasks by either teachers or learners. In other words, we examine the relationship between authors' intentions for the task and the affordances and opportunities that the task provides; that is, we are interested in the bridge between curriculum intention and pedagogic implementation that can be provided through static text. This requires imagination and anticipation about learners' and teachers' engagement with tasks. Our approach is to draw on existing research where possible and also to take a scholarly perspective to task design issues by drawing on professional experience and published tasks. Many of the tasks presented in this chapter were shared and discussed by participants during the conference for ICMI Study 22 on which this book is based.

5.1.2 Shape of the Chapter

We consider three interrelated aspects of the design issues of text-based tasks: (1) nature and structure of such tasks, (2) pedagogic/didactic purpose of their design (i.e., intentions), and (3) intended/implemented mathematical activity as embedded in them (i.e., affordances and opportunities for learning). Although the chapter considers these headings, the relationships between the task designer, a teacher, a task, the mathematics within the task, and the learner are important at every stage. In particular, the relationship between task and teaching is like two sides of a coin because both contribute to the context of the learners' mathematical activity.

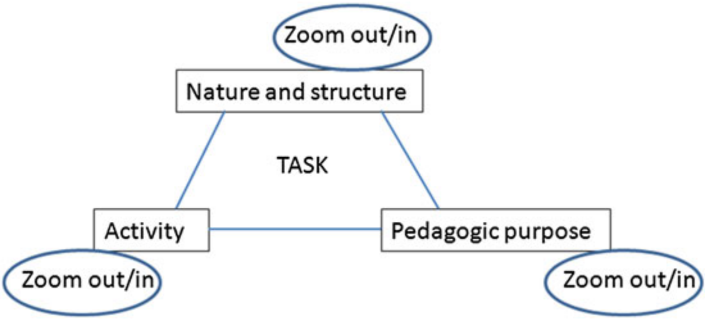


Fig. 5.1 Task design intention triangle

The three interrelated perspectives can be seen as a triangular structure with nodes (Fig. 5.1). Each node can be considered by zooming out and thinking about the overall educational context and how this affects task design and also by zooming in to the imagined interaction between one learner and the task.

Throughout the chapter, we present and compare examples of tasks to illustrate variation within each node and how principles of design play out within each node.

5.2 Nature and Structure of Tasks

We consider design principles related to three aspects of the nature and structure of tasks: (1) the types of text materials in which tasks are found, (2) the authorship, authority, or voice of the task, and (3) the mathematical content of the task. Within each of these aspects, particular issues related to design are evident.

5.2.1 Different Kinds of Text Materials

We start by making distinctions among different kinds of text-based materials. *Learning management systems* are those in which learning is presumed to be managed either by sequencing (such as in a traditional textbook series) or by a planned interplay between formative assessment and instructional tasks of various kinds. In some online curriculum packages (e.g., *I Can Learn* in the USA), the learner is essentially in his/her own private classroom using a software system that provides tasks and then gives mechanistic feedback to both the learner and the teacher about the learner's interaction with the task. Based on that feedback, the learner might move forward to new tasks on new concepts or might engage in tasks designed to offer remediation. The overall topic sequence is thus managed by a (sometimes virtual) teacher and/or possibly learners themselves.

In textbooks, the sequence is fixed according to a designed narrative suggested by the authors, with tasks potentially designed to build on each other and with careful consideration to necessary prerequisites. Teachers often choose to modify the textbook sequence based on perceived needs of their learners or mandated curriculum goals and must consider what assumed knowledge their students may not possess in a revised sequence and adjust tasks accordingly. In an online system, sequence might be varied according to learners' responses but a designed narrative controls those variations. In *task banks*, collections of varied tasks are published for which the teacher (or even the learner) is the effective learning manager and makes decisions about who does what and in what order; the individual tasks themselves may not be linked by a narrative (e.g., Yerushalmy (2015, Chap. 7, this volume); SMILE, n.d.). *Freestanding* tasks are those that do not form part of a curriculum package but are supplementary or fulfill a special purpose. For example, the NRich website provides extension tasks accessible by students, teachers, and parents that are intended as curriculum supplements (nrich.maths.org); the COMPASS (Common Problem Solving Strategies as Links Between Mathematics and Science) (2013) project (Maaβ, Garcia, Mousoulides, and Wake, 2013) provides interdisciplinary tasks within a European setting that can be used within mathematics and science classrooms. Freestanding tasks are typically designed to be self-contained, without relying on completion of previous tasks; if prerequisite knowledge is needed, then that information would need to be provided to a potential task user.

Task collections might exist in printed form as banks or books for learners with or without teacher guidance, in multimedia form such as paper and digital and/or physical materials for learners with or without teacher guidance, or in the form of guidance for teachers with materials for tasks, but no text for learners (e.g., Numicon at: global.oup.com/education/content/primary/series/numicon/). We do not consider the latter type in detail here as there are no given text-based tasks for learners unless the teacher constructs one, but our remarks about construction and use of text-based tasks apply also to teacher-made text-based tasks.

There are general observations about task design that apply across these multiple kinds of text. However, we also acknowledge that tasks designed to be included in mandated curricula material or those adopted by a governing body (e.g., school, district, ministry of education) may be designed under content and pedagogy constraints that do not exist for designers of supplementary materials or for teachers who design tasks for use with their own learners (Gueudet, Pepin, and Trouche, 2013). For instance, in tasks within mandated curricula, there may be a focus on problems addressing particular processes (e.g., reasoning, graphical representations) or particular solution approaches (e.g., written explanations); tasks may be designed to be facilitated by a teacher, with appropriate teacher guidance provided for implementation of the task. In contrast, in collections of tasks for supplementary use, there may be more of a focus on inquiry or exploratory approaches or multiple solution pathways. Similarly, a task designed for a specific purpose, such as introducing learners to engineering as a career choice, would adhere to different principles than a teacher-designed task to help a class learn a particular mathematical idea. The first requires a zoomed-out view of the design intention triangle, perhaps focusing on the value of engineering or the types of problems engineers solve, while the second will be a zoomed-in view at the classroom level, perhaps focusing on skills and understandings learners need for an assessment or to provide evidence of meeting an established curricular goal.

5.2.2 Authorship, Authority, and Voice

As designers plan and develop tasks, several issues come into play: (1) how will designers interact with each other and with the ultimate users of their tasks, namely, teachers and learners; (2) how will they position authority for evaluating the correctness or completeness of a task, namely, within the task or within the user of the task; and (3) what voice is used, namely, whether the task is addressed to a teacher or to a learner. We consider each of these issues in turn.

5.2.2.1 Authorship

Task designers work together in various formats to author mathematical text-based tasks:

- Substantial teams working with a long development process that includes field trials to obtain teacher input. Large-scale curriculum development projects, such as the School Mathematics Project in the UK, the many Standards-based curriculum projects in the 1990s in the USA, or the Chinese government teams developing national curriculum texts, have used this format. Also this format is used for many projects with more specific aims, such as numeracy recovery. The Canada-based project JumpMaths includes information about the effects of experience on its genesis (https://jumpmath.org/cms/).
- Author teams working on short time frames using design principles imposed by publishers or authorities. For example, official textbook production in China in 1960 took place within a 1-year cycle; there were serious learning problems within the textbooks that had to be changed (Li, Zhang, and Ma, 2009). Anecdotally, we know of one US state which requested a new official textbook in 6 months.
- Project teams working within particular agreed principles for pedagogical and epistemological coherence. For instance, in the COMPASS project, a large team works across multiple European countries to develop interdisciplinary tasks, with specific principles, such as a project-centered approach, an inquiry-oriented pedagogy, and an integration of information technology (Maaβ et al., 2013).
- Individuals or teams developing innovative or idiosyncratic materials with a specific focus or for use under specific conditions. For instance, Staats and Johnson (2013) created specific interdisciplinary tasks for use in college algebra and Movshovitz-Hadar and Edri (2013) developed social justice tasks to focus on values education in Israeli classrooms.

- Teams of teachers producing editable materials or task banks in a dynamic process. For instance, a group of lower secondary teachers in France are working together to produce a text easily adapted by all (*Sesamath* as described in Gueudet et al., 2013); a group of Israeli teachers are designing tasks using *Wikitext* (Even and Olsher, 2012).
- Individual teachers or small local teams disseminating their ideas. The proliferation of Internet resources has made it possible for teachers to post lessons and tasks for use by teachers anywhere in the world, at no cost or for a nominal fee (e.g., http://www.teachmathematics.net/; http://www.teacherspayteachers.com/).

A team which has come together because of an underlying shared belief and agreed-upon design principles, such as a team designing tasks that use a particular software (e.g., Geogebra, Cabri) or have a particular curriculum aim (e.g., Realistic Mathematics Education in the Netherlands), is presumed to have epistemological and conceptual coherence in their work. In contrast, a Wiki-type team might produce materials with variable principles (e.g., the French team for *Sesamath*). Even when materials have initially been developed by teams with specific design principles, the move from the design stage to more widespread use and adoption through commercial publication can create constraints or pressures that force adaptations or modifications in tasks to satisfy publishing needs.

An example of innovative text-based task development was the Resources for Learning and Development Unit (RLDU, n.d.) in which teams of about 10 teachers worked together to design tasks which they trialed and adapted. The final tasks were published in a format that implied certain pedagogic principles, namely, that tasks would be presented in a learner-friendly format and that learners would engage with mathematics through exploration and inquiry without being told exactly what to do or how to do it. Authorship for the text of the task rested with the teacher team, but authority for the mathematics rested in the explorations of the students (Llinares, Krainer, and Brown, 2014). Further information can be obtained from Laurinda Brown who coordinated and edited the resources (laurinda.brown@bristol.ac.uk).

Consider the RLDU task in Fig. 5.2. Although there is a sequence of questions to facilitate engagement with the task, there are no numbers or measures within the task. Thus, as written, there is not enough information for learners to answer the questions; rather there is the comment, “Bring a bicycle into the classroom!” This comment suggests the task designer wanted teachers to use a physical bicycle to facilitate inquiry; learners were expected to use the bicycle to explore mathematics and possibly to consider differences in answers to the questions for bicycles of different sizes. The questions should be relatively easy for learners to understand, but the mathematics is only accessible through exploration.

