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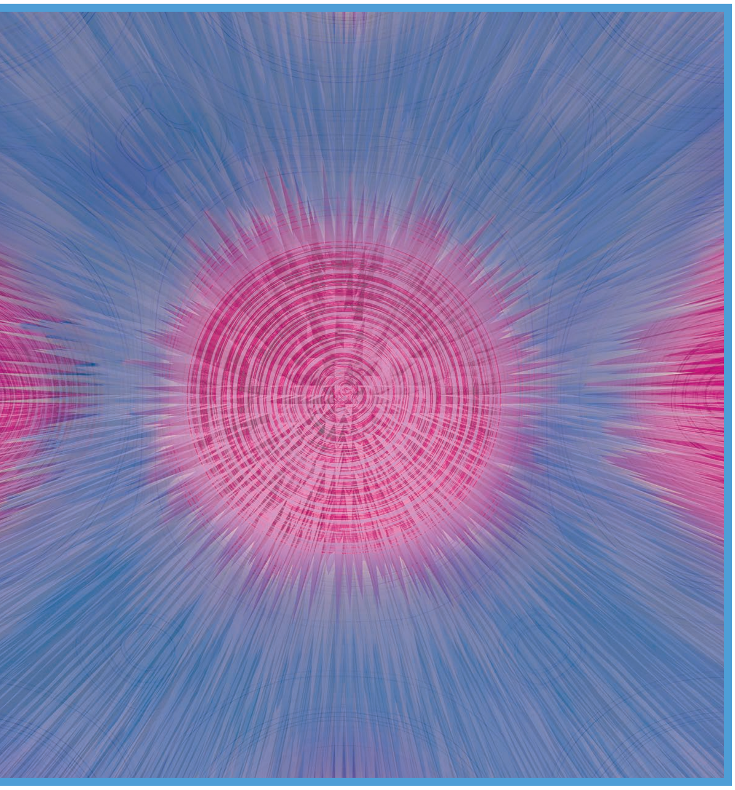
Hypercomplex Signal Processing in Digital Twin of the Ocean

Theory and application



The digital twin of the ocean (DTO) is a groundbreaking concept that uses interactive simulations to improve decision-making and promote sustainability in earth science. The DTO effectively combines ocean observations, artificial intelligence (AI), advanced modeling, and high-performance computing to unite digital replicas, forecasting, and what-if scenario simulations of the ocean systems. However, there are several challenges to overcome in achieving the DTO's objectives, including the integration of heterogeneous data with multiple coordinate systems, multidimensional data analysis, feature extraction, high-fidelity scene modeling, and interactive virtual–real feedback. Hypercomplex signal processing offers a promising solution to these challenges, and this study provides a comprehensive overview of its application in DTO development. We investigate a range of techniques, including geometric algebra, quaternion signal processing, Clifford signal processing, and hypercomplex machine learning, as the theoretical foundation for hypercomplex signal processing in the DTO. We also review the various application aspects of the DTO that can benefit from hypercomplex signal processing, such as data representation and information fusion, feature extraction and pattern recognition, and intelligent process simulation and forecasting, as well as visualization and interactive virtual–real feedback. Our research demonstrates that hypercomplex signal processing provides innovative solutions for DTO advancement and resolving scientific challenges in oceanography and broader earth science.

Introduction

The digital twin, originally envisioned as an "information mirroring model," involves the use of physical models, sensors, historical operational data, and knowledge to simulate complex, interdisciplinary, and multiscale processes [1]. This approach aims to unite digital replicas to assess the corresponding physical entity's entire lifecycle. Digital twins have found applications in diverse domains, including smart cities, security surveillance [2], maritime transportation [3], and digital earth. As digital twin

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technology advances, its scope of applications continues to expand, notably including marine environments, driving the development of the DTO [4].

The DTO fundamentally utilizes modern information technology within the marine domain to create a comprehensive and interdisciplinary simulation platform for continuous and detailed monitoring and virtual–real interactions of the ocean. The main goal of the DTO is to create a comprehensive and interdisciplinary simulation platform that unites digital replicas, forecasting, and what-if scenario simulations for assessment of the ocean conditions under different scenarios. This enables the study of intricate phenomena, precise predictions, virtual reconstruction of historical and current ocean states, and the evaluation of socioeconomic impacts arising from factors like climate change, natural disasters, coastal development, and human activities.

As the ocean is a complex system, analyzing data related to interactions among land, ocean, and human activities involves various data.

The initial idea behind the DTO is to translate complex ocean systems digitally into real-time data signals and analyze these data intelligently to understand, simulate, and forecast ocean situations. As the ocean is a complex system, analyzing data related to interactions among land, ocean, and human activities involves various data. It encompasses scalar fields like temperature, salinity, and dissolved oxygen concentration; vector fields such as ocean currents, winds, and waves; and tensor fields like sediment deformation flow. These marine environmental factors are captured by diverse data types and characterized as spatiotemporal (ST) coupling, high dimensionality, multisource heterogeneity, and dynamic evolution [5]. This brings challenges in integrating analyses for understanding fundamental mechanisms, enhancing prediction trends, and simulating and controlling the behavior of the overall ocean system. However, the distinct signal characteristics impose notable challenges upon signal processing techniques, especially for high-dimensional, multisource, and massive ocean data. This obscures the key signals in ocean changes in response to more complex coupling influences from atmospheres, land, and human activities [6]. Advanced methodologies are demanded to integrate, process, and represent ocean data to extract crucial insights.

In the domain of signal processing, the hypercomplex signal, an extension of the analytic signal to multidimensional cases, finds applications in fields such as digital imagery, hyperspectral imaging, and geophysics [7]. The mathematical basis of hypercomplex numbers, encompassing quaternions, octonions, and multivectors [8], along with techniques such as geometric calculus, Fourier transform [9], and filtering, offers powerful tools for multidimensional signal representation, information integration, model construction, visual expression, and feedback control mechanisms [10]. In particular, within the context of the DTO, the utilization of hypercomplex signal processing techniques offers substantial benefits in terms of improved accuracy, reliability, and efficiency.

Despite the growing importance of hypercomplex signal processing in DTO-related domains, a comprehensive review of its accomplishments is currently lacking.

This article seeks to offer an overview of this field, covering both theories and applications. For the theoretical part, we hope to describe signal processing methods using geometric algebra, quaternions, and hypercomplex machine learning. For the application part, we provide practical applications of hypercomplex signal processing methods in data representation and information fusion, feature extraction and pattern recognition, and intelligent process simulation and forecasting, as well as visualization and interactive virtual–real feedback in the DTO construction. By combining both theory and applications, we aim to showcase the flourishing development of hypercomplex signal processing in the emerging field of the DTO. This article contributes by being the first to explore the feasibility of using hypercomplex signal processing methods for the DTO, accompanied by a comprehensive review to offer innovative solutions to the demands and challenges faced by oceanography in the context of digital twin advancements.

The article is organized as follows: The section "DTO" provides a brief introduction of the DTO and its bottlenecks. The section "Hypercomplex Signal Analysis in the DTO" describes the inherent relationships between the DTO and hypercomplex signal processing. The section "Theories of Hypercomplex Signal Processing in the DTO" introduces the theory of hypercomplex signal processing in the DTO. The section "Application of Hypercomplex Processing in the DTO" presents the application of hypercomplex signal processing in the DTO. Discussions and conclusions are provided in the section "Conclusions."

DTO

Major components

The development of the DTO system should encompass four key components; the desired effect of each part is illustrated in Figure 1.

Data representation and information fusion

The DTO framework includes the representation of data from diverse data sources, including remote sensing and in situ observations, deep ocean sensing and photogrammetry, ocean buoys, and shipboard observations, as well as the model simulation of marine environment factors. The vast volume and variety of ocean data necessitate a unified approach to ensure real-time accessibility, reliability, and standardization [11]. By integrating data from multiple sources, the DTO can leverage the full potential of its assorted data streams. This not only addresses technical needs but is also a foundational element for enhancing the DTO's capacity for detailed analysis and effective prediction. The adept integration of diverse information significantly augments the DTO's functionality and applicability in various oceanic and environmental scenarios, standing as a crucial aspect of the evolution of the DTO paradigm.

Feature extraction and pattern recognition

The marine system is a complex entity characterized by a mixture of scales and factors. The feature extraction and pattern recognition component of the DTO extracts diverse spatial–temporal features and evolutionary patterns from massive ocean data. It employs advanced signal processing, AI, and geophysical modeling techniques to identify and analyze key elements and patterns amid sparse observations. This component processes raw data, highlighting essential features, and utilizes advanced algorithms to detect crucial patterns and anomalies, driving informed decision-making and forecasting. These powerful functionalities support marine environmental monitoring, resource management, and disaster prevention.