5.2.2.2 Authority for Mathematics

As task developers design a task or a collection of tasks to be included in a textbook, the manner in which the task is written can determine where authority lies for evaluating the mathematics that is the product of the task. In considering textbook

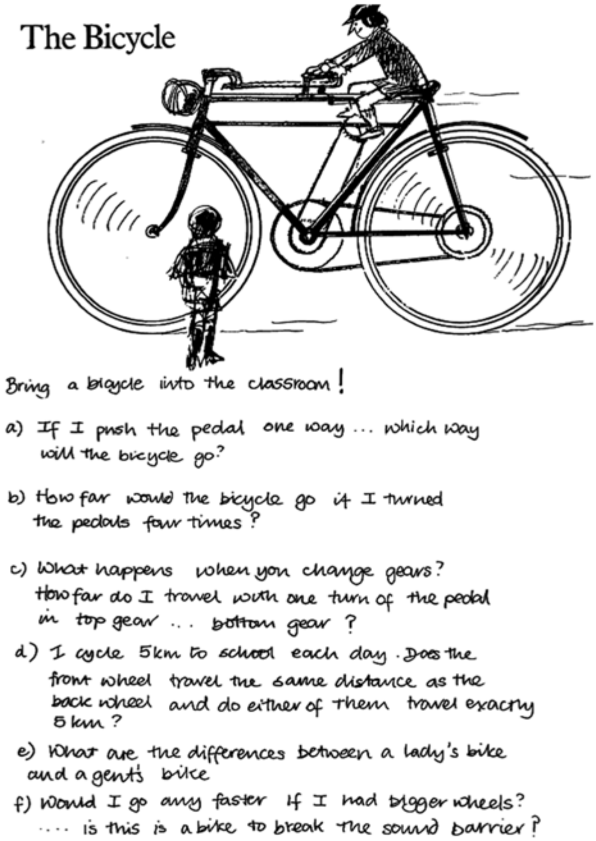


Fig. 5.2 An RLDU (Resources for Learning and Development Unit) worksheet (n.d.)

authority, Herbel-Eisenmann (2009) cites research indicating that textbooks often derive their authority from the structure of the text itself as well as the pedagogy that results or the political and cultural context. Authority for mathematics is ultimately within mathematics itself at the school level: most results can be checked by working backward or using different tools or searching for implications or contradictions, so long as learners are working within the usual conventions. However, mathematical authority is often ceded to the textbook authors who provide an answer book or the teacher who checks answers. Rarely do text-based tasks support the development of the self-checking learner, using mathematics to verify solutions. Developing self-checking learners could be accomplished by including regular explicit self-checking strategies or implicit strategies in which a later action modifies an earlier action or tasks that embed immediate feedback or tasks with a solution obtained from multiple approaches. These are possible using digital resources that show implications of incorrect reasoning and interactive software that allows for adjustment and self-correction as an integral activity. Some curriculum materials have attempted to build in features to facilitate this self-monitoring. For instance, curriculum materials for secondary learners developed by the University of Chicago School Mathematics Project have Quiz Yourself questions at different points within a lesson for learners to stop and check their comprehension, with answers to these short tasks at the end of the lesson. Throughout recent decades, various teams have developed programmed learning suites, either in hard copy or digitally, which provide multiple pathways, triggered according to diagnostic evaluation of learners' success so far. Responsibility for diagnosing common errors might be with the designers (Anderson and Schunn, 2000) but could be valuable for the learners themselves. For example, the German textbook series *Mathewerkstatt* contains tasks that enable learners to self-check their work and diagnose potential errors, ensuring that learners do not proceed too far without feedback relative to their mathematical progress (Hußmann, Leuders, Prediger, and Barzel, 2011a). In some of Swan's tasks, learners diagnose errors in the work of other anonymous learners.

5.2.2.3 Voice of a Task

In designing tasks, developers must consider how to address the ultimate user of the task and what messages are conveyed through different usages of language. When analyzing *voice* of text-based tasks, differences might be found between text which is directed at the teacher or at learners in the classroom or at learners who are assumed to be studying on their own. Zooming in to the learner's perspective, the main voice used in tasks is imperative. Learners are told what to do: *look at*, *write*, *solve*, *measure*, *find out*, and so on. Instructions may be supplemented with questions, some of which trigger action or application (such as *how many* …?) and others which trigger reflection, conjecture, and generalization (such as *what do you think would happen if*…?). There is evidence from a study of 400 learners that those who are used to imperative texts will scan the text looking for *what to do* (Shuard and Rothery, 1984, p. 114). When examining textbooks, we have found that the imperative tone dominates; when the teacher then refers to the textbook during instruction, such as "what does the textbook say?", the authority of the textbook as a means to resolve issues or questions is reinforced (Herbel-Eisenmann, 2009).

In some texts, there is no direct instruction for tasks but an assumption that some action will be carried out, triggered by a format to be completed or some objects to be contemplated. Such tasks may be used for young children who may be unable to read, but also be used with older students to encourage inquiry and exploration. The implied instruction may be to *complete* or *fill in the gaps*. In all texts, we might ask whether the learner is positioned as a compliant learner or as a co-creator of knowledge. Typical worksheets or workbook pages for very young learners provide a printed writing frame in which the learner merely fills in answers or uses color, arrows, and so on to indicate connections or classifications. In these cases, dialogue with a teacher shapes whether the experience is merely compliant or seen as the creation of meaning. Another way to frame this issue is to identify whether the overall mathematical narrative is delivered by the text or is created by the learner.

Fig. 5.3 Worksheet from RLDU (n.d.)

For example, consider the *worksheet* from the RLDU in Fig. 5.3. There are no questions or instructions provided for the task. However, learners who have previously worked with tasks from this resource know they are to create their own questions and explorations from the figure. Thus, the task positions the learner as creator of the narrative, possibly working with others and in dialogue with the teacher.

Another example of how voice can shift authority is demonstrated by Wagner (2012) who analyzes two of his own tasks using Rotman's (1988) distinction between imperatives that require learners to write (*scribble*) and those that require them to think. The first of his tasks is traditionally imperative; the first two parts tell the learner what to do and the final part invites some thinking.

This polygon is drawn on 1 cm dot paper.

1. Find its area by dividing it into a rectangle and two triangles.
2. Find its area by dividing it into two triangles.
3. Show another way you can divide the polygon into two triangles.

The second task uses the first person, as if a real person is talking and sharing a solution but in a problem-solving sequence used to find the area of an unnamed shape as if it is the only method. So, authorship of the second task is more overt than in the first task but still focuses mainly on *scribbling* with a little attention to thinking and no room for another learner's own ideas.

- I knew it was a trapezoid because the arrow marks showed that it had exactly two parallel sides.
- I identified the bases and the height. I noticed that the 6.8 cm side length was extra information that I didn't need.
- I used the formula. (Wagner cites from Small, Connelly, Hamilton, Sterenberg, and Wagner, 2008, pp. 144–145.)

In an example from Korea, a solution uses the phrases "Let's think" and "Let's find" to suggest that the learner work alongside an unknown person (Lee, Lee, and Park, 2013). Thus, the voice invites the learner to be a co-creator of the mathematical solution.

Fig. 5.4 Collection of sequences to be completed

a.	7,	...,	25,	34,	...,	52,	61,	...,
b.	...,	1.6,	2.5,	3.4,	...,	5.2,	...,	7.0,
c.	1.7,	2.6,	...,	...,	5.3,	...,	7.1,	...,
d.	...,	0.16,	...,	0.34,	0.43,	0.52,	0.61,	...,
e.	-7,	2,	11,	20,	29,	...,	...,	...,

5.2.3 Nature of Mathematics

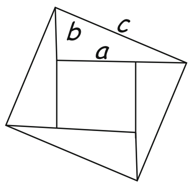
The examples in Figs. 5.2 and 5.3 strongly imply that mathematics provides tools for posing questions and understanding phenomena and also that mathematics can emerge from enquiry. By contrast, the teacher-designed sequence task in Fig. 5.4 implies that mathematics consists of symbolic objects that are acted on according to some rules and that mathematical activity consists of mental reasoning (in this case calculation) and writing and then seeking and expressing patterns. Task design influences the nature of mathematical activity and therefore the learners' perception of what it means to do mathematics. If the task in Fig. 5.4 is left at the stage of *filling in the blanks* with no reflection on connections between the five sequences, some opportunities to learn will have been missed as learners may simply think of mathematics as doing computations. However, if learners look for similarities and patterns among the sequences, then they are able to develop understanding of the structure and regularity within numbers.

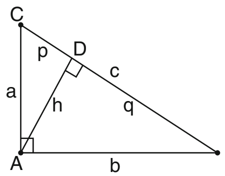
Comparison of text-based tasks from different educational cultures can be of value in highlighting deeper implications about the nature of mathematics as presented in the task and text. Although ontology is important, in the context of this chapter we cannot separate this from epistemology and, hence, the intended or assumed nature of mathematical activity. For example, in the Resources for Learning and Development Unit (RLDU), an overall intention was to develop a style of mathematics that focused on mathematization, posing mathematical questions, collaboration, and problem-solving; zooming in, the learner was encouraged to experiment, make conjectures, and search for or create mathematical procedures to carry out their enquiries. The mathematical activity that can be imagined being initiated by these tasks would involve practical materials, physical activity, discussion, application of techniques, and definitions. In addition, the nature of the tasks naturally lends itself to learners working collaboratively in small groups. Thus, pedagogy is not only implied but also structured by the materials, as in the note to bring a bicycle to class.

To explore how different views of the nature of mathematics play out in task collections, we look at three examples of comparative studies. The first example shows that different perspectives on the nature of mathematics can be related to the notion of authority previously discussed. Gueudet et al. (2013) compared features of two different French textbooks. Differences between them can be considered to create, rather than to reflect, differences in school mathematics cultures. One textbook, *Helice* (Chesné, Le Yaouanq, Coulange, and Grapin, 2009), was of a traditional type written by four experts; the other, *Sesamath* (Sesamath, 2009), was developed collaboratively by a group of 57 teachers using digital open-access tools. As a result of the manner in which the books and their related tasks are created, *Helice* has more overall consistency in its presentation of mathematics and mathematical activity and more coherence of conceptual development. *Sesamath*, while possibly empowering teachers, offers a fragmented and atomistic approach to concepts. For instance, *Sesamath* presents a counting or enumeration approach to finding area; in contrast, *Helice* presents a conceptual approach to understanding area as a conserved property of 2-d shapes by having students decompose and recompose shapes. In addition, *Helice* offers problems with a variety of solutions but *Sesamath* offers one expert solution. Learners therefore experience mathematics either as a connected whole with several possibilities for action or a fragmented collection of limited actions.

In the second example, the focus of tasks might inculcate different ways of reasoning. Chang, Lin, and Reiss (2013) compared a Taiwanese and a German textbook series according to principles of continuity, accessibility, and contextualization and the ways that content was structured. They took different approaches to proof and, more importantly for our purposes, the tasks that followed the proof. Figure 5.5 illustrates the Taiwanese visual-algebraic approach which was followed by several visual-algebraic tasks that used the theorem; the German approach was deductive and followed by tasks with a focus on area.

There is a subtle but important difference when we zoom in to the learner's perspective. In the Taiwanese approach, learners extract the relationships from a diagram directly but must have prior knowledge of how to determine the area of a triangle and the area of a square; in the German approach, learners apply a priori formal knowledge to a diagram using knowledge of similar triangles embedded

<table>
<tr><th>Taiwanese Example</th><th>German Example</th></tr>
<tr><td>


</td><td>


</td></tr>
<tr><td>
$$c^2 = 4 \times \left(\frac{a \times b}{2}\right) + (a - b)^2 = a^2 + b^2$$
</td><td>
$a^2 = p \times c$
$b^2 = q \times c$
$h^2 = p \times q$
therefore
$a^2 + b^2 = c^2$
</td></tr>
</table>

Fig. 5.5 Comparison of Taiwanese and German typical approaches to Pythagoras' theorem (Adapted from Chang et al., 2013, p. 308)

within the large right triangle. Although the study offered these as examples of differences in overall mathematics pedagogy, this contrast highlights how differences in task design can engender different learner experiences of geometric reasoning through different kinds of mental activity.

The third comparison we offer contrasts the treatment of the additive relations between small whole numbers for young children in the Chinese textbook and a Portuguese textbook (Sun, Neto, and Ordóñez, 2013). Analysis of control and variation in the tasks for this concept shows significant differences in opportunities to learn. In the Chinese textbook (Fig. 5.6), the focus on each page is representing a part-part-whole relation visually, physically, and in alternative equivalent symbolic forms. Thus, the Chinese textbook presents mathematics as a variety of formal representations of some actions, and this implies that mathematical activity consists of carrying out physical actions, forming mental images and expressing them in numerical instantiations.

In the Portuguese textbook, which we cannot show here, the focus was on active methods (such as "doubling plus 1") applied to several sums. If we look at one

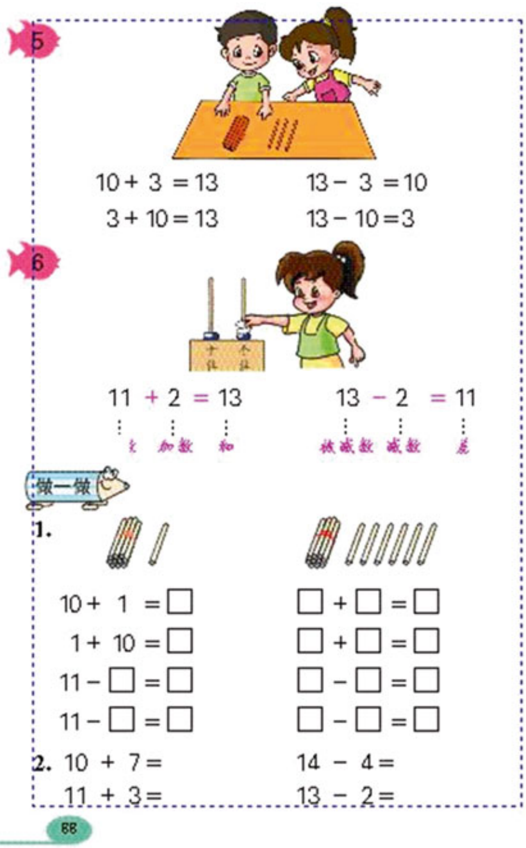


Fig. 5.6 Page from Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, p.68)

page as a text-based task, in the Chinese approach the object of learning and the underlying concept is the additive relation, with connections made between addition and subtraction and reasoning with tens and ones. In the Portuguese textbook, the object of learning appeared to be a different procedure applied to different problems on every page.

To understand what can be learned from one task, it is useful to consider the immediate sequence. In the Chinese textbook, the subsequent tasks also focus on additive relations, while in the Portuguese textbook a variety of methods which can be used in different cases are presented, leaving the teacher to make the connections. Sun also shows this difference applies to some US textbooks (Sun, 2011).

These three comparative examples suggest three issues related to task design in terms of how learners may view what it means to do mathematics:

- Is the learner encouraged to explore and compare different solution methods, or must the learner apply one given method?
- Is the learner expected to apply a priori knowledge or to apply mathematical reasoning (such as expressing relationships) to access new ideas?
- Do the choice and sequence of examples prioritize conceptual understanding (such as through relating actions to symbols or comparing representations) or prioritize methods for reaching a solution?

5.2.4 Summary

In this section, we have discussed the nature of text-based tasks, the view of mathematics they imply through their structure and expectations, and the authority they assume based on the nature of their authorship or the voice they employ. We have not intended to imply that any particular set of design principles in these aspects is inherently better than any other. Text-based tasks are prepared and static. A major question to be considered is: "How can tasks shape an experience of mathematics that is dynamic and dialogic and sees the learner as a sensemaking creator of connections, insights, and solutions?" Some of the tasks already presented could provide the opportunity for such dynamic dialogue, but only if the associated pedagogy supports this. In the next section, we consider the pedagogic issues in task design.

5.3 Pedagogic/Didactic Purpose of Text-Based Tasks

The pedagogic intent of a task also influences how that task might be designed by its developers. In this section, we consider how cultural differences in purpose influence design, how learners are made aware of purpose, how developers ensure a coherent purpose within a collection of tasks, and how new knowledge is integrated with existing knowledge.

5.3.1 Cultural Differences in Purpose

Throughout this chapter, we assume the aims of mathematics education are multifaceted, so that learners become knowledgeable about concepts, competent with procedures, and capable and willing to select, adapt, and use mathematics in a variety of familiar and unfamiliar contexts and problems. An overarching question is whether and how text-based tasks can contribute to all these aims. The relative importance of these aims is to some extent cultural, and there is some evidence of cultural differences in how aims are translated into text-based tasks and used to promote learning.

In several countries, lessons typically have three levels: the first level is basic facts and formula; the second level is to make connections between these; and the third level is for learners to apply some higher-level thinking to problems. The third level of classroom mathematical activity is hardest to achieve and depends on progression embedded within task sequences. One example of a three-level task lesson from Taiwan follows:

1. Have a triangle with three angles and put the angles together to form a straight line with 180°.
2. Show that the exterior angle sum is 360°.
3. Then use the exterior angle sum to prove the sum of the interior angles (National Academy for Educational Research (Taiwan), 2009).

The first-level task is informal and involves constructing a demonstration of the interior angle sum; the second-level task requires some reasoning about angles, applying previous knowledge that there are 360° in a full rotation; the third task requires a different kind of reasoning, involving formal proof. It is the responsibility of the teacher to make the links between the three levels so that it does not have the appearance of circular reasoning. In cultures where the emphasis is on efficient actions of teachers and learners, it is hard to introduce the messier aspects of problem-solving in which solutions may not be arrived at through the optimal use of time, effort, and method. In the given example, there is a further difficulty, namely, that the practical, spatial reasoning expected for the first two tasks gives way to formal proof for the third task, a shift which is recognized as a major learning obstacle and pedagogical challenge (e.g., Bell, 1976).

In the texts available in our discussions at the ICMI Study Conference, we identified different emphases on mathematical behavior. Learners were expected to develop efficiency (e.g., Dindyal et al., 2013), abstraction (e.g., Chang et al., 2013), and applications (e.g., Maaβ et al., 2013) or investigate social problems (e.g., Movshovitz-Hadar and Edri, 2013) in various cultures. There are variations within cultures as well. No text can do any of this on its own; in most cases, the effects of the texts are mediated by the actions of the teacher. Ensuring that teachers understand the pedagogic purpose of a task is another design issue that has cultural variations.

In some systems (e.g., China, see Ma (1999)), the teacher guide is understood to be the authority for pedagogic knowledge and the national curriculum includes very detailed information about curriculum and pedagogic purpose and design. That is, the guide provides information to teachers about the *curriculum vision* so that teachers understand the overall goals and how materials fit together; such understanding is important in building *curriculum trust* so that any adaptations made by the teachers are consistent with the overall pedagogical and epistemological vision of the materials (Drake and Sherin, 2009).

In other systems, teacher guides exist but might be ignored or used in different ways, particularly if the pedagogic purpose for tasks or their sequence is not clearly laid out in the guide. For example, it might be assumed, incorrectly, that merely using a textbook in its given order assures coverage of the curriculum and coherence with the designers' intent (Thompson and Senk, 2010). Another reason teachers might ignore teacher guides is when they expect to interpret and structure the curriculum for themselves, using textbooks as one of several resources and determining their own pedagogic purpose for tasks for use with their students.

Differences in pedagogic purpose also play out in how various cultures design tasks to address learner diversity. In Japan, one task might be offered to an entire class with the emphasis on collaboration, recognizing that different learners will learn from this process in different ways. In the UK, it has been common to have different but parallel textbook series aimed at learners whose level of attainment differs, so that those with lower prior attainment have textbooks in which the conceptual and cognitive material is less challenging. In Sweden, the law requires equal access for all, so it is seen to be against the law to differentiate between learners by placing them in different curriculum tracks. As we write, there is debate about the interpretation of these laws (Lundberg, personal communication, 6 January 2014), but at present it means that there is no differentiation of textbooks, except for those with varied communication abilities.

One implication of non-differentiated materials is that tasks need to be designed to enable learners with previously low attainment to gain higher-level understandings and also for those with high understanding to extend their knowledge. This means that mathematics cannot be presented as a linear accumulation of ideas with assumptions about prior learning, but instead task design needs to develop concept images and dispositions that will be sustainable across a range of mathematical activity and enable learning at several levels. That is, tasks need to be designed so there are multiple entry points, with options for extensions and adaptations.

To illustrate what we mean by zooming in to learners' experience, consider the following task:

- *Find* $9+7=\underline{\quad ? \quad}$.

As written, this is a fairly closed task and students generally know the fact or they have to work it out. Now consider the following adaptation:

- *Write as many number sentences as you can for 16.*

This version of the task addresses a similar ultimate goal but has many points of entry. Some learners may start by writing $15+1=16$, $14+2=16$, and so on. Depending on judgments of learners' potential and their past achievements, the teacher can ask students to use more than two addends, more than one operation, etc.

The point is that all learners in a classroom could experience success with the task, some with simple number sentences and others with more complex ones. In the process, students are investigating number relationships and completing many more computations than would have occurred from the original task. When elementary learners have been given such a task, it is not uncommon to have pairs of learners write upward of 20 number sentences in a relatively short time span. If pairs of learners check the work of other pairs, learners have opportunities to consider other potential ways to combine numbers in appropriate number sentences.

Similar adaptations of closed tasks might occur in many contexts. Rather than consider a task with a single answer and one way to obtain that answer, teachers might adapt tasks to encourage multiple answers or multiple pathways to an answer. The intent of such tasks is to provide access to diverse learners, and many teacher guides provide information about the pedagogic purpose of such adaptations through examples of adapting tasks for students who need remediation or extension. Such features of text-based tasks, especially the expectations of organization of work, are most likely to be identified through textbook comparison studies in which assumptions and expectations about ways of working can become more clear (e.g., Pepin and Haggarty, 2001; Stylianides, 2009, 2014).

5.3.2 How Purpose Is Presented to Learners

In the task at the end of the previous section, all learners are investigating number relationships with differing levels of difficulty. We now zoom in again to examine more possible purposes of tasks and how these might be expressed to learners.

In their classic study of children reading mathematics, Shuard and Rothery (1984) present five main purposes for mathematical texts:

1. Teach concepts, principles, skills, and problem-solving strategies.
2. Give practice in the use of concepts, principles, skills, and problem-solving strategies.
3. Provide revision of 1 and 2 above.
4. Test the acquisition of concepts, principles, skills, and problem-solving strategies.
5. Develop mathematical language, for instance, by broadening the pupils' mathematical vocabulary and their skill in the presentation of mathematics in a written form (pp. 5–6).

Shuard and Rothery's five purposes apply to texts in their entirety. Applying these five purposes to individual tasks would imply that such tasks might address individual purposes, such as the various desirable goals and outcomes presented in Chap. 2. A more helpful approach would be to use these purposes as parameters for task sequence intentions, so a task might incorporate some revision content, some new concept content, some relevant language, and so on, and a task sequence might present all these purposes in a developmental order. To some extent these purposes would be teacher and learner specific, or even topic specific, so that tasks that support learning to resolve right-angled triangles would look very different to tasks that

Table 5.1 MATH taxonomy categories (From Smith et al., 1996)

Group A	Factual knowledge (A1) Comprehension (A2) Routine use of procedures (A3)
Group B	Information transfer (B1) Applications in new situations (B2)
Group C	Justifying and interpreting (C1) Implications, conjectures, and comparisons (C2) Evaluation (C3)

Table 5.2 Categorizations to analyze assessment tasks for mathematical processes (From Thompson et al., 2013)

Reasoning and Proof The item directs students to provide or show a justification or argument for ***why they gave that response***
Opportunity for Mathematical Communication The item directs students to communicate ***what they are thinking*** through symbols, graphics/pictures, or words
Connections The item is set in a real-world context outside of mathematics The item is *not* set in a real-world context, but explicitly interconnects two or more mathematical concepts (e.g., multiplication and repeated addition, perimeter and area)
Representation: Role of Graphics A graphic is given and must be interpreted to answer the question The item directs students to make a graphic or add to an existing graphic
Representation: Translation of Representational Forms Students are expected to record a translation from a verbal representation to a symbolic representation or vice versa Students are expected to record a translation from a symbolic representation to a graphical (graphs, tables, or pictures) representation or vice versa Students are expected to record a translation from a verbal representation to a graphical representation or vice versa Students are expected to record a translation from one graphical representation to another graphical representation

support learning to prove properties of triangles. Another approach is that of the MATH taxonomy derived by Smith et al. (1996) from examination questions but which could be applied to opportunities afforded by individual text-based tasks for learning. Their categories are outlined in Table 5.1.

A more detailed categorization designed to analyze assessment tasks is that of Thompson, Hunsader, and Zorin (2013). In Table 5.2, we give a summarized form to show the range of foci that can be present in a task.