Intelligent processing simulation and forecast

A fundamental aim of the DTO initiative is to significantly enhance the utility of extensive oceanic datasets and develop understanding of the oceanic mechanisms behind the data. To realize this objective, the DTO methodically integrates the model simulation and intelligent analysis to convert comprehensive datasets into practical, actionable knowledge. This innovative approach lays the foundation for detailed simulations of complex ocean systems effectively interlinking terrestrial, marine, atmospheric, and socioeconomic elements. The comprehensiveness of this strategy not only augments our understanding of oceanic systems and its dynamic behaviors, a pivotal aspect in achieving accurate ocean forecasting, but also arms stakeholders with sophisticated tools to analyze and react to various hypothetical scenarios. For instance, it could allow users to simulate the consequences of reclamation projects [12] or extensive ship route extension in the Arctic seas [13] for the ocean.

Visualization and interactive virtual–real feedback

The core of the DTO lies in the reciprocal relationship between a virtual model (ocean simulations) and its real-world equivalent (the ocean), enabling mutual impacts. Interaction occurs through coupling of observations and measurements within their respective virtual and physical domains. Both virtual and physical entities develop independently and dynamically within their environments, interconnected by real-time information exchange. Users interact with the DTO in a realistic, immersive setting, where the virtual model is refined with real-world observations and, in turn, identifies areas in the real ocean requiring targeted observations. This interactive framework incorporates advanced technologies like geovisualization, immersive visualization, holographic displays, and virtual/augmented reality [14]. These technologies enhance data exploration, synthesis, presentation, and analysis, offering immersive services and fostering collaborative interactions, thereby transforming user engagement with ocean environments.

Characteristics of the DTO

Within the DTO, the data collection and integration component cumulatively gathers data from various sources like satellite observations, buoys, automated stations, and sensor networks. Advanced techniques like AI, physical modeling, high-performance computing, and simulation capabilities then process these data, extracting features and patterns of the ocean environmental factors. It provides scenario-based simulations and forecasts to promote marine information creation, manipulation, and analysis to deepen understanding of the ocean. In this context, the DTO transcends conventional information systems, presenting a more intricate landscape for data collection and processing when compared with traditional geographic information systems (GISs) and simulation models. The virtual reality interactive system involves the presentation and communication of simulation results to various stakeholders and the general public through advanced visualization techniques, including 3D rendering, animations, and augmented reality [15]. Its integration with other systems, such as globe climate models [16], land use planning systems [17], and disaster management systems [18], facilitates a comprehensive approach aimed at enhancing our understanding of the potential consequences of actions and decisions [19]. This approach plays a pivotal role in addressing critical environmental challenges, including those resulting from climate change, biodiversity loss, and related issues.

While the DTO is poised to advance scientific comprehension of oceanic phenomena, its overarching goal is to create an open and accessible platform that unifies various data sources, models, and tools. This collaborative platform is intended to cater not only to the scientific community but also to a broader audience, including policymakers and stakeholders in the blue economy sector. A typical DTO system encompasses the collection and integration of diverse data sources, the development of a real-time virtual model utilizing advanced modeling techniques, the simulation and analysis of diverse scenarios, and the effective visualization and communication of results to form the virtual–real feedback.

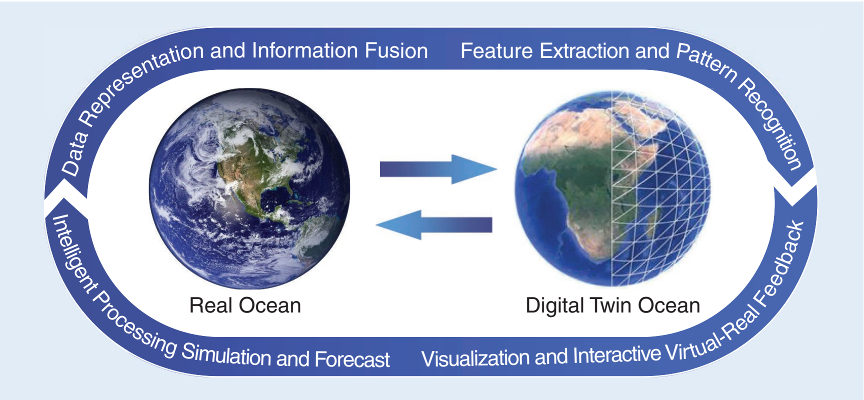

FIGURE 1. The concept model and major components of the DTO.

These integrated components collectively contribute to provide science-based decision support systems for global and regional marine management in many cross-cutting issues such as climate change, marine spatial planning, and ecosystem resilience.

Current bottleneck of the DTO

Although the DTO has made significant progress in recent years, it still faces the following bottlenecks.

Integration of data with different coordinate systems

The integration of diverse observation data in the DTO from different spatial and temporal scales is the first step to construct the DTO. Due to variations in data collection range and sensor types, the data obtained by different sensors have different coordinate systems. For example, the DTO needs to integrate global satellite imagery and various terrain observations, including fixed hydrological stations and mobile buoys or ship-based systems. Achieving accurate ocean replication means integrating data from different coordinate systems simultaneously. However, converting different coordinate systems leads to precision loss and computational complexity. The DTO needs a unified coordinate framework capable of accommodating and transforming various coordinate types for accurate analyses and predictions in oceanic and environmental contexts.

The DTO, characterized by its ability to simulate "what-if" scenarios in a realistic marine environment, offers accurate feedback within a universal platform.

Integrating multisource heterogeneous data

Integrating heterogeneous data, which encompass various data types such as vector, raster, trajectory, and ST fields, is a significant challenge in the DTO [4]. Accurately representing the real ocean in the DTO requires integrating and analyzing ocean factor data, evolution processes, and interaction mechanisms across multidimensions. Although traditional GIS processing has a relatively fixed workflow for these data types, the DTO's data need be dynamically updated and analyzed based on different analytical tasks. The fusion of different data types is a significant bottleneck in the DTO. In summary, the successful integration of multisource heterogeneous data in the DTO relies on the establishment of a unified storage system and elastic data types. Building a unified data model with elastic data types and content enables the establishment of a unified storage system, maximizing the DTO's potential for detailed analysis and effective prediction in oceanic and environmental scenarios.

Modeling and analysis lie in fidelity scenes

The ocean's complex nature, characterized by multiscale, multiinterface, and multiprocess attributes, presents challenges such as data sparsity, variable indexing, and partial observations. This necessitates a departure from traditional marine modeling methods. The DTO framework, encompassing both dynamic and static ocean scenes, demands accurate representation of features like currents and formal expression of parameters like gradients and vortices. This requires a sophisticated integration of statistical, dynamic, and AI models to address data sparsity, undefined mechanisms, and irregular shapes within a high-dimensional framework, all while adhering to scene constraints. Frequent queries on variable correlations and data segmentation into different zones are needed, but the lack of integrated data hinders integrated analysis. Current ocean status simulations, primarily based on differential equations or dynamic models, have limitations that also affect the DTO's performance.

Unified data and interfacing flow for virtual–real interaction

The DTO, characterized by its ability to simulate "what-if" scenarios in a realistic marine environment, offers accurate feedback within a universal platform. Unlike traditional software systems, the DTO's visualization and virtual–real feedback should be closely integrated with the analysis process, continually operating in a dynamic loop. This leads to the continues update of various data types across different stages of information flow, leading to frequent data transformations. To minimize performance loss from this transformation, it is crucial to have adaptable data models and structures supporting the unified flow of data–computation–analysis. Furthermore, the DTO's visual analysis and interactions are tightly linked with hardware, necessitating an interfacing control flow in addition to the data flow. One of the DTO's current challenges is the effective unification of these flows and the implementation of straightforward, integrated intelligent interaction control.

Hypercomplex signal analysis in the DTO

The relationship between the DTO and signal processing

The DTO and signal processing share significant parallels. The DTO has advantages in high dimensionality by surpassing conventional 2D or 3D constraints and seamlessly integrating diverse and multiscale ocean data. It could offer comprehensive scene modeling across scales and utilize intelligent models for intricate analyses to uncover hidden correlations in vast datasets, boosting predictive and decision-making abilities in a unified system. Hypercomplex signal processing extends traditional methodologies into higher-dimensional spaces through the utilization of hypercomplex numbers. This framework provides a unified structure for high-dimensional signals and structured multisource information, rendering it an ideal mathematical foundation for the integration of multidimensional ocean environmental factors. The signal analysis and AI methods developed within this framework confer distinct advantages for the intricate handling of high-dimensional signals.

Hypercomplex signal processing

Hypercomplex signal processing is an emerging and significant technology within the field of signal processing, presenting distinct advantages for managing intricate signals characterized by high dimensions. Hypercomplex signal processing encompasses crucial methods, such as correlation, Fourier transform, convolution, adaptive filtering, wavelet transform, and linear canonical transform, operating within the hypercomplex domain. These methods provide a new set of robust tools that facilitate a more profound understanding and analysis of complex signal attributes. The profound insights derived from these tools are crucial in the understanding and management of complex signals [20].