Although these categorizations apply to assessment tasks, they can also be related to the purpose of learning tasks. For example, if some assessment tasks focus on translation between representations, this kind of mathematical action needs to be met throughout lessons and also needs to be accompanied with a coherent theory that connects translation with some desirable learning outcomes. While using any of these categorizations to ensure that a learning management system addresses the associated desirable learning outcomes, it is not the case that merely setting a task that requires a particular mathematical approach ensures learning. The three-part task sequence given previously about triangles gives no help to learners who cannot see how to proceed. By contrast, the text-based tasks in Burn's approach (e.g., *A Pathway into Number Theory*, 1982) promote guided learning by anticipatory dialogue. Burn's opening four questions are (p.18):

1. Look at table 1.1 [below]. If the same pattern was extended downwards, would it eventually incorporate any positive integer $\{1, 2, 3, \ldots, n, n+1, \ldots\}$ that we might care to name?
2. What is the relation between each number in table 1.1 and the number below it?
3. Give a succinct description of the full set of numbers in the column below 0.
4. If you choose two numbers from the column below 0 and add them together, where in the table must their sum lie?

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
etc.			

Burn's "answers" for these tasks are unusual as they often set up new ideas for learners and encourage dialogue with the text. As an answer to question 1, he gives formal definitions and notations for natural numbers, integers, rational numbers, real numbers, and complex numbers. For question 2, he writes *four less and four more* and for question 3, *three multiples of 4*. Question 4 he answers with $4n+4m=4(n+m)$. So the first answer situates what is meant by *integers* in the context of different classes of number. The second and third confirm the learner's reasoning. The fourth indicates that it is time to shift to symbolic representations and shows how such representations can be a tool to express reasoning. In this fashion, Burns leads the reader through a number theory course in which the learner's activity initiates the closest thing to a dialogue that one can get from a static text. Note also the fact that only two of these *questions* are imperatives; even then, the first instructs the learner to *look* before posing a question. In terms of the previous categorizations of Thompson et al., Burns' questions provide opportunities for students to communicate, make connections, and interpret a graphic.

It would be possible to dismiss Burns' approach as a teaching style relevant only for adult learners or those who are studying mathematics through choice. However, The School Mathematics Project (SMP) in England experimented widely with dialogic teaching for 11-/12-year-old learners of average attainment from the early 1960s. We do not have good measures of readability that take into account the need to interpret mathematical ideas and multilingual classrooms, but it is likely that learners with restricted literacy would find a text-based dialogic approach hard to understand and successive versions of SMP materials reduced the reading requirements and hence the dialogic opportunities significantly (there were multiple editions, now all out of print).

You have been given some meter strips of paper and a meter stick and a table to enter your findings. Fold the strip into 2 equal lengths; measure the length of one piece in centimeters and write the measurement in the table. Fill in any other cells that you can in the same row of the table. Look at the column heading to decide what to write. Now fold strips into other numbers of equal lengths and continue to complete the table.

Equal pieces (n)	Fraction of a metre: $\frac{1}{n}$	Measurement in cm	Percentage of a metre	Decimal fraction of a metre	Calculate $1 \div n$
2					
4					
8					
5					
10					
3					
6					

Fig. 5.7 An example of a task to format new knowledge

Anticipation of learners' responses is a key idea in designing tasks to promote mathematical dialogues with the text, and published examples are often trialed before publication. Burns anticipates responses in his tasks previously described (1982). A Portuguese textbook (Gregório, Valente, and Calafate, 2010) provides examples of a new method intended to be used in that page, but the first example ($1+2=1+1+1=3$) is ambiguous and could be taken by learners to mean individual counting, rather than an instance of $n+(n+1)=n+n+1$. Here, anticipation has not led authors to imagine alternative interpretations. Some texts have specific examples in which there is a thinking frame to help learners think about a solution. Then an actual solution is written in a different font to model the type of response that the learner would be expected to provide; such a method is another way to engage in dialogue between a learner and a static text.

The task in Fig. 5.7 was given to a class of 11-year-olds with diverse prior knowledge. The task shaped the mathematical activity in such a way that the teacher was able to identify particular problems of understanding. It also provided a structure within which learners could work together to show each other how to measure accurately, how to use the algebraic information, how to express a fraction of 100 as a percentage, and so on. Because the class had been enculturated into making conjectures, connections within rows were identified by learners. However, if this had not happened, the teacher could have used a digital version of the table, bringing appropriate columns adjacent to each other so that conjectures could be made. While all learners were working on connections between decimal fractions, vulgar fractions, and percentages, new learning varied from learner to learner depending on what they already knew and could do.

The task in Fig. 5.7 demonstrates a particular strength of text-based tasks, namely, that they offer formats that bring particular features together so that comparisons and connections can be made to show relationships, equivalence, and so on. The provision of tables, grids, sequences, columns, and so on to organize mathematical data can draw attention to connections between different representations or different instances. Comparisons and connections could be engineered to ensure that a critical feature of a concept is foregrounded and that data are structured so that patterns and relations can be sought and conjectures made. Formatting the outcomes of activities is one way that text-based prepared tasks can provide scaffolding for new insights and relational ways of thinking. Texts developed by the University of Chicago School Mathematics Project contain guided examples, with blanks to help students get started on a solution. The well-known use of ratio tables in RME (Corcoran and Moffett, 2011; Van den Heuvel-Panhuizen, 2003) and bars in Singapore (Hoven and Garelick, 2007) shows how consistent use of images in text-based tasks can scaffold understanding.

One issue, however, in all these task examples is that the purpose for the task is not always made clear to students. Particularly when tasks have engaged students in inquiry and discovery, there is a need to bring closure or summary to ensure that students take from their engagement with the task the expected learning objectives. So, opportunities to summarize learning are an essential feature of task design and associated pedagogy.

5.3.3 Coherent Purpose in Collections of Tasks

The role of the teacher with regard to text-based tasks is to mediate between the text and the learner. If that role is passive, the teacher is neither augmenting nor limiting what is offered by the text, whether compliant or dialogic. Of course, there is no guarantee that a learner will use a static text interactively, even if there are interactive prompts such as those in Fig. 5.2. By contrast, teachers who assume responsibility to provide conceptual and pedagogic coherence through their teaching inevitably mediate tasks through the construction of classroom cultures in which tasks direct and shape existing forms of mathematical activity. Between these two extremes, published collections of tasks can themselves provide conceptual and pedagogic coherence through the consistent application of design principles. Firstly, we look at a coherent approach to epistemology and pedagogy.

Herbart (1904a, 1904b) suggested that teachers should raise learners' interest before formal teaching. His approach contrasts with classical texts in which a formal definition is provided first. In Herbart's model, the learner's first task is to think imaginatively about the phenomenon; in the classical model, the learner's first task is to decode the definition and possibly imagine some examples of it. The role of *direct apprehension*, i.e., being provided with a situation or image that embodies a concept, is more than merely motivational; it suggests that mathematics is a process of abstraction of structures, properties, and relationships from specific contexts, *2a.* *Find a number in Z_{10} that you can add to 6 to get 0 mod 10. Such a number is the additive inverse of 6 in Z_{10}.*

2b. *Find a number in Z_{10} that you can add to 2 to get 0 mod 10. Such a number is the additive inverse of 2 in Z_{10}.*

2c. *What is the additive inverse of 3 in Z_{10}? Explain.*

2d. *What is the additive inverse of 3 in Z_8? Explain why this answer is different than the answer you got for the additive inverse of 3 in Part c.*

Fig. 5.8 Task sequence to scaffold conceptualization of a new idea leading to a definition (Hart, 2013, p.340)

whereas the definitional approach suggests that mathematics consists of instantiations and use of abstract ideas. An example of direct apprehension is the use of a domino rally to introduce proof by induction. Love and Pimm (1996, p. 375) note that starting with an exploratory task involving inquiry also has implications for authority because the work starts with learners' activity or with the learner's mental model, whereas starting with a definition implies an external authority. They contrast texts which consist of exposition of given ideas with texts which develop meaning through learners' construction.

As Hart says (personal communication, 24 July 2013), "A good definition encapsulates a core idea … But, in terms of learning and task design, they seem to be more effective if they come after instruction, not before. After wrestling with an idea, figuring it out, seeing how it naturally comes up when trying to solve interesting problems, then you say, well, let's call this idea ___, and then define it." Applying this perspective to task design, he provides carefully structured sequences of examples that, on reflection, can be treated as data for conjecture and conceptualization of a new idea (Fig. 5.8). As Hart notes, the sequence of numbers used is critical as part of the design process. Numbers that are special cases and do not assist in developing a generalization are not appropriate as examples in the development of the concepts.

To establish in learners the habit of automatic reflection on collections of examples, this kind of task would need to be used regularly (see also Watson and Mason, 2006). It is more usual, however, for a sequence of procedural questions to be treated by learners and teachers as a sequence of isolated cases. When cases are used to encourage looking for patterns, the specific instances must be chosen with care to ensure they lead to the desired generalization and do not generate a misconception. For example, consider attempts at conceptual understanding of division using the examples in Fig. 5.9.

Note how the task embeds practice of division and encourages a comparison between division and subtraction that may connect them via a *repeated subtraction* procedure. As with some other tasks in this chapter, practice is not only associated with fluent use of procedures but also with insights into relationships. There is a potential problem, however. In the two cases, the divisor and the quotient have the same value. So, learners may correctly write $16 \div 4 = 4$ and $25 \div 5 = 5$, respectively,

16 – 4 = 12	25 – 5 = 20
12 – 4 = 8	20 – 5 = 15
8 – 4 = 4	15 – 5 = 10
4 – 4 = 0	10 – 5 =5
	5 – 5 = 0

Rewrite each set of subtractions as a division.

Fig. 5.9 Attempts to connect conceptual and procedural views on division (Adapted from Zorin, Hunsader, and Thompson, 2013)

for the two examples but reverse the meanings of the divisor and the quotient. Modifying the example to be 20–5 = 15, 15–5 = 10, 10–5 = 5, and 5–5 = 0 avoids the potential confusion because the appropriate division sentence is 20 ÷ 5 = 4, and the divisor and quotient cannot be switched.

In the previous two tasks, there is an imperative approach to what needs to be done, but not how to do it. In the Chinese textbook and *Helice* series mentioned earlier in this chapter, an important feature is multiple approaches to each mathematical situation. In a German textbook described by Barzel, Leuders, Prediger, and Huβmann (2013), a consistent set of characters regularly display preferred approaches to solving problems throughout the series, for example, "Till likes to try numbers and to begin a table. Pia likes to explore patterns and to describe a situation algebraically". This embeds the idea that there are alternative approaches to problems that might be valid.

Zooming out to the overall context, another way in which published task collections can establish cultures of mathematical activity is through the inclusion of assessment tasks whose design aligns with the curriculum and pedagogic aims (Thompson et al., 2013). In the cases just considered, method is less important—so long as it is correct—than reflection on the outcomes. If both the formative and summative assessment tasks provide the same coherence and consistency as the intended curriculum, then even if the teacher and learners approach mathematics with a test-focused lens, broad aims might be achieved. Such consistency can be achieved through aligning curriculum, pedagogy, and assessment so that the expected forms of reasoning, the expected communication methods, the connections between and within topics, the representations used, and the connections between them are evident. Consistency also requires that similar things are prioritized and foregrounded in assessment tasks as in the curriculum, not only in conceptual and procedural aspects but also in the nature of mathematical activity.

For example, consider the development of an assessment task as in Fig. 5.10 showing how apparently similar tasks can make different demands on learners, emphasizing first counting and representation, second the action of sharing and a representation, and third the action of sharing and producing two related representations. In the final version, the connection between pictorial and symbolic representation has to be explicit, thus providing insight into conceptual understanding by having to change register, as described by Duval (2006). This task involves more than simple translation, because in each representation some process has to take place, and the processes are different. In words, some imaginary modeling needs to

Adaptation 1. Five friends have 20 pieces of candy to share equally. How many pieces of candy will each friend get? Write a number sentence to show how many pieces of candy each friend will get.

Adaptation 2. Five friends have 20 pieces of candy to share equally. How many pieces of candy will each friend get? Write a number sentence to show how many pieces of candy each friend will get. Use the picture to explain your thinking about the problem.

Adaptation 3. Five friends have 20 pieces of candy to share equally. Draw a picture to show how many pieces of candy each friend will get. Write a number sentence to represent your picture.

Fig. 5.10 Three adaptations to a basic division task to engage learners in different mathematical processes (From Thompson et al., 2013, p. 406)

be done; in the arrays of sweets, some methods of enumeration have to take place. The varied forms of the assessment can align with the nature of the tasks incorporated as part of instruction. Mathematics is presented as being about expressing relations between quantities in alternative representations. In addition, mathematical activity consists of following instructions to draw these different representations, with the implication that comparison will take place to support cognition. These versions show how assessment tasks within a learning management system can express overall aims of the system and also give learners direct insight in the associated values of that system. Appropriate assessment tasks are essential to ensure that collections of tasks have a coherent purpose overall, both instructionally and in evaluative aspects.

Of course, high-stakes assessment tasks structure purpose and pedagogy to some extent, but a recurrent problem is how such tasks can recognize and even measure the development of mathematical behavior. Dindyal et al. (2013) have attempted to make problem-solving activity assessable by using a practical worksheet, similar to worksheets or data recording sheets used in laboratories. By providing a format in which learners can record the stages and processes of problem-solving (see Fig. 5.11), teachers not only enculturate learners into the habits of exploratory group work but also are able to monitor competence and progress in relevant behavior. This is another situation where textual formats can scaffold mathematical enquiry and insights.

Collections of tasks need to have a consistent approach to the conceptual development of the content. We have already compared how different ideas about learning addition can be enacted throughout a text, presenting either the additive relationship or addition techniques. We also gave an example in Fig. 5.5 of how geometric reasoning can be differently enacted. In the respective textbook series as a whole, these differences are sustained; the one based on similarity assumes

Devise a plan
Write down the key concepts that might be involved in solving the equation.

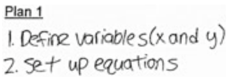


Fig. 5.11 Outline of part of the practical worksheet with student's response (From Dindyal et al., 2013, p. 319)

that this idea has been understood earlier and the relevant notation already adopted. In the use of mathematical notation, there is little room for variation throughout a task collection as notation tends to be standardized, but in the use of images, development of mental images, inner language, and promoted action, there is more room for variety, sometimes based on cultural expectations.

At times, clashes of images need to be anticipated in collections of tasks within a textbook or between textbooks in a series. For instance, difficulty can occur when learners have depended on a balance model for solving linear equations but are then introduced to a "find roots" approach for quadratics, thus fragmenting their knowledge of solving equations. A similar problem for younger students is having an array understanding of multiplication and then trying to use multiplication for scaling quantities. Text can introduce particular images, but these have to be used in a coherent manner.

5.3.4 Embedding New Knowledge with Existing Knowledge

So far we have discussed tasks for learners to introduce them to new ideas, problems and procedures, and assessment tasks that might follow and convey a particular view of what is valued in mathematics. An intermediate range of tasks can be designed to connect new to existing knowledge, help learners recognize the value of that knowledge, and make it available for future use. Such tasks are particularly important when the main teaching/learning mode is exploratory and divergent because the tasks help relate the exploration to conventional knowledge. There are various ways to address this range of tasks, as described in more detail in Chap. 2: in Realistic Mathematics Education, *vertical mathematization* describes the necessary process of transforming methods used for individual problems into tools for future use (Treffers, 1987); Brousseau refers to *institutionalization* as a process of legitimizing the work done in conventional mathematical terms (1997). In Japan a process of *neriage* (kneading) takes place to bring students' into a coherent whole (Takahashi, 2011). In these approaches, the teaching is vital. In the KOSIMA project (Hußmann, Leuders, Prediger, and Barzel, 2011b), this exploratory process has

Ratios and Fractions!

Consider this statement: The ratio of boys to girls in a class is 1 **:** 2. $\frac{1}{3}$ of the class are boys, $\frac{2}{3}$ are girls. The table below is based on statements like these. Can you arrange the given numbers into the right place to make similar statements correct? You must use all the numbers and once only. For example, for the statement I have just made the given numbers would be: 1, 1, 2, 2, 3, 3.

Numbers	Ratio of boys to girls	Fraction of class that are boys	Fraction of the class that are girls
3, 3, 5, 5, 8, 8			
3, 3, 4, 4, 7, 7			
4, 4, 5, 5, 9, 9			
1, 2, 3, 4, 4, 6			
1, 2, 3, 3, 3, 6			

Fig. 5.12 Formatted task to help students connect ratios and fractions (From Jim Noble, personal communication)

been a specific focus for research in determining how to coordinate the individual efforts of learners with the intended conventional knowledge and how to support this through the text. Note that all these approaches suggest that the teacher's intellectual input needs to relate to students' activity.

Barzel et al. (2013) offer three components of knowledge organization necessary for learners to incorporate their experience of exploratory tasks into a repertoire of conventional knowledge: *systematization* (structuring results and connecting them to other knowledge), *regularization* (transforming into the conventional repertoire), and *preserving* (writing in an accessible form). These processes do more than *institutionalization* by also focusing on personal conceptual development and recording. Tasks can align divergent experiences toward shared understanding of concepts and procedures by embedding technical language and conventional symbolism, relating definitions to recognition, and providing individuals with opportunities to express in words and symbols.

A good teacher can provide knowledge organization experiences by orchestrating students' ideas as seen in the task designed by teacher Jim Noble (personal communication, 8 April 2014) when he found that some of his students were confused by information in a textbook that ratios could be expressed as fractions, e.g., 1:2 could be expressed as 1/2. He created a formatted task to help them compare meanings of ratios and fractions (Fig. 5.12). Note that his use of fractions differs from that in the textbook they were using.

Published tasks can also provide these reflective perspectives. KOSIMA includes many strategies for knowledge organization as specific tasks, for example, after studying parallel and perpendicular pairs of lines, students are offered several statements from which to choose *the best* descriptions of parallelism and perpendicularity (Barzel et al., 2013).

5.3.5 Summary

In this section, we have talked mainly about the purposes a designer might have for tasks to address learning or assessment and have given examples of how these purposes are turned into design parameters for collections and sequences of tasks. In the next section, we focus more systematically on desirable forms of mathematical activity and whether these can be shaped by text-based tasks.

5.4 Intended/Implemented Mathematical Activity

The final node in the task design intention triangle relates to the mathematical activity of the task. Design issues related to mathematical activity relate to principles about learning and mathematical aims of particular tasks. We discuss both of these issues in turn.

5.4.1 Principles About Learning

We cannot talk about connections between tasks and learning without some theories about how pedagogy shapes mathematical learning and hence design frameworks as described in Chap. 2. Here, we add to the arguments in Chap. 2 by providing examples of these principles in action in individual test-based tasks and what they might look like on a page or screen.

The idea behind a cognitive conflict approach is that learners are presented with situations that conflict with ideas they already have, so that their ideas have to be modified to incorporate new experiences. Tasks have to bring these conflicts to learners' attention by leading them to become stuck if they continue with the old idea. Learning in this theoretical frame means to adapt, alter, or extend a previous notion, so tasks have to present opportunities to use previous notions and then find contradictions or puzzles that need to be resolved. Many other examples of tasks evincing the resolution of cognitive conflicts created through paradoxes that can be integrated in school curriculum can be found in Movshovitz-Hadar and Webb (2013) and also in Swan (2006).

As an example of how cognitive conflict can be used to extend learning, Barabash (personal communication, March 2014) points to potential conflicts between learners' early conceptions of the tangent concept that do not hold in more advanced settings. In the Israeli curriculum, the concept of *tangent* is introduced in geometry as a tangent to a circle and then in precalculus as a tangent to a parabola. For both the circle and the parabola, the tangent is intuitively understood or actually defined as "if a line has common points with a curve, then it **either** intersects it (in two points!) **or** is tangent to it." However, this result is only true provided the curve is seen as convex and smooth; developing the concept of tangency on this basis is not

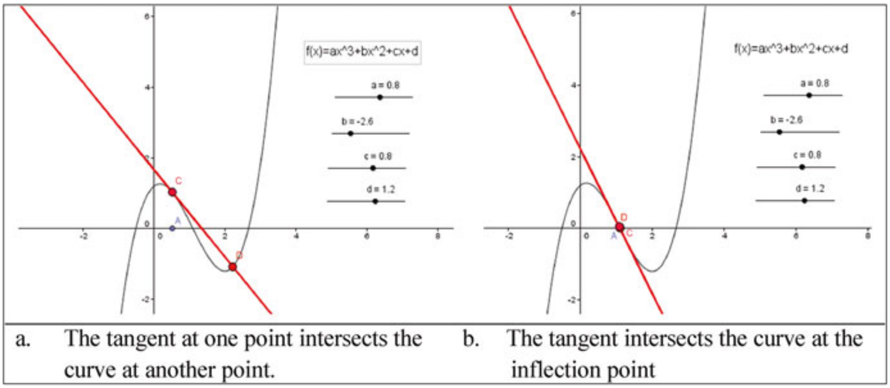


Fig. 5.13 Illustrations in which the concept of tangent conflicts with formal definition as a line that intersects a *curve* at only one point (diagrams from Barabash, personal communication). (**a**) The tangent at one point intersects the *curve* at another point. (**b**) The tangent intersects the *curve* at the inflection point

always valid as illustrated in Fig. 5.13a, b when the curve has power higher than 2. Thus, the means for introducing tangent in early instances and with the given definition are in conflict with later, more advanced perspectives on the concept. The task is to redefine tangency given these two juxtaposed examples. In presentational terms, the graphs can be compared side by side to identify what is the same and what is different.

In Variation Theory (VT), the idea is that learners will notice what is varying against an invariant background (Marton, 2014). Mathematics tasks should be designed so that the desired key idea (known as the *critical aspect* in the theory) is varied and learners can see this and the effects of such variation in successive examples. A full application of VT requires the initial identification of a critical aspect to be learned, and it is this that will then be varied. Such identification takes place through a phenomenographic analysis of learners' work in a particular context, so it could be argued that no static text can fully use VT. Nevertheless, the theory does draw attention to the importance of organizing variation in learners' experience, the *space of learning*. Dynamic digital environments are very useful for this type of variation as variations of a parameter of an object and variation in a representation of the object can be seen at the same time or soon after each other. For instance, imagine learners using a graphing tool to graph $y=x$, $y=2x$, $y=3x$, $y=-x$, $y=-2x$, and so on. Learners should quickly be able to determine that all lines of the form $y=mx$ go through the origin and that the value of m determines the slant and steepness of the line. In static environments, near-simultaneous or adjacent presentation can have a similar effect. The presentation of the sequence of examples needs to make clear what features of a concept are varying in order to show relationships between different aspects of the mathematical idea. The learner is to recognize and generalize the relationship between variables.

ATD (Anthropological Theory of Didactics) and RME are both ways to engineer a need for a formal mathematical idea. Fujii gave an example of such a task (2015, Chap. 9, this volume) in which learners were given various distance-time relationships and asked to identify which relationship represented the *fastest*. It is impossible to give a typical example of an RME task in text, because the nature of the resources varies widely. One type of task is to present a picture, such as a stack of oranges, and invite questions to be posed and generalizations made about piles of oranges of various heights and widths (Dickinson and Hough, 2012). Another type is to offer a realistic problem and a format for the findings of the problem-solving process, such as a bar diagram, which can then become a model for future reasoning (Van den Heuvel-Panhuizen, 2003).