Hypercomplex signal processing extends traditional signal processing into higher-dimensional spaces by employing hypercomplex numbers. This includes the use of geometric algebra, quaternions, octonions, and other forms of hypercomplex numbers. This extension enhances the ability to capture multidimensional signal characteristics. For example, a quaternion hypercomplex number system that extends 2D complex numbers into four dimensions can naturally represent color images and rotations in 3D space [4]. This allows for the capture of multiple, superposed attributes. Another rising trend is the use of geometric algebra in signal processing. Within the geometric algebra framework, the dot product and cross product among vectors can be generalized to multidimensions with a geometric product that possesses unique reversibility. Moreover, the structured algebraic computations in hypercomplex signal processing offer a unified framework for high-dimensional signals and structured multisource information, which enhances the accuracy, reliability, and efficiency for the processing of complex signals [21]. The unique aspects of hypercomplex signal processing, such as unified high-dimensional signal representation, meaningful feature description, and structured algebraic computation in high-dimensional spaces, make it a valuable asset for numerous scientific and engineering domains, including earth science.

Considering the current limitations in the DTO, the integration of hypercomplex signals could significantly boost its modeling capabilities, especially for complex tasks like multisource data integration, complex feature extraction, coupled ocean dynamics simulation, and interactive virtual–real feedback. For example, in fidelity modeling, the multidimensionality nature of hypercomplex signals facilitates a more direct integrated modeling of complex ocean phenomena like wave propagation, wind dynamics [22], etc. It offers a high-dimensional space for integrating multisource data. State-of-the-art hypercomplex machine learning algorithms can also be leveraged to provide more accurate simulations. By capitalizing on its inherent multidimensional properties, hypercomplex signal processing can enhance the DTO's predictive capacity by revealing latent correlations and patterns within extensive, high-dimensional datasets.

Framework of hypercomplex signal processing for the DTO

In addressing the fundamental challenges of the DTO and exploiting the capabilities of hypercomplex signal processing in the realm of high-dimensional data, we propose a hypercomplex signal processing framework for the DTO, as illustrated in Figure 2.

Initially, we focus on the multisource heterogeneous oceanic big data inherent to the DTO, comprising in situ observation data, model simulation data, and sensor/satellite data. These datasets inherently involve vectors, scalars, and trajectory networks as mathematical representations. By leveraging the strengths

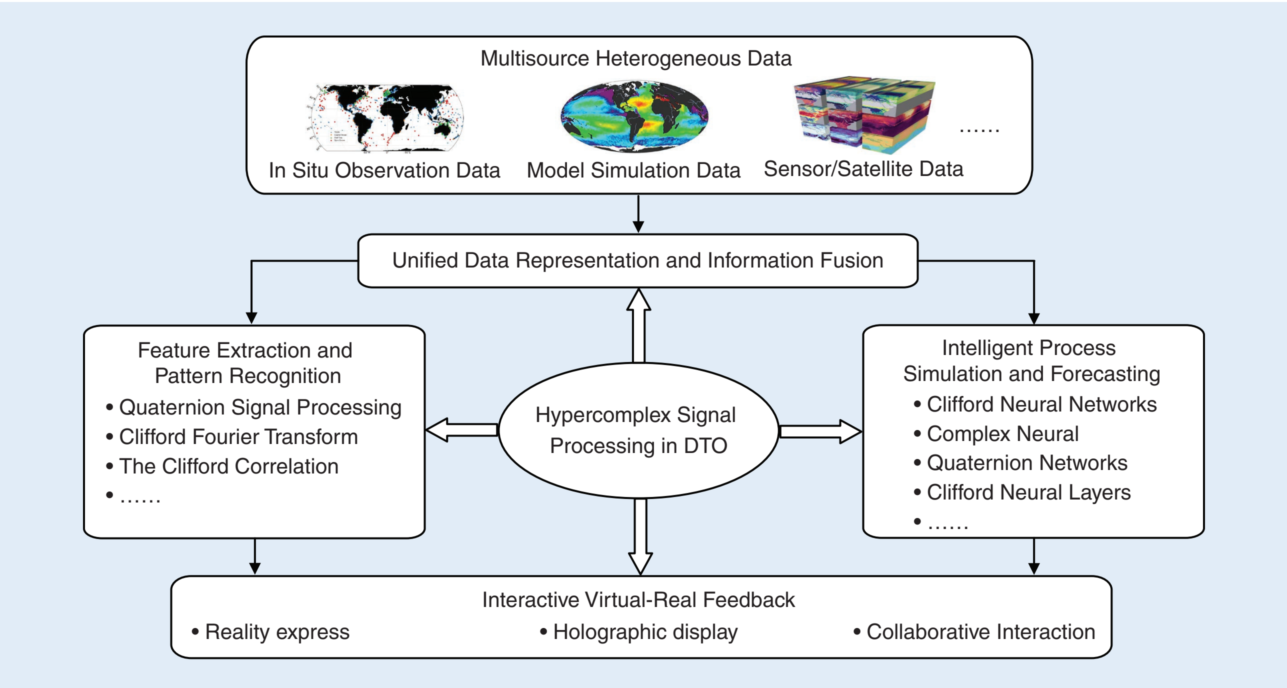

FIGURE 2. The hypercomplex signal processing framework for the DTO.

of geometric algebra in the unified expression and analysis of multidimensional data, we establish a geometric algebra-based approach for unified data representation and information fusion. This model enables the cohesive expression and management of diverse coordinate systems and data types. Subsequently, we introduce hypercomplex signal processing techniques, such as quaternion signal processing, Clifford Fourier transform, and Clifford correlation. These techniques are employed for feature extraction and pattern recognition from the aforementioned multisource heterogeneous oceanic big data.

Further, we introduce approaches like Clifford/complex/quaternion neural networks and the recently proposed Clifford neural layers for intelligent process simulation and forecasting. This application of hypercomplex-based AI analysis effectively utilizes the coupling relationships among multidimensional variables. While ensuring the accuracy of process simulation and forecasting, it also significantly reduces the model's parameters, thus enhancing model stability. Finally, we explore the applications of hypercomplex signal processing in the DTO's interactive virtual–real feedback aspects, including reality expression, holographic display, and collaborative interaction.

Our review of hypercomplex signal processing applications within the DTO framework, including data models, analytical models, and interactive expressions, offers insights for developing innovative DTOs. These DTOs are distinguished by a unified mathematical foundation, enhanced adaptability, and user-specific customization.

Theories of hypercomplex signal processing in the DTO

In this section, we will discuss some theories of hypercomplex signal processing that are helpful for the DTO.

Hypercomplex algebra

The hypercomplex algebra $\mathbb{A}$ is typically defined in the following manner:

$$\boldsymbol{a} = a_1 + a_2\boldsymbol{i}_2 + \ldots + a_n\boldsymbol{i}_n \in \mathbb{A}, \quad a_1, \ldots, a_n \in \mathbb{K} \tag{1}$$

as an n-dimensional $\mathbb{K}$-vector space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with an associated multiplication rule, comprising the unit element 1 $(1 \cdot \boldsymbol{a} = \boldsymbol{a} \cdot 1 = \boldsymbol{a})$. The multiplication rule delineates each connection among the imaginary units $\boldsymbol{i}_2, \ldots \boldsymbol{i}_n$ and implies the specific properties inherent to the algebra.

The concept of hypercomplex numbers encompasses various algebraic systems, such as geometric algebra and quaternions, which can be represented by different mathematical spaces. Quaternions, for instance, can be represented by a 4D vector space, where each quaternion can be expressed as a linear combination of a real number and three imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, with $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. Geometric algebra $\mathcal{G}_{p,q,r}$ is also known as the Clifford algebra of a n-dimensional vector space with a metric of signature (p, q, r), where p, q, and r represent the number of positive, negative, and null basis vectors, respectively, with $p + q + r = n$, formed by combining the vector spaces like real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$. All of these structures can be viewed as special forms of a hypercomplex algebra, which provides a unified framework for studying and representing different algebraic systems.

Geometric algebra

Geometric algebra offers a unified and consistent language for integrating constructs like complex numbers and quaternions into a single framework. This approach aids in developing new geometric methods that clarify physical interpretations without relying on specific coordinate systems [23]. Such a unified language is especially beneficial in DTO modeling for accurate prediction and representation of oceanic phenomena.

The fundamental principle of geometric algebra is to uniformly represent scalar, vector, plane, and other geometric objects as multidimensional "elements" or "multivectors." By using operators such as outer products and inner products, geometry transformations, projections, reflections, rotations, and other operations can be performed. The elements at different dimensions, named scalar A_0, vector A_1, bivector A_2, trivector $A_3, \ldots$, n-vector A_n, can be combined to form a new kind of entity called a multivector M, i.e., $M = A_0 + A_1 + A_2 + \ldots + A_n$. Each element of a multivector has an associated grade.

Geometric algebra is a comprehensive framework that seamlessly integrates geometric representations with algebraic operations. The geometric product, a foundational operation in geometric algebra, unifies inner and outer product operations, embodying the characteristic of invertibility. This methodology is particularly beneficial in managing high-dimensional and multichannel signals, offering enhanced precision in delineating signal structures and properties. Additionally, it simplifies the implementation of geometric transformations, making processes like signal rotations, translations, and other spatial manipulations more intuitive and efficient [24].

In this context, we will first discuss how geometric algebra can integrate complex numbers and quaternions. Then, we will demonstrate the utility of geometric algebra in facilitating a coordinate-free representation in data acquisition, highlighting its effectiveness in streamlining and improving the data gathering process in the DTO.

Complex numbers and quaternions are commonly used to present ocean wave dynamics, attitudes, and trajectories of buoys or unmanned underwater vehicles [25], [26]. However, the mathematical backgrounds of the real numbers, complex numbers, and quaternions are not directly compatible. Geometric algebra, which embeds real numbers, complex numbers, and quaternions in a unified framework, may provide a simple and clear mathematical foundation for the DTO.

A complex number can be represented in the form of $a + b\boldsymbol{i}$, where $a, b \in \mathbb{R}$ and $\boldsymbol{i}$ is the imaginary unit, satisfying $\boldsymbol{i}^2 = -1$. The quaternion system extends the complex numbers with three imaginary units. A quaternions are generally represented in the form $a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$, and $a, b, c, d \in \mathbb{R}$.

Noteworthily, any unit bivector from $\mathbb{G}^n$ has similar algebraic properties to those of the imaginary unit of complex numbers, i.e., $(\boldsymbol{v}_i\boldsymbol{v}_j)^2 = -\boldsymbol{v}_i\boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_j = -1 = \boldsymbol{i}^2$. The basic elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of the quaternions can be identified with the unit bivectors in a three-grade multivector space $\mathbb{G}^3$, i.e.,

$$\begin{aligned} \boldsymbol{v}_1\boldsymbol{v}_2 = \boldsymbol{i}, \boldsymbol{v}_2\boldsymbol{v}_3 = \boldsymbol{j}, \boldsymbol{v}_1\boldsymbol{v}_3 = \boldsymbol{k} \\ \boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_1\boldsymbol{v}_3 = \boldsymbol{v}_1\boldsymbol{v}_3\boldsymbol{v}_1\boldsymbol{v}_3 = -1 = \boldsymbol{ijk}. \end{aligned} \tag{2}$$

Thus, the complex numbers can be embedded into a subalgebra of $\mathbb{G}^2$, which is formed with scalars and bivectors. A quaternion can be identified as a multivector consisting of a scalar and three bivectors from $\mathbb{G}^3$.

In the development of the DTO, the aggregation of high-dimensional ST data is an essential task. During this process, data frequently undergo operations such as translation, rotation, and scaling. In traditional methodologies grounded in real-number systems, these fundamental operations each exhibit distinct constitutive forms. This diversity contributes to the complexity and instability of combined operations, posing challenges in maintaining consistency and accuracy in geometric processing [27]. To address these issues, theories such as quaternions have been introduced to unify various mathematical constructs and solve the problems in 3D space. However, further development is needed to address the limitations of these theories in high-dimensional analysis.

In geometric algebra, translation, rotation, and scaling can be unified with the versor [28], as the following sandwich form:

$$V \left\{ \begin{matrix} \text{point} \\ \text{line} \\ \text{plane} \\ \vdots \end{matrix} \right\} V^{-1} \tag{3}$$

where V is the k-versor, which is an element of the geometric algebra that can be obtained by multiplying k vectors using the geometric product:

$$\text{versor: } V = \mathbf{v}_k \dots \mathbf{v}_2\mathbf{v}_1. \tag{4}$$

The inverse of V is of course simply obtained by the inverse vector factors in opposite order: $V^{-1} = \mathbf{v}_1^{-1}\mathbf{v}_2^{-1}\dots\mathbf{v}_k^{-1}$.

Within the context of the DTO, the application of the versor in geometric algebra serves as an efficacious means for representing dynamic sensor positions and coordinating disparate systems. This extends to intricate endeavors such as the analysis of high-dimensional data and the simulation of dynamic processes. The versor's capacity to accurately depict and update relational dynamics within the DTO framework is noteworthy, particularly its distinctive reversibility feature. This aspect of versors furnishes invaluable tools for the resolution of directional parameters associated with the motion of objects, thereby enhancing the precision and efficacy of the DTO's analytical capabilities.

A typical example on how geometric algebra can benefit the DTO is the integration of multiple coordinates. Employing drones for terrain mapping of coastal–landward boundaries constitutes a fundamental aspect of DTO scene modeling. This data acquisition task requires the utilization of diverse coordinate systems, encompassing terrestrial, marine, and aircraft attitudes, to accurately measure both land and ocean terrains. Conventional GISs mandate the conversion of all coordinate types, inclusive of both absolute and relative systems, into a unified coordinate framework. However, this method has practical difficulties in the DTO, owing to complexities, time constraints, and the occasional impracticality of frequent coordinate transformations. For instance, in unmanned aerial vehicle (UAV) surveying, it is common to capture only relative relationships, posing a significant challenge in dynamically reconstructing actual flight trajectories.

As illustrated in Figure 3, for precise terrain mapping of coastal–landward boundaries, the terrestrial part typically employs a coordinate reference system (X, Y, Z), while the ocean part uses a relative coordinate system $(\Delta X, \Delta Y, \Delta Z)$. UAVs collect a series of maritime data at different times $(t_1, t_2, \cdots, t_n)$, which necessitates conversion based on the marine local reference system, followed by transformation to the terrestrial reference system. Conventionally, this process involves a series of matrix operations, including rotation and translation, which have inherent limitations in terms of parameter complexity, computational efficiency, and stability.

Within the framework of geometric algebra, terrestrial reference systems can expressed using geometric algebraic notations as (e_1^i, e_2^i, e_3^i), and marine reference systems can expressed using geometric algebraic notations as (e_1^g, e_2^g, e_3^g). The use of rotor operators enables the transformation of marine references to terrestrial references as R_g^i. Additionally, data obtained by UAVs can be transformed from their local references to marine references as $R_{t^1}^g, R_{t^2}^g, \dots, R_{t^n}^g$ and then to terrestrial references using rotor operators as $R_g^i, R_{t^m}^g$, where $m = 1, 2, \dots, n$. In this geometric algebra framework, which unifies rotations and translations within a single mathematical operator, the geometric algebra further extends the usual vector-space operations by defining an invertible product of

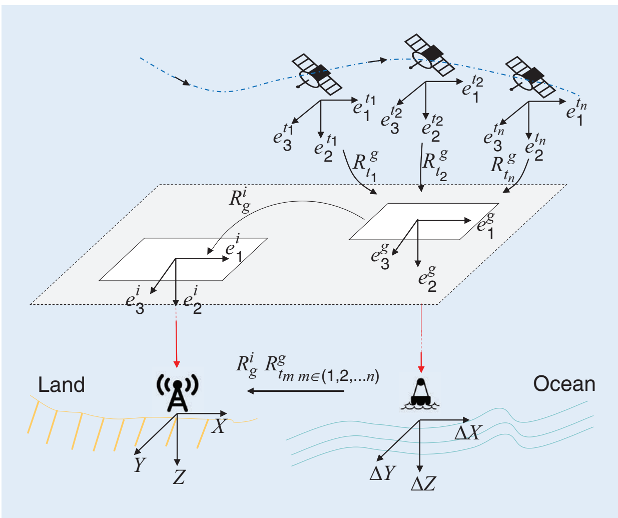

FIGURE 3. Multiple-coordinate sensoring of the ocean with geometric algebra.

vectors, the geometric product. This introduction of an invertible vector product allows for the definition of vector division and, consequently, directional derivation [29]. Such an advanced mathematical framework leads to more robust solutions in ocean data processing and analysis.

Quaternion signal processing

In contrast to conventional complex numbers, which comprise a real part and an imaginary part, quaternions exhibit supplementary degrees of freedom, rendering them particularly suitable for the representation and processing of multidimensional data. A quaternion is a number in a 4D space with one real part and three imaginary parts. The quaternion $\boldsymbol{q}$ can be expressed as

$$\boldsymbol{q} = a_0 + a_1 \cdot \boldsymbol{i} + a_2 \cdot \boldsymbol{j} + a_3 \boldsymbol{k} \tag{5}$$

where a_0, a_1, a_2, a_3 are real numbers, and $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are imaginary units. Quaternions are effectively utilized in color image representation, particularly for RGB images, where each pixel is represented as a quaternion with its scalar part as zero, and red, green, and blue components correspond to the pixel colors [30]. This representation extends to other color spaces like luminance–chrominance, expressible in pure quaternion form [31]. Traditionally, color image processing involved handling 2D RGB matrices and a similarity relation matrix to normalize the gray color space for pixel smoothness, but this approach often led to information loss and failed to capture the intrinsic relationship among color image pixels. Quaternion color image processing, on the other hand, treats color pixels as vectors in a multidimensional space, better reflecting the image's color correlations and allowing complex numbers to be depicted as quaternion four-tuples [32]. With the signal represented by a quaternion, an adaptive filtering algorithm can be established using the minimum mean-square error method [22] or the minimum mean kurtosis method [34]. These filtering algorithms are more accurate and stable for the representation of non-Gaussian data, which is a common situation in marine science. A more detailed overview can be found in [35].