In all these theoretical frameworks, the main pedagogic purpose is to learn new mathematical concepts and methods. In the first two (cognitive conflict and variation theory), this is generally achieved by working on given examples and then comparing and reflecting on the outcomes. In the second two (RME and ATD), learners have often to be exploratory and exert some mental effort to suggest solution paths. Most of the tasks presented so far can be viewed as examples of one or more of these approaches, but Fig. 5.3 offers nothing but the opportunity to mathematize a situation. The teacher could use this figure to develop the need for the idea of proportionality or as a context for applying proportional reasoning. By contrast, Fig. 5.4 is a relatively closed task that can be treated merely as practice in completing linear sequences, but a teacher could then encourage reflection and comparison of outcomes to develop algebraic understanding of linearity, possibly using learners' conjectures to do so.

Tasks on their own are unlikely to address all the complex aims of the mathematics curriculum, particularly those that are about developing mathematical behavior, and the authors' intentions depend on associated pedagogy. Note that the author might be the teacher (as in Jim Noble's task in Fig. 5.12) and may have produced the text-based task to support complex pedagogic aims. The extent to which the teacher understands and supports the pedagogic aims of the text influences the manner in which adaptations are conducted in order to maintain those aims—the issue of curriculum vision and trust (Drake and Sherin, 2009) discussed previously.

5.4.2 Aims Enacted Through Individual Tasks

In earlier sections, we have sometimes described the mathematical activity prompted by a task. We now systematize this from the perspective of desirable mathematical activity. In Chap. 2, there is a call for more focus on the grain size of frameworks as research in task design would benefit from further clarity about different levels of activity. Because the purpose of tasks is to promote mathematical action, we look at grain size from the point of view of actions:

Grain Sizes of Mathematical Actions

i. Basic actions
ii. Transformative actions

iii. Concept-building actions
iv. Problem-solving, proving, and applying
v. Interdisciplinary activity

Compliant and passive learners expect to undertake basic actions of type (i) and can begin to get stuck with transformative actions of type (ii). A common pedagogic approach to overcoming these difficulties is to routinize type (ii) actions by providing rules for transformation, such as *change the side, change the sign* as a routine when solving linear equations, or *FOIL* (first, outside, inside, last) as a routine for multiplying two binomials. Type (iii) actions have been available to learners in many of the task examples presented so far, sometimes explicitly and sometimes implicitly. Where these are implicit, explicit pedagogy and the development of appropriate cultures of classroom mathematical work can ensure they take place. Actions of type (iv) are expected in most national curricula and statements of educational aims, but an overreliance on routinizing type (i) and type (ii) actions and a lack of explicit focus on type (iii) actions can make type (iv) actions, and hence type (v) actions, hard to achieve. We also note that some would say the outcomes of type (iv) activity are essential components of concept building, and we would agree. However, here we are focusing on how a designer needs to imagine what the learner is actually going to DO in response to the task, rather than only imagining what MIGHT be possible with supportive teaching.

It is helpful to think in terms of desirable mathematical behavior that needs to be promoted by tasks. Cuoco, Goldenberg, and Mark (1996) describe *habits of mind*, or what several people call the *verbs of mathematics*, as starting points for tasks (i), (i), and (iii). Similarly, Schoenfeld (1985) and Mason, Burton, and Stacey (2010) provide ways to think about tasks of types (iv) and (v). For this chapter, our main goal is to indicate behavior which can be triggered by text-based tasks, how this can happen, and what remains the domain of pedagogy, particularly the creation of certain classroom cultures.

In Table 5.3, we elaborate on actions at different grain sizes but do not claim that these are mutually exclusive or that the table is complete.

This approach to thinking about what learners need to do omits some aspects of mathematical experience, such as:

- The need to talk, write, and listen to mathematics
- Reasoning for different purposes, e.g., to conjecture, persuade, and prove
- The need to use mathematical feedback, such as self-correcting, monitoring overall sense, understanding comments from others, and appreciating a need for consistency
- Seeing mathematics as part of citizenship—information for understanding the world
- Relating mathematical work to other human values

We see these aspects as pertaining to all grain sizes of mathematical actions. None of these can be embedded in learners' experience solely by indicating them in text or including an opportunity in a task; there has to be the associated pedagogy to ensure they happen. For example, Simon's design of a questioning sequence used

Table 5.3 Actions for different grain sizes of mathematical work

Grain size	General focus	Examples of specific actions
i	Basic actions	Calculating, doing procedures, stating facts
ii	Transformative	Organizing, rearranging, systematizing, visualizing, representing
iii	Concept building	Sorting, comparing, classifying, generalizing, structuring, varying, extending, restricting, defining, specializing, relating to familiar and intuitive ideas
iv	Problem-solving, proving, applying	Conjecturing, assuming, symbolizing, modeling, predicting, explaining, verifying, justifying, refuting, testing special cases
v	Interdisciplinary activity	Incorporating other epistemologies, identifying variables and structures, recognizing similarities, comparing familiar and unfamiliar knowledge

to move Erin toward understanding the traditional fraction division method depends on being responsive to her (described in Chap. 2). He tries to identify, through observing patterns in talk, when she has developed a new schema, so the next question he poses must provide reflection on this new schema toward abstraction. In this way, Simon structures a task that sounds like type (i) but has the effect of moving through to type (iii) as she extends the domain of application and then generalizes her ad hoc and visual reasoning into an algorithm. Although it would be possible to write the questions as a sequence in text, it would not be possible to hold them until the right pedagogic moment arises for them to be effective in changing understanding. Human dialogue is necessary, even though the outcome will be a calculation procedure.

5.4.3 Complex Aims Enacted Through Large-Grain-Size Tasks

It is possible to shape experiences related to more complex activity through providing tasks in textual form. Movshovitz-Hadar and Edri (2013) developed an approach to help teachers bring values into the mathematics classroom (Fig. 5.14).

By reverse engineering the outcomes of the project (as is also applied to educational tasks by Amit and Movshovitz-Hadar, 2011, p. 176), the authors present four issues related to design that make this approach manageable for teachers and learners:

1. Tasks are based in the mathematics curriculum and designed to last one class or homework session.
2. They include a clearly phrased introduction followed by two kinds of short questions: (i) mathematical exploration or thinking and (ii) dialogue to clarify values using mathematical and other perspectives.
3. Editing to avoid obstacles.
4. Advice about the social mode of working: group, individual, discussion, and so on.

In 2007, minorities (Arabs, Druze, and Circassians) were one fifth of the population in Israel. Despite this, only 6.2% of all civil service employees in this year were minorities. Over the years, the Israel government has made decisions (in 2004, 2006, and 2007) to promote suitable representation of minorities in the civil service, setting 10% as a target for the percentage of employment of minorities in the civil service.

1. What do you think about the goal that was set by the government?
2. The Ministry of Housing and Construction had 741 employees in 2007. Had the target set by the government been achieved, how many members of minorities would have worked in the Ministry of Housing and Construction?
3. Twelve employees in the Ministry of Housing and Construction were minorities in 2007. What percentage of all employees in the ministry were minorities?
4. What do you have to say about the two results you obtained?

Fig. 5.14 Values task for Israeli classrooms (From Movshovitz-Hadar and Edri, 2013, p. 382)

Consideration of values-focused tasks raises the question of the starting points for task design and for presentation of the task in text. Is priority given to the context and the mathematical perspectives that emerge from it? Or is priority given to a mathematical idea and the context is then built around it? Design might have problem orientation, concept orientation, or context orientation (Nikitina, 2006). For example, within variation theory the critical aspects of the mathematical concept have priority; however, when interdisciplinary work is an explicit aim, priority might be given to context. In the COMPASS project, Maaβ et al. (2013) report on their development of design issues for interdisciplinary tasks. This was an international project, so thought had to be given to different prevailing pedagogic attitudes and ICT use in the participating countries. The aim was to produce digital and paper-based resources for dissemination beyond the development process, and these had to communicate clearly the key mathematical and scientific ideas so that teachers and learners could see how these emerged from their work. Each task had to make reference to the appropriate national curriculum to encourage teachers to use it. The most appropriate pedagogy for interdisciplinary tasks of this type would have been extended exploratory project-type work, but the designers also provided more structured versions to support teachers who were not confident enough to undertake long tasks. Task designers gave considerable thought to how complex materials could be made teacher friendly and easy to use. Further issues, many of which emerged during the design research process with teachers, included:

- The need for an overview of the lesson sequence, with tasks and subtasks.
- Tasks presented so they could be used directly in lessons without transformation.
- Questions needed for guiding learners.
- Clear links between subtasks and the big contextual questions.
- Possible solutions.
- Information about adapting tasks for different learners.
- Different materials (e.g., task sheets, solutions, background information) had to be easily distinguished at first glance.

What are the main sources that cause severe water shortage in Europe? How do different harmful elements pollute freshwater? What methods / strategies can be used to save water?	**Task 4: Desalination World!** Students learn about potential future sources of drinking water. Students, through a hands-on activity, learn about the desalination process.
What methods can be used to provide people with freshwater? How much freshwater can be conserved if strategies and attitudes at a personal level are adopted?	**Task 5: Building a new Desalination Plant!** Investigate which is the optimal place to build a new desalination plant in order to serve in a fair way the needs of four cities. Students explore different quadrilaterals and their properties (diagonals, collinear points, angles)

Fig. 5.15 Examples from cross-curriculum task in COMPASS (2013© 2010–2011 COMPASS project reuse under Creative Commons Licence)

One such project is about water shortage. The subtask materials are too complex to give in full here. The guiding questions and two of the subtasks are shown in Fig. 5.15.

The issues of design for COMPASS are useful guidelines for the production of any multimedia curriculum package. They address key questions in interdisciplinary work about the balance between context for mathematical learning and the use of mathematics to support learning in other disciplines.

Staats and Johnson (2013) tackled the problem of algebraic competence at college level through a novel interdisciplinary approach. They adopted the typical social science pedagogic method and provided tasks coauthored by a mathematics teacher, a disciplinary specialist, and a creative writer. The creative writer prepares an engaging presentation of a scenario. This is followed by explicit learning goals for both disciplines for class discussion, with scaffolding questions and a short supportive bibliography. We do not have room here to present a full narrative but will summarize the content of one module (for more details see z.umn.edu/icmi22). The core of the module is a short story titled *Indebted,* in which a young man wrestles with the question of how to pay for his college education. The young man visits his grandfather, who suffers from Alzheimer's disease. The grandfather hoped to contribute to his grandson's education but instead had to use his savings for his own care. The young man considers mathematical scenarios associated with indebtedness, such as rapidly rising college tuition and the per capita value of the national debt. Finally, he signs his college loan papers. The learner recognizes the social and emotional dimensions of the problem as well as the mathematical issues.

So far in this section, we have talked about tasks that address the largest of our grain sizes but include aspects of smaller grain size, mainly type (i) and type (ii) with other types to obtain models for prediction purposes. This is not a surprise as most of the research in task design addresses complex aspects of mathematical work. However, there is much to be understood about the design of tasks that scaffold basic, transformative, or concept-building activity. Each of the actions in Table 5.3 could be triggered by an imperative, such as *put in order*; *classify according to …*; *give three contrasting examples of …*; and *prove that ….* As the professional development procedures of lesson study and learning study have shown,

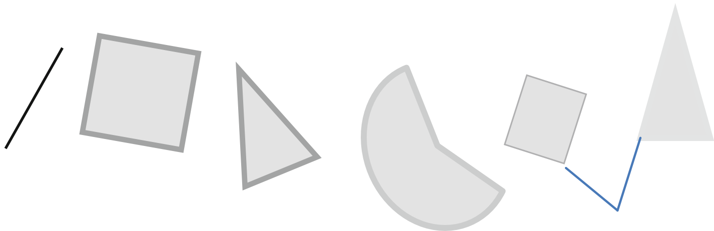


Fig. 5.16 Marks left by *solids* from static contact with a surface or by rolling; the task is to determine which solid could have left the given mark (Barabash and Guberman, 2013, p. 297)

the actual examples used and their availability to learners make a difference in opportunities to learn. We give some insight here, and there are others throughout this chapter.