Clifford signal processing

In this section we will mainly discuss two typical Clifford signal processing theories, the Clifford Fourier transform and correlation analysis, and their possible applications in the DTO.

In the DTO, a significant challenge was the integration of scalar x vector, tensor, multitensor, and nontensor fields. Current research commonly employs the orthogonal decomposition of longitude and latitude components to represent vector fields as separate scalar fields [36]. While this approach is clearly related to the separation rules of longitude and latitude, it is inadequate for representing the self-characteristics of the factor and interaction across different factors.

In the geometric algebra framework, a more comprehensive approach is achievable by processing multivector signals as a whole using unified operators. This could eliminate the need for signal separation and simplifies the processing of complex signal structures. For instance, the Clifford Fourier transform can be used for template matching and vector field segmentation, as demonstrated in Yuan et al. [37]. This method highlights its effectiveness in integrating patterns influenced by diverse geophysical phenomena, leading to improved latent information extraction. The unified processing approach offered by geometric algebra improves the robustness of feature extraction and analysis in DTO constructions, providing a more efficient framework for handling complex signals.

Another typical application of Clifford analysis is the Clifford correlation analysis. Extracting correlation relations among various Earth observation fields stands as a fundamental task for the DTO. Clifford convolution efficiently captures the interconnections among diverse field data by concurrently quantifying their similarity and dissimilarity through the utilization of the geometric product [38].

Hypercomplex machine learning

Hypercomplex machine learning is an advanced form of AI that uses hypercomplex structures to analyze intricate data [39]. Here we will provide a review on several commonly used hypercomplex machine learning models as follows.

Clifford neural networks

Neural networks in the Clifford domain empower the execution of operations such as enhancement, rotation, and expansion, effectively transcending the limitations associated with the Euclidean metric in real-valued neural networks. Researchers have made significant advancements in extending the traditional models of recurrent neural networks in the real-valued domain to the Clifford domain, which involves the transformation of conventional multilayer perceptrons into Clifford multilayer perceptrons, support vector machine in Clifford algebra, geometric Clifford algebra networks, and recurrent neural networks based on Clifford algebra [40]. Here we exemplify with geometric algebra convolution, as illustrated in Figure 4. Unlike traditional convolution operations conducted in the domain of real numbers, the inputs and kernels in geometric algebra convolution are multivector, and the operations among them are substituted with geometric products.

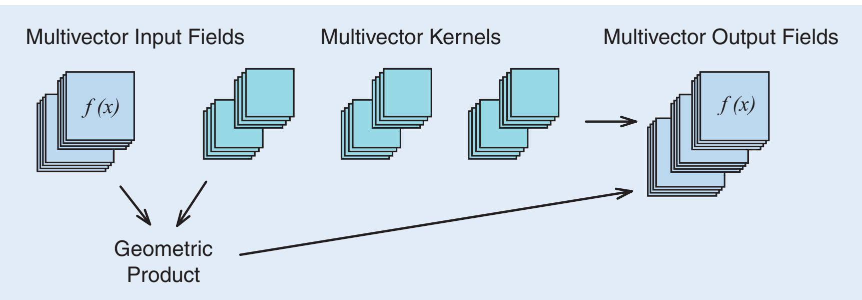

FIGURE 4. The architecture of Clifford convolution.

Complex and quaternion neural networks

Complex-valued and quaternion neural networks significantly advance the analysis of multidimensional data. The complex-valued networks incorporate elements such as convolutional feed-forward, advanced complex batch normalization, and complex weight initialization strategies [41]. They adeptly encode multidimensional data using complex number-based structures, thereby expanding the capabilities of complex neural networks. Quaternion-based deep learning architectures are a natural extension of complex neural networks [42]. When applied to oceanographic data, these neural network methodologies hold the potential to provide promising avenues for the processing and analysis of extensive, multidimensional marine environmental data. This includes data such as color and high-dimensional satellite or underwater imagery and is particularly effective in enhancing data analysis and visualization for the DTO.

Clifford neural layers

Traditionally, neural networks have been restricted to finite-dimensional Euclidean spaces or finite sets, but recent advancements have broadened their range to include operators within infinite-dimensional function spaces. The Fourier neural operator (FNO) model enhances this by replacing the kernel integral operator with a convolution operator defined in Fourier space, simplifying it into a multiplication operation through a discrete fast Fourier transform. This model has shown its effectiveness for the DTO and highlights the potential of physics-inspired deep neural operators [43]. This model emerges as a cost-effective alternative to high-resolution numerical ocean models, demonstrating its practical applicability and efficiency in the field.

The FNO model outperforms traditional convolutional neural networks in computational efficiency and in capturing the multiscale structure. Additionally, its extension within the geometric algebra framework has been documented by Brandstetter et al. [44]. The work presented in this article highlights that existing methods frequently overlook the interconnections among distinct fields and their internal constituents, even though these connections often exhibit correlations. By examining the temporal evolution of these correlated fields from the perspective of multivector fields, we can effectively address these limitations.

The resultant Clifford neural layers, with their universal applicability, are particularly valuable for DTO applications, such as fluid dynamics, weather forecasting, and the comprehensive modeling of oceanic systems. Moreover, the implementation of Clifford neural layers significantly enhances the generalization capabilities of neural surrogate models for partial differential equations [44], crucial for accurately simulating and predicting oceanic phenomena [45]. Thus, these advancements in the FNO model and Clifford neural layers can significantly contribute to the development and refinement of DTO models, enhancing our understanding and prediction of oceanic systems.

Application of hypercomplex processing in the DTO

Multidimensional representation and information fusion

The development of the DTO requires a dynamic, multidimensional data platform that integrates diverse data sources into a comprehensive, real-time observation system. The rapid evolution of ocean data observation systems has led to a proliferation of data sources, formats, and protocols, presenting substantial challenges in data integration and fusion. Oceanic variables in coastal regions, represented through ST field data encompassing phenomena like sediment transport, nutrient cycling, and fluid dynamics, typically include scalar, vector, and tensor field data types. Other data formats, such as buoy data, create 3D space-time trajectories as they follow the currents. The comprehensive representation of ocean data necessitates the efficient integration of these multimodal data types. However, challenges are exacerbated by the irregular sampling areas of ocean observation data, which require diverse geometric structures for data representation.

A significant proportion of traditional data models predominantly rely on geometric representations, employing basic geometric entities like points, lines, surfaces, and volumes to represent complex data. This approach leads to intricate data transformations and the exclusion of essential high-dimensional information, including ST correlations and interactions. This highlights the need to overcome challenges in implementing digitalization, intelligence, and automation in the DTO, particularly in dealing with the increasing amount of multimodal marine data. This highlights the need for innovative methods to overcome these challenges and enable more effective DTO implementation.

Geometric algebra offers significant advantages in the unified expression and analysis of multidimensional data. The multivector structure allows for a more intuitive representation of complex geometric relationships and transformations. A key strength of geometric algebra is its ability to directly represent and compute with subspaces (like lines, planes, and volumes) as first-class entities. This capability facilitates a more natural description of multidimensional phenomena encountered in engineering, physics, and computer vision.

Notable research in this area includes the development of a multivector-based hierarchical construction pattern and representation method for vector/ST field/network data, utilizing the mathematical structure of geometric algebra for consistency by Yuan et al. [46]. This approach supports a multidimensional GIS scene data model with unified storage, representation, and computation structures. The multivector tree (MVTree) data structure was designed for the expression, organization, storage, retrieval, and computation of these multimodal data, facilitating the integration of complex geographic scenes [47]. The use of the HDF5 format allowed efficient representation and storage of large amounts of data [48]. The introduction of the BRNO-ST index, a hierarchical tensor-based framework for ST data, and an integrated storage structure for geographic networks contribute to the development of a comprehensive DTO data model, effectively unifying diverse data representation and fusion.

For the effective integration of vector, ST field, and trajectory data within the DTO framework and to maintain consistency with the current ocean information system, a geometric algebra-based DTO data model is employed, as illustrated in Figure 5. This model encodes original observed point data using an operator-based approach, allowing for the algebraic construction of various data types. These diverse data types are stored within unified multivector data structures. Within this framework, geometric algebra operators are utilized for data manipulation. The vector data within this model capitalize on the Grassmann structure inherent in geometric algebra [47]. This structure facilitates the effective representation and manipulation of vector data. Conversely, ST field data are represented through tensors. These tensors, along with associated tensor operators and analytical methods, are employed for data manipulation and visualization, enhancing the interpretability and utility of the data [49].

In terms of networks and trajectories, both nodes and k-walk routes are represented within the $Cl(n,n)$ space. This representation allows for the computation of topological relationships using the Clifford adjacency matrix. Additionally, network routes are expanded and traversed using the oriented meet product, a method that further refines the analysis and representation of network data [50]. This harmonization of expression, storage, and computation within the multivector structure significantly streamlines the organization, representation, and analysis of complex geographical scenes. By unifying these diverse data types within a singular structure, the geometric algebra-based DTO model offers an efficient and coherent framework for managing and analyzing the multifaceted data inherent in oceanographic studies.

Feature extraction and pattern recognition

The DTO framework fundamentally requires an effective description and simulation of the ocean system. This involves extracting features and identifying patterns from various datasets, such as time series and remote sensing images. For time series data, commonly utilized techniques encompass time series decomposition, spectral analysis, autoregressive integrated moving average, cross correlation analysis, and singular value decomposition. These methodologies are instrumental in identifying periodic patterns, dominant frequencies, trends, and seasonality [38], offering vital insights into the ocean's influence on climate, ecosystems, and extreme events [52]. Multispectrum remote sensing imagery is distinguished by its high dimensionality and extensive interdimensional coupling, challenged by noise from overlapping geographical regions and variable ocean conditions. These complexities hinder the effectiveness of traditional algorithms in recognizing, classifying, and extracting features from small objects.

In response, the field of hypercomplex signal processing is garnering increasing attention for its aptitude in handling multicomponent data. The adoption of hypercomplex algebraic structures, such as quaternions and Clifford algebras, has marked a significant advancement in the field of time series and image processing. Techniques like quaternion singular spectrum analysis, quaternion frequency-space deconvolution, and hypercomplex Fourier transform have been effectively employed in both synthetic and real-world scenarios. Recurrent correlation neural networks are extended for processing hypercomplex-valued data, providing high-capacity associative memories for grayscale image storage and analysis [53]. Extreme learning machines on general hypercomplex algebras provide superiority in processing multidimensional data compared with their real-valued counterparts on time series prediction and color image autoencoding [39]. In the field of remote sensing data processing and analysis, Clifford algebras have emerged as a powerful tool, offering a natural and efficient approach to extract spatial-spectral features in multidimensional images. This hypercomplex signal processing technique integrates information from hyperspectral remote sensing

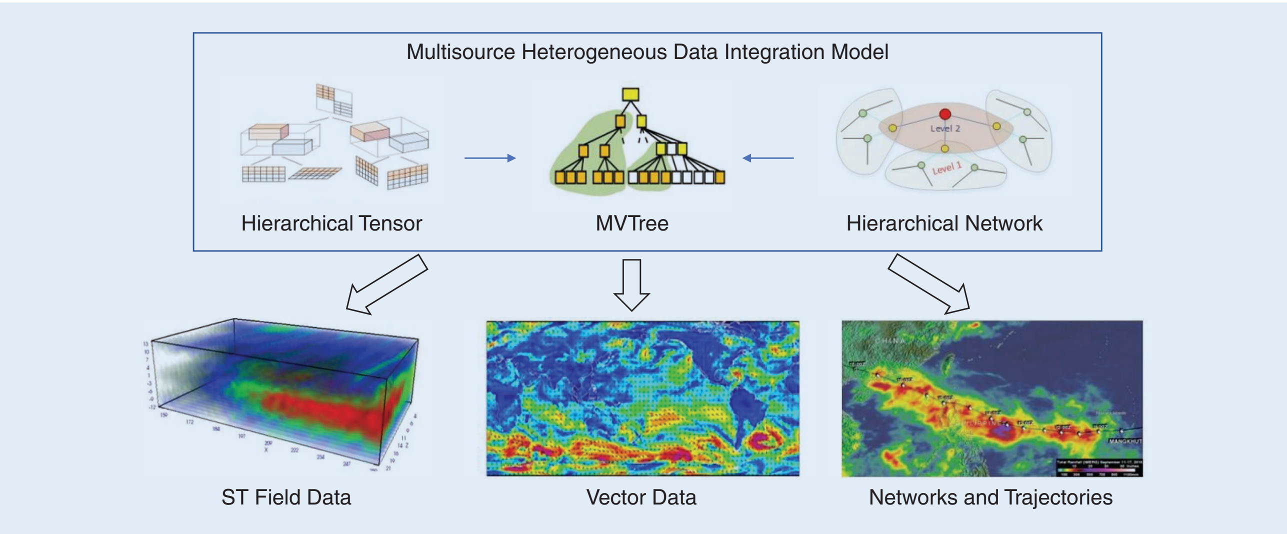

FIGURE 5. The geometric algebra-based unified data model in the DTO.

images into multivector components, demonstrating superior performance in multidimensional image analysis, including segmentation and spectral analysis [55].

Hypercomplex methodologies have been effectively utilized within the DTO framework, yielding significant advantages. A prime application is in marine monitoring systems, particularly for ship detection in optical satellite imagery against complex ocean backdrops. Research has highlighted the robustness of hypercomplex techniques against rotational, scaling, and translational transformations in ship imagery [56]. These methods maintain accuracy across a variety of ship sizes and are resilient in diverse maritime environments, including ship wakes, and varying ship dimensions [56]. In other applications, such as kinematic attitude estimation for floating offshore wind turbines, quaternion-based formulas have been developed. These formulas offer a singularity-free approach for computing large-angle rigid body rotations. Furthermore, quaternion algebra-based direction-of-arrival algorithms have demonstrated high resolution and enhanced performance in estimation accuracy and angular resolution. This is particularly evident in scenarios involving coherent underwater sources in low signal-to-noise ratio conditions. Additionally, the use of biquaternion-based models for vector hydrophone output data has proven effective in managing multicomponent information [57]. This approach has led to significant improvements in memory space utilization, showcasing the practicality and efficiency of hypercomplex methods in diverse marine and oceanographic applications.

In response, the field of hypercomplex signal processing is garnering increasing attention for its aptitude in handling multicomponent data.

Intelligent process simulation and forecasting

The DTO differs from traditional virtual or augmented reality by offering intelligent functionalities that go beyond basic digital imitation. This advanced capability is largely driven by the integration of AI-based methodologies. The DTO is notably characterized by its use of big data and AI to significantly enhance the scope of knowledge discovery in AI oceanography. This integration allows for more accurate predictions and detailed virtual reconstructions of both the historical and present states of oceanic environments.

In the field of AI, machine learning models have emerged as crucial tools, particularly in their ability to autonomously process and interpret nonlinear relationships within large, high-dimensional datasets. In the DTO, using ocean models to simulate dynamics is essential. Accurately estimating model parameters is crucial for accurately representing ocean processes. Machine learning, employed for extracting patterns and predicting optimal parameters from comprehensive observational datasets, has proven more effective than traditional methods based on physical–empirical relationships [58]. Such capabilities make them indispensable for the development and refinement of DTO models. Significant advancements in machine learning, such as computer vision, neural language processing, and scientific applications, have considerably enhanced the capabilities of the DTO system. These developments have markedly improved the accuracy and comprehensiveness of simulations and predictions of oceanic environments.

In the pursuit of enhancing the fidelity and dependability of DTOs as reflections of real-world ocean systems, the integration of domain-specific expertise into machine learning architectures is paramount. Machine learning provides a flexible analytical framework that can be effectively augmented with specialized knowledge or adherence to fundamental physical laws, such as conservation of mass and energy. This integration will make DTO models that are not only data-driven but also firmly grounded in established scientific principles [59].

In the DTO, accurately simulating oceanic dynamics is essential for maintaining the system's accuracy and trustworthiness. Advances in AI for science have introduced various machine learning methods, such as nonlinear dynamics sparse identification, which are crucial in this field. These techniques have an advantage at revealing underlying dynamic processes from observational data by developing interpretable control equations. This is particularly relevant in areas like fluid dynamics, nonlinear systems, and stochastic processes. Within the DTO, these methods, notably sparse identification of nonlinear dynamics, are employed to precisely model complex ocean behaviors, including fluid stretching and rotation, based on oceanic data. Control equations for sea ice evolution, derived from satellite remote sensing imagery, have significantly improved the understanding and prediction of Arctic ice growth, melting, and movement processes [60]. Notably, predictions from these AI-based control equations have been shown to be more accurate than those from conventional models [61]. These methods effectively capture the dynamic processes of ST systems, even with limited observational data, and exhibit good interpretability and generalization capabilities.