The example comes from Barabash and Guberman (2013). In writing textbook tasks about shapes, they had to choose between (a) allocating a lesson to each solid in turn and then asking for comparison or (b) looking at all relevant solids together and finding their common and special properties. They chose (b), an approach which brings type (i), (ii), and (iii) actions together, rather than (a) which climbs up from (i) to (iii). The authors embedded the approach in a problem situation in which an intruder has left traces of certain solids in the form of stains made from static contact or traces left by rolling solids (see Fig. 5.16), thus working with type (iv) actions. Their overall aim is to develop *mathematical insight*, which is akin to the organization of knowledge described by Barzel et al. (2013). Their approach is shaped at the start of the work by the expectation that mathematics is going to provide the analytical tools to identify the solids. Students explore with the solids, make conjectures, and then refine their conjectures.

Note that we have addressed all grain sizes from the zoomed-in view of learners' experience, rather than from a zoomed-out view of curriculum aims. How learners can be "ramped" from simpler tasks to more complex tasks, where their previous experience has not prepared them for complexity, is a related problem. One useful approach is to use a gradient of "novice", "apprentice", and "expert" tasks (MARS, 2014). These descriptions were developed for designing assessment tasks but could be used equally well for the development of complex mathematical habits of mind.

5.5 Visual Features of Text-Based Tasks

Text-based tasks are planned, prepared, and presented to learners visually; they are not the tasks that arise within teacher-learner dynamics. Most of our discussion is about the work that goes into planning and preparation, but the experience for the learner is firstly visual. For this reason, research about learners' experiences with text-based tasks needs to take into account many of the same perceptual impacts as might be considered by graphic designers. However, we found little research which related perceptual impact to mathematical cognition. In the graphic design literature, we read that the learners' attention can be directed in sequence to particular features, as in statistical representations, but making mathematical sense of those features and coordinating with other features are not widely discussed across mathematics. In a few textbook comparisons, researchers draw attention to the use of color, pictures, text boxes, and so on and whether these are used for cognitively specific purposes or whether they are merely to make a page appear attractive to young learners. Thompson et al. (2013) classify the use of graphics as was previously shown, but we repeat a shortened form here: no graphic; graphic does not illustrate inherent mathematics; graphic explicitly illustrates inherent mathematics; graphic has to be interpreted to answer a question; and make a graphic.

When we talk of visual features as part of a task, we are not interested in the use of color, pictures, or position merely for visual attraction but at how those features contribute to learners' mathematical activity by enabling coordination of the eye and brain. In the static environments that are our focus in this chapter, we are also interested in how pictures and diagrams can suggest action. For example, in Fig. 5.6 the pictures suggest both collation and separation using place value. In Fig. 5.10, the sweets suggest systematic enumeration.

In Fig. 5.6, the pictures have a deliberate cognitive purpose in that they illustrate actions which can contribute to an understanding of the symbolic representations that follow immediately. In Chap. 2, Fig. 2.7 demonstrates a similar set of pictures and diagrams. Symbolic statements are placed next to each other when they relate to each other in particular structural ways or follow each other in a deliberately varied fashion. Without the need for mediation through speech, a learner who deciphers the page from top to bottom and left to right has the information they need to complete the suggested statements. In Fig. 5.4, the layout confirms for the learner that they need to fill in some blank spaces, and when this is done there are some relationships to be found. In other words, the layout encourages comparison, conjecture, and generalization between sequences rather than merely completing them separately. However, visual similarity does not always imply mathematical equivalence and learners have to sort out when it does and when it does not.

Learners have to make a distinction between pictures and diagrams. For example, Curcio (1987), among others, describes learners being over-influenced by the shape of graphs when matching them to situations that might be generating the relevant data and points to confusion between words such as higher, faster, and lower and the shape of associated graphs. In several studies, learners are seen to react visually to diagrams that need to be interpreted symbolically (Dörfler 2005; Radford, 2008). In geometrical reasoning, a diagram has to be understood not as an accurate case but as a representation of a system of relations and properties. Moreover, both diagrams and pictures can introduce simplifications or elaborations which could mislead novice learners. For example, a vertex could look as if it is a right angle when it is supposed to be general; a learner might assume that the base of a triangle has to be parallel to the page edge. Love and Pimm (1996, p. 380) point out that dynamic digital technology will help learners to understand that a single example is an instance of a class of figures and that a specific example can be manipulated to address these potential misconceptions. Puphaiboon, Woodcock, and Scrivener assume a dynamic environment, saying "The graphic representation must portray the relationship between the graphical parts in time and space to reinforce cause and effect relationships" (2005, p. 3). Static images have to embed this dynamic relationship and be understood as instances of a variable class so that they direct the learner toward the salient features of the class.

Tufte shows a variety of ways in which quantitative and dynamic data can be represented through static diagrams, such as through labeling, encoding, relating data to familiar scales, etc. (1997, p. 13). He claims that good design enables attention without clutter so that "clear and precise seeing becomes as one with clear and precise thinking" (p. 53). Shuard and Rothery (1984, p. 61) draw attention to the use of arrows in text to indicate some movement and action, the static equivalent of mouse clicks and dragging; the use of arrows is a convention that learners know from outside the classroom but that might have a special meaning and use inside mathematics. Diagrams make a difference to learning; so long as the diagram and its associated text are near each other so the eye can move back and forth, relative position does not matter (Shuard and Rothery, 1984, p. 53). Some recent research using eye tracking to determine if experts and novices *read* tabular data differently has shown no differences, but each participant in the study seemed to have a personal pattern of engagement with such data (Crisp, Inglis, Mason, and Watson, 2011). More work is needed in this area to find out how learners acquire and coordinate information from a mathematical text.

The use of color for specific mathematical purposes became established in nineteenth-century geometry teaching, for example, Byrne's edition of Euclid (1847) color to draw attention to different objects and quantities whose relations could then be understood spatially. Fig. 5.17 gives a sense of the use of color to compare objects.

Color is widely used in the teaching of algebra to draw attention to like terms or to provide spatial patterns to be generalized (e.g., showing $(x+y)^2=x^2+2xy+y^2$ with appropriate shading), as a way to draw attention to area as the space inside a closed 2-d shape and so on. Indeed color is used for a mixture of mathematical and

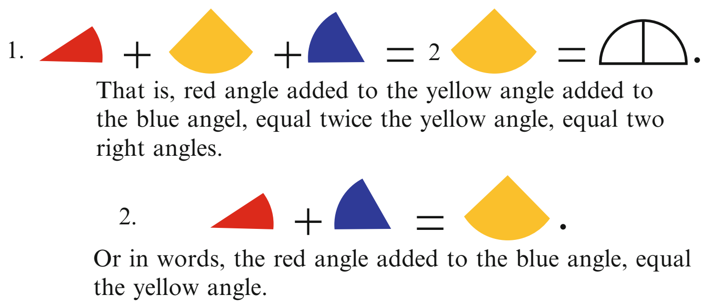


Fig. 5.17 Use of *color* to relate objects (Byrne, 1847, p. x)

attentional reasons to support the distinctions learners make when understanding a mathematical idea, but it is also used purely for visual variety. We wonder how learners learn when to use color cognitively and when it has no mathematical purpose. The use of specific color words disadvantages color-blind learners (up to 10 % of boys and 0.5 % of girls can be color blind). In Fig. 5.17, the words *red*, *yellow*, and *blue* will confuse a significant number of people, so although different shades may be perceived, it is better to refer to them in some other way, as is done in Japan (Ohtani, personal communication, 22 May 2014). The key idea is how the text draws learners' attention to examples and how they relate to each other. As well as the use of emphasis through color and layout, the control of variability and the near-simultaneous presentation of variation are key factors in the kind of attention that is necessary for type (ii) and (iii) activities. Tasks in Figs. 5.5 and 5.9, among others, demonstrate the use of juxtaposition to elicit conjecturing and generalization.

Thinking about the page as a whole, Kress and Van Leeuwen (1996) use the metaphor of an art gallery or museum (see Yerushalmy 2015, Chap. 7, this volume) to think about where to position items to direct the learner in a logical or developmental order while making ancillary information and elaborations available through hyperlinks. Although difficult to replicate on the printed page, such links could easily be provided in digital text. Cognitive load theory, which is concerned with finding the optimal number of ideas that can be handled to understand a concept while not oversimplifying it, also has a role to play in the preparation of a page. The learner should be able to follow a pathway through the text that allows access to core ideas, possibly through various representations and instances, without becoming too distracted by irrelevant details; in essence, the learner needs to distinguish the core idea from other material and cannot learn to do that if it is always presented in isolation (Love and Pimm, 1996). In cognitive load theory, researchers are concerned with whether the content is intrinsically necessary for the object of learning or germane to it or extraneous (e.g., Paas, Renkl, and Sweller, 2003). We would argue that the grain size of the pedagogic intentions determines whether these loads are desirable or not. For type (i) activity, only intrinsic content is necessary; for the other types of activity, mixtures of intrinsic and germane and even extraneous content are desirable. In variation theory, it is suggested that background variables (i.e., those that are not the critical aspect for learning) should be kept invariant but need to be present to enable variation in the critical aspect to be observed in the foreground. An alternative is to have variation of many features but invariance of a key feature. Examples could be a collection of contextual problems that all have the same underlying structure when the structure is the intended object of learning or a collection of quadratics that all have the same roots when roots are the intended object of learning.

An associated factor is whether collections of exercise questions in grid form are done horizontally or vertically and whether it matters. The example from an old algebra text in Fig. 5.18 shows that it does matter whether the learner follows rows or columns. The authors of the exercise seem to be aware of the value of organizing

Fig. 5.18 From *Elementary Algebra*, Part 1 (Godfrey and Siddons, 1915, p. 43, Cambridge University Press)

1.	$\frac{3}{11} \times 5$	4.	$\frac{5}{12} \times 3$
2.	$\frac{5}{11} \times 5$	5.	$\frac{7}{12} \times 3$
3.	$\frac{x}{11} \times 5$	6.	$\frac{x}{12} \times 3$

Find the sum of:

1.	$a + b$ and $a - b$	2.	$2x - a$ and $3x + a$
3.	$-x + a$ and $x + a$	4.	$2x + a$ and $3x + a$
5.	$a - 3b$ and $a + 2b$	6.	$2a - b$ and $3a - b$

Fig. 5.19 From W. Baker and A. Bourne (1937, p. 19)

variation in examples, and the numbering order encourages comparisons, so that the role of the numerator can be reflected upon.

In Fig. 5.19, the authors appear to be aware of the importance of variation, but the layout and order provide several variations between successive questions, so that little reflective awareness is available. If the first two questions had been: $a + b$ and $a–b$; $-b + a$ and $b + a$, there would have been something to notice and justify which could have supported conceptual learning. A type (i) task could have become a type (i), (ii), (iii), and even (iv) task in this way.

The amount of writing may be an issue for some learners. Shuard and Rothery (1984) studied 400 learners' reactions to versions of a task with more or less writing required and found that reactions varied (p. 130). Learners' reactions may be due to past experience, causing Shuard and Rothery to conclude that people need help to learn how to read mathematical text. Reactions to an exposition presented in comic strip style also varied inconclusively, but there is now a range of popular resources of this style on the internet.

The use of background effects such as grids, frames, fills, and so on can enhance attention and avoid the *flatness* of appearance of the page (Cuoco, 2001), in which every part of the text appears to have a similar status. However, care has to be taken that such effects do not mislead readers. For example, presenting rectilinear shapes on squared grids can mislead learners into counting boundary squares to find the perimeter.

In addition, the choice of font, or variation of fonts, and length of lines of print can influence learners' attention and reading capability. Such features can even influence the ways in which teachers and their learners engage with the text, an interesting example of this being the use of handwriting in the RLDU materials (e.g., Fig. 5.2) establishing a sense of mathematics as a human and exploratory endeavor. However, in Shuard and Rothery's study of 400 learners aged 11–12 using handwritten text, some found this font friendly and helpful, but others found “funny writing” harder to read (1984). Literacy in the language of instruction is an omnipresent issue as are broader issues about language and mathematics.