Within the DTO framework, data often exhibit high dimensionality, accompanied by complex interdependencies among their various dimensions. Traditional machine learning models, typically confined to the real number domain, struggle to adequately address the interdependencies among different fields and their internal components. In response, complex-domain machine learning methods have gained traction in DTOs due to their proficiency in handling high-dimensional data and capturing interrelated data relationships. For example, in the processing of synthetic aperture radar (SAR) imagery, it is noted that most conventional machine learning models, designed for real-valued inputs, fail to exploit the phase information intrinsic to the complex-valued data of polarimetric SAR images. Employing complex-valued convolutional neural networks for classifying marine oil spills in polarimetric SAR images has improved performance [62]. Additionally, a neural network operating in a 5D geometric algebra space effectively captures algebraic–geometric dependencies in time series data, outperforming traditional vector algebra-based models in generalization and prediction accuracy [63].

Furthermore, the hypercomplex number-based machine learning models require fewer parameters compared with their real-valued counterparts, enabling the training of equivalently capable models on smaller hardware with reduced operational complexity. This leads to a significant reduction in the number of operations required to achieve similar levels of accuracy. Consequently, such lightweight models substantially enhance computational efficiency without sacrificing accuracy, thereby facilitating more efficient processing and analysis of the DTO. A notable example of this progress is the recent integration of geometric algebra into machine learning algorithms. This is exemplified in the development of neural partial differential equation surrogates designed for solving partial differential equations [64]. By embedding multidimensional objects as multivector structures, these methods significantly improve both the accuracy and computational efficiency of model solutions. Additionally, the adoption of multivector representations in conjunction with Clifford convolutions and Fourier transform within deep learning frameworks has proven to be highly effective. As demonstrated in Brandstetter et al.'s research [44], Clifford neural layers show broad applicability and are well suited for various domains, including fluid dynamics, ocean state forecasting, and physical system modeling. This advancement signifies a major leap in the application of geometric algebra in machine learning, contributing to the enhanced modeling capabilities within the DTO framework.

Visualization and interactive virtual–real feedback

The DTO's unique challenges lie in its dynamic virtual–real feedback for the visualization and interaction components. Continues iterations existing in the DTO make the efficient management of large datasets, creating accurate visual representations, and designing specific interaction methods critical.

Geometric algebra has been proven to have better performance in handling complex geometric entities and real-time visualization tasks. One of the major strengths of geometric algebra is its ability to represent a wide range of geometric entities, from points, lines, planes, and higher-dimensional objects, within a unified algebraic framework. This simplifies computations and manipulations, making it particularly suitable for complex visualization tasks [65]. Geometric algebra's versatility enables efficient representation and manipulation of complex oceanic environments. For instance, it can be used to accurately extract rotations and streamlines from vector fields [66] and model and visualize oceanic phenomena, such as currents, wave patterns, and underwater topography. A detailed review of the application of geometric algebra in vector field analysis and visualization is provided by Ausoni and Frey [67]. Geometric algebra can distinguish between inner and outer rotations, which is crucial for analyzing oceanic vortices [9]. Moreover, geometric algebra can be used to visualize differential equations in a geometric manner, providing a clear physical interpretation for each computational component. Klausen's [68] work on visualizing Stokes' theorem with geometric algebra provides a prime example of such an approach. Extending this method to visualize equations like the shallow water wave equations or even more intricate oceanic equations is a promising direction for future research.

Large-scale dataset management is critical for successful visualization in the DTO. Hypercomplex representations for signal processing are beneficial due to their hierarchy. Yuan et al. [69] employed conformal geometric algebra in a novel method for feature extraction, utilizing k-means clustering on extensive point cloud data. This approach can effectively extract geometric features from extensive point cloud data leading to high accuracy and compression ratio. Additionally, several studies have developed efficient algorithms for rendering large-scale datasets that can adapt to significant changes in scale and perspective. Benger developed an efficient algorithm for rendering large-scale datasets that can adapt to significant changes in scale and perspective [48]. This algorithm employs an adaptive hierarchical structure composed of data blocks, which are arranged based on their visibility.

Hypercomplex signals also play a significant role in streamlining the computationally intensive operations. Rendering tasks, such as shadow generation and ray tracing for extensive light sources, require complex intersection calculations that are challenging due to the complexity of marine environments and the need for accurate representation of light and shadow effects. Belon and Hildenbrand [70] developed a GPU-based method for smoothly interpolating the normal information of the conformal geometric model, represented by its vertices and faces. This method has proven its capability in processing models with millions of vertices efficiently using conformal geometric algebra, achieving practical processing speeds. Moreover, the application of geometric algebra and the Plücker space further enhances the DTO's capability in rendering. Utilizing these mathematical frameworks allows for the accurate computation of n-dimensional shadows, which is beneficial in creating realistic and detailed visualizations within the DTO environment, where accurate shadow rendering can significantly impact the overall visual quality and the user's perception of the modeled oceanic space.

According to the virtual–real feedback problem, Aristidou and Lasenby [71] have made strides in this area by applying a geometric algebra-based inverse kinematics solver to control hand postures, monitored through minimal optical motion capture. This solver incorporates physiological constraints to limit movements to feasible and natural ranges, demonstrating an advanced approach to human interaction. Such methods introduce innovative interaction techniques that enable human correct movements with minimal processing time, making them particularly effective for real-time hand motion tracking and reconstruction. Thus, they ensure smooth and accurate interactive experiences. Other advancements in high-dimensional visualization and interaction, underpinned by motion capture and dynamic

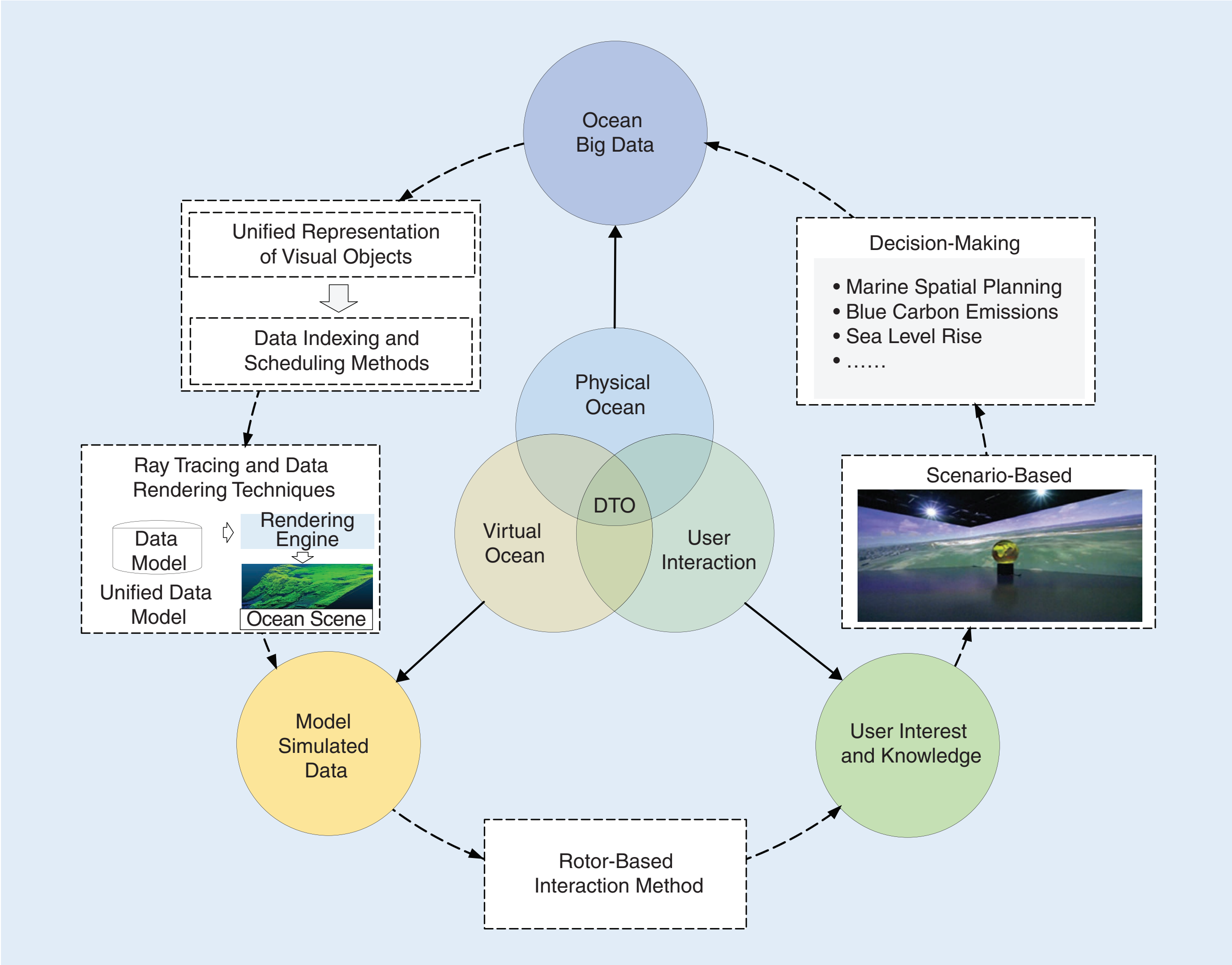

FIGURE 6. Visualization and interactive data analysis in the DTO.

interactive methods, may also significantly enhance the interactive capabilities of DTO systems.