In the discussion so far, we have recognized the difficulties designers have in enacting their intentions through their design. In every case, we have focused not on what WILL be learned but the opportunities made available to learn by each task. We now turn to how designers need to anticipate teachers’ use of tasks (see also Chap. 3).

5.6 Teachers’ Use of Tasks

Design and use are like two sides of a coin and both are influenced by the educational system, assessment system, culture, and other contextual features. The job of design is to communicate to teachers and learners through text the mathematical intentions; the teacher’s role includes modification to enable learners to connect to the core ideas and learning goals (Love and Pimm, 1996; Rezat, 2006; Tzur, Zaslavsky, and Sullivan, 2008). Assuming that the designer is not the teacher, we ask whether the design assumes that teachers can use the task as written, or whether teachers will need to adapt the task for their context. The latter assumes that teachers have the motivation, time, and knowledge to make adaptations, whereas the former masks the fact that teachers are likely to make adaptations anyway, deliberately or not. Teachers are integral actors in the whole process of design when they use published tasks in their classrooms, with the tools, cultural expectations, and norms of classroom life. An important aspect of professional learning is to become critical users of an externally designed task. There has always been debate about this. Wittmann (1995) wanted to preserve task design as a specialist process undertaken by those who have time and experience to develop tasks that are to some extent *teacher proof*. Stein, Grover, and Henningsen (1996) point out that there will always be adaptation of tasks in use, at least because of classroom dynamics and at most because of teachers who alter the goals and demands of tasks. These issues are explored further in Chap. 3 of this volume.

Prestage and Perks (2007) offer a collection of task-adapting tools, with which busy teachers can develop complex tasks from textbook resources: change a given, add a mathematical constraint, change representations, and so on. Whereas designers have more time, experience, and access to research than busy teachers, teachers have more local knowledge but need design adaptation tools of this kind. Swan (2006) also provides design heuristics that could be used by teachers to create and adapt tasks:

- Is a statement always, sometimes, or never true?
- Interpret, match, and classify different representations of similar objects.
- Diagnose and correct examples of common mistakes.
- Resolve cognitive conflicts.
- Create new problems by reversing given problems or varying givens.

The final type can be extended by turning givens into variables, a technique also suggested by Prestage and Perks (*ibid.*). In an elementary example, given that $8=3+5$, tasks could be created to find x where $x=3+5$; $8=x+5$; $8=3+x$ or x and y where $x=y+5$; $8=x+y$ and so on. This sequence shifts learners from a number fact to dealing with an unknown, to dealing with variables and, finally, $z=x+y$ could be an exploration of structure. Watson and Mason (1998) collected generic task design heuristics by providing a range of actions that can be applied to mathematical objects: classifying, ordering, defining, constructing, varying, reversing, exemplifying, and so on. Using these, teacher task design can be a repertoire of in-the-moment strategies rather than a time-consuming process.

Lee et al. (2013) observed how teachers modified tasks and noticed they would typically change the givens or the context or the demands of the question. One example they shared is as follows: a rectangle of paper is folded in half and then cut along the diagonal of the new shape; the textbook asks for the name of the resulting shape and where equal angles might be found. One teacher made the task more open ended by asking students to find the properties of the resulting shape. In their research, the mathematical knowledge of the teachers played an important part in their decisions to improve text-based tasks. In contrast, Lundberg and Kilhamn (2013) show how problems inherent in a published task derailed a teacher who relied on the textbook to prepare learners to solve some ratio problems. The problem involved mixing lemon squash using juice, water, and sugar and asked for measures in liters. They report widespread confusion about whether the sugar contributes any volume to the drink, confusing everyday knowledge and mathematical assumptions. They also report that teachers resorted to ad hoc additive methods rather than setting up a multiplicative equation as the authors expected.

In several papers referred to in this chapter, teachers have been involved throughout the design process (e.g., Hußmann et al., 2011a; Movshovitz-Hadar and Edri, 2013). In many countries, teachers are involved in collaborations that produce banks of tasks, shared among teachers (e.g., Sesamath, Wikitext, SMILE). There is a growing use of digital sharing which enables individual teachers to adapt the text, the examples, and the language for their own learners. This massive growth of resources places an increasing burden on teachers who design or selectively choose tasks rather than rely on the authority of an unknown author from the web. It is safe to assume that the reason for proliferation of such resources is teachers' dissatisfaction with commercially published materials and inability to find published tasks which address precisely their teaching goals. There is also software which supports teachers' creation of worksheets, sometimes from banks of individual questions, and video resources. This could be seen as rejection of the authority of textbook authors, publishing houses, or outside designers.

Designers, by contrast, are often concerned with how to make their intentions explicit to teachers and what support to provide to enable tasks to be used as intended. It is likely there needs to be a professional development component in the textual presentation of the task, and teachers need time to prepare themselves to use a task fully. Even though there is a need for research about whether, how, and why teachers pursue the explicit intentions of task designers, such research will always be contextualized within the normal pedagogic practices of the research sites. So, such research might be seen as local, specific evaluation.

Other designers might want tasks to provide something that can be used directly by learners without teacher mediation. When these are published as collections, such as in the textbook or as a package of worksheets, there may be consistent, strong messages about how to study. For example, readers might be asked to predict answers or reflect on key ideas that arise in a task. In a package of tasks that is developed over time with classroom trialing, the question of how, whether, and why learners take up these messages would be a key aspect of evaluation.

One particular aspect of teacher adaptation that may be of general concern is when tasks are adapted so that learners of different capabilities can work with them. Such adaptations can simplify or extend the learning goals or simplify access or both. Designers might indicate ways in which tasks can be adapted that maintain the core learning afforded by the task.

Moving from the use of individual tasks to the sequencing of tasks, again much depends on the prevailing culture. For example in the UK, the roles of good mathematics teachers are to provide the curriculum and cognitive mathematical coherence and to adapt text not only to engage learners but also to help them organize their knowledge (in the sense offered by Barzel et al., 2013). Ancillary materials and teacher guides might be available but are not necessarily used consistently. The use of teacher guides and textbooks in some cultures is seen as central to professional practice; for example, the *kyozaikenkyu* phase of Japanese lesson study uses these, while Chinese teachers claim to learn most from the teacher guides (Fan, 2013), and *concept study* is a growing practice in Canada (Davis, 2008). Some schemes provide no learner textbook but offer teacher-friendly guides and resources such as photocopiable masters, apps, and online resources. Teachers who engage in these schemes have to engage with the guidance, possibly collaboratively, and develop their own teaching from the scheme. By contrast, there are schemes which provide detailed lesson scripts or videos to copy. There is, therefore, a spectrum of practice and expectations, ranging from using the task sequence provided so as to adhere to the implied theories and goals of learning and development behind such sequencing, to teachers developing their own sequencing and populating it with tasks from a variety of sources and media. The relations between tasks and teaching are pluralistic and situated.

5.7 Conclusion: What Text-Based Tasks Can and Cannot Do

To introduce this section, we present a thought experiment as an extension to an example given earlier from Taiwan about plausible pedagogic approaches to the interior angle sum of any triangle, assuming that learners understand what angles are.

- Teachers could state the property and then give learners various triangles with two angle measures so that learners use the property to find the measure of the third angle.

- Teachers could have learners draw various triangles, measure all the angles, and then put the sum of the measures of the angles at the front of the room, presumably having most sums close to 180°.
- Teachers could construct a triangle using geometric software, have learners measure the angles, place the measures in a table, and then drag one vertex of the triangle and record the angle measures as they update, again finding that all sums are 180°.
- Teachers could have learners draw various triangles, tear the triangles into three pieces without tearing through the angles, and then place the angles adjacent to each other to demonstrate that the three angles form a straight line.
- Learners could be told that angles round a point add up to 360°, given a tessellation of the plane by congruent triangles, and asked to use logical reasoning to deduce the angle sum of any triangle.

The underlying mathematics concept is the same in all five tasks, but the nature of the mathematical activity embedded within each instantiation influences the degree to which learners develop sensemaking, reasoning, and a justification that such a relationship is true for all possible triangles. The version of the task used by the teacher depends to some extent on a curricular and pedagogical vision of learning. It is possible to imagine all of these presented as written text to learners, especially the prepared triangles in the first suggestion. However, could learners follow the instructions (especially in the fourth version), and, if so, would they come to the conclusion that the angle sum is 180° without a further lesson phase of regularization, systematization, and verbalization as described by Barzel et al. (2013)? In this thought experiment, nothing needs to be prepared on paper apart from a bank of examples in version 1 or the materials for version 5.

So why do we need text-based tasks at all? Most teachers cannot initiate all mathematical activity from their own creativity and resources, due to a range of workplace limitations. As we have shown, tasks can offer engagement in mathematical processes and opportunities to demonstrate, practice, and apply knowledge. They can offer suggestions for action, in an order, with some intentions for learning, with a range of visual and verbal stimuli in planned positional relation to each other (possibly using hypertext). Text-based tasks can offer models of structuring questions and prompts at all grain sizes of mathematical activity, planned sequences of tasks, conceptual focus and development, representations, pedagogic assumptions, and triggers to organize knowledge and can also provide simultaneous or sequential representational variety, possibly using hyperlinks. At the level of complex tasks, text can provide realistic resources which would be hard for individual teachers to find or construct and can bring together documentary resources for enquiry tasks. Text-based tasks can introduce teachers and learners to new ways of engaging in mathematics even if these are not taken up. We have also shown that text can provide formats which structure mathematical information and make comparisons and connections available for learners and teachers. Text can offer visual repetition of useful images and layouts. Text can also provide frames and methods of self-evaluation.

Many of these features can be provided digitally, but it is important to note that text can offer learning management systems in which tasks are presented in an order based on mathematical and educational principles; static text can offer immutable structure, data, images, and layouts; static text can be used in a variety of on-screen and off-screen modes of working. Static text cannot provide direct haptic experience of mathematical change nor instant feedback from all possible learners' actions nor models of continuous variability in mathematical and other phenomena. Furthermore, text cannot provide the important phases of learning that take place through interaction and mathematical reflections on what has been done by a particular set of learners. In other words, to echo what was said earlier in Chap. 2, tasks are only one element of a complex interactive learning ecology.

5.7.1 A Potential Research Agenda

The design principles described throughout this chapter and illustrated with varied examples from a range of international sources suggest areas where future research might be conducted. Research into textbook design and use is being undertaken widely and addresses concerns about the relationships between curriculum authorities, publishers, author teams, teachers, pedagogy, and learners from many perspectives and has led to international conferences (International Conference on Mathematics Textbook Research 2014) and several publications (e.g., Thompson and Usiskin, 2014).

Where individual tasks are concerned, teachers' use (Chap. 3) and learners' perspectives (Chap. 4) make critical contributions to the act of design. In thinking about the actual words, diagrams, and appearance of text-based tasks, we can ask: *how do differences in authority and voice in text-based tasks influence learning* and *how do visual aspects of text-based tasks influence attention and learning*? There is little robust research about how these aspects of text-based tasks influence learning. More attention to these, such as is undertaken in Learning Study, might help answer the question: *what different conceptual experiences arise from different task treatments of the same concept*? Working on such comparisons will generate more knowledge about the relationships between task and learning.

We can also ask: *what are the relationships between grain size of tasks, types of mathematical activity, and learners' mathematical development*? We are not convinced that these are always fully matched in practice. In structuring this chapter, we offered a triangular, interdependent relationship between the nature and structure of a task, its purpose, and the resulting mathematical activity for consideration. This structure has given a way to think about design and selection of tasks that places the task at the heart of the connection between teaching and learning.

The questions above should all be seen in the context of more general research about design principles, implementation, teacher knowledge, learners' perspectives, and digital affordances as described in other chapters.

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