Figure 6 shows an example of visualization and interactive data analysis in a DTO. Observation and simulation data are collected and integrated into the system for unified management. Data management is related to data services, and the two collaborate to respond to the data demands through data analysis tasks. The core computing module compares and analyzes multisource observation and simulation data through the hypercomplex mathematical architecture and submits the results to the unified data model. The unified data model integrates the management of different types of original and feature data (such as raster, vector, and tensor) to eliminate the additional pressure on the rendering engine caused by different organization forms. The rendering engine and interactive system visualize the data according to user needs and feedback on user operations to the data model to update the data visualization. The same geometric algebraic-based architecture is used to organize the inputs, computational processes, and outputs between the computational model and the data model to reduce the extra cost of data reorganization across the visualization. Consequently, a more mechanically unified data rendering and interaction architecture can be applied to the development of the DTO, thus achieving parallel processes between computation and rendering for user interaction.

Conclusions

This article provides demonstrations of hypercomplex signal processing in the context of the DTO, covering various aspects including the representation and integration of heterogeneous data from multiple sources, feature extraction and pattern recognition, and intelligent process simulation and forecasting, as well as interactive virtual–real feedback.

A thorough analysis reveals that hypercomplex numbers, especially geometric algebra, are highly effective in providing a unified representation and spatial analysis for multidimensional objects. This feature is pivotal in handling the varied data sources within the DTO, establishing it as a robust mathematical framework for data representation and information fusion. Hypercomplex signal processing techniques demonstrate proficiency in processing multicomponent data, thereby offering valuable tools for feature extraction and pattern recognition within the DTO's framework of analyzing multisource heterogeneous data. Furthermore, machine learning methods within the hypercomplex domain are adept at capturing complex interrelationships among data elements. These methods are proven to outperform traditional models in several aspects, such as the learning of multilevel structures, resistance to sensitivity, the model's robustness, and the operation's simplicity. In the context of the DTO, such methods exhibit substantial potential for applications in fluid dynamics, ocean state forecasting, and modeling of physical ocean systems. Additionally, the use of hypercomplex numbers, particularly through the lens of geometric algebra, significantly enhances data visualization and interactive analysis within the DTO. This enhancement is achieved by unifying the representation and computation of both scalar and vector field data, thus offering a more comprehensive approach to data handling and analysis in the DTO environment.

Additionally, the use of hypercomplex numbers, particularly through the lens of geometric algebra, significantly enhances data visualization and interactive analysis within the DTO.

The potential applications and benefits of hypercomplex signal processing signify a groundbreaking and innovative field for resolving complex issues in earth science and enriching our understanding of the planet. Its ability to skillfully manipulate high-dimensional, nonlinear, and noisy data opens up new possibilities for advanced simulations, accurate predictions, and intelligent decision-making across diverse geoscience domains. Moving forward, research endeavors should focus on advancing and refining hypercomplex signal processing techniques. This involves integrating them with other earth science methodologies and fully harnessing the potential of big data analysis to generate more intricate and comprehensive digital twin models. Furthermore, investigations into unexplored areas such as hypercomplex time-frequency analysis and hypercomplex machine learning could not only offer new insights into the DTO but also potentially contribute to a broader range of earth science and natural science disciplines.

In the future, our focus lies in integrating hypercomplex signal analysis for the DTO across three key domains: constructing the unified ocean data management model for multisource data fusion, developing ocean AI for enhancing the accuracy of marine mechanism simulation, and stablishing ocean visualization and intelligent interaction systems for convenient and efficient applications.

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References

[1] M. W. Grieves, "Digital twins: Past, present, and future," in *The Digital Twin*. Cham, Switzerland: Springer-Verlag, 2023, pp. 97–121.

[2] H. Kim and J. Ben-Othman, "Eco-friendly low resource security surveillance framework toward green AI digital twin," *IEEE Commun. Lett.*, vol. 27, no. 1, pp. 377–380, Jan. 2022, doi: 10.1109/LCOMM.2022.3218050.

[3] S. Lee, S. Lee, and H. Kim, "Differential security barriers for virtual emotion detection in maritime transportation stations with cooperative mobile robots and UAVs," *IEEE Trans. Intell. Transp. Syst.*, vol. 24, no. 2, pp. 2461–2471, Feb. 2023, doi: 10.1109/TITS.2022.3172668.

[4] G. Chen et al., "Toward digital twin of the ocean: From digitalization to cloning," *Intell. Mar. Technol. Syst.*, vol. 1, no. 1, 2023, Art. no. 3, doi: 10.1007/s44295-023-00003-2.

[5] Q. Chengcheng and C. Ge, "Big data science for ocean: Present and future," *Bull. Chin. Acad. Sci.*, vol. 33, no. 8, pp. 884–891, 2018, doi: 10.16418/j.issn.1000-3045.2018.08.018.

[6] M. Muruganandam et al., "Impact of climate change and anthropogenic activities on aquatic ecosystem—A review," *Environmental Res.*, vol. 238, Dec. 2023, Art. no. 117233, doi: 10.1016/j.envres.2023.117233.

[7] T. Bulow and G. Sommer, "Hypercomplex signals-a novel extension of the analytic signal to the multidimensional case," *IEEE Trans. Signal Process.*, vol. 49, no. 11, pp. 2844–2852, Nov. 2001, doi: 10.1109/78.960432.

[8] A. Cariow and G. Cariowa, "An unified approach for developing rationalized algorithms for hypercomplex number multiplication," *Electric Rev.*, vol. 91, no. 2, pp. 36–39, 2015, doi: 10.15199/48.2015.02.09.

[9] R. Bujack, G. Scheuermann, and E. Hitzer, "A general geometric Fourier transform," in *Quaternion and Clifford Fourier Transforms and Wavelets*. Basel, Switzerland: Birkhäuser, 2013, pp. 155–176.

[10] S. Kumar and U. Rastogi, "A comprehensive review on the advancement of high-dimensional neural networks in quaternionic domain with relevant applications," *Arch. Comput. Methods Eng.*, vol. 30, no. 6, pp. 1–28, 2023, doi: 10.1007/s11831-023-09925-w.

[11] P. Wang, T. Wang, J. Sun, X. Zhang, and T. Song, "Dual-quaternion-based dynamics modeling for a rigid–flexible coupling satellite," *J. Guid., Control, Dyn.*, vol. 46, no. 7, pp. 1298–1313, 2023, doi: 10.2514/1.g007000.

[12] B. Zeng, Y. Sun, W. Song, Z. Wang, and X. Zhang, "Recurrence of the green tide in the Bohai Sea, China: A green tide caused by coastal reclamation projects," *J. Sea Res.*, vol. 191, Feb. 2023, Art. no. 102333, doi: 10.1016/j.seares.2022.102333.

[13] M. M. Bennett, S. R. Stephenson, K. Yang, M. T. Bravo, and B. De Jonghe, "The opening of the transpolar sea route: Logistical, geopolitical, environmental, and socioeconomic impacts," *Mar. Policy*, vol. 121, Nov. 2020, Art. no. 104178, doi: 10.1016/j.marpol.2020.104178.

[14] A. Çöltekin et al., "Extended reality in spatial sciences: A review of research challenges and future directions," *ISPRS Int. J. Geo-Inform.*, vol. 9, no. 7, 2020, Art. no. 439, doi: 10.3390/ijgi9070439.

[15] M. N. Kamel Boulos, Z. Lu, P. Guerrero, C. Jennett, and A. Steed, "From urban planning and emergency training to Pokémon Go: Applications of virtual reality GIS (VRGIS) and augmented reality GIS (ARGIS) in personal, public and environmental health," *Int. J. Health Geograph.*, vol. 16, Feb. 2017, Art. no. 7, doi: 10.1186/s12942-017-0081-0.

[16] P. N. Edwards, "History of climate modeling," *Wiley Interdisciplinary Rev., Climate Change*, vol. 2, no. 1, pp. 128–139, 2011, doi: 10.1002/wcc.95.

[17] J. Fu, Z. Bu, D. Jiang, G. Lin, and X. Li, "Sustainable land use diagnosis based on the perspective of production–living–ecological spaces in China," *Land Use Policy*, vol. 122, Nov. 2022, Art. no. 106386, doi: 10.1016/j.landusepol.2022.106386.

[18] S. M. Khan et al., "A systematic review of disaster management systems: Approaches, challenges, and future directions," *Land*, vol. 12, no. 8, 2023, Art. no. 1514, doi: 10.3390/land12081514.

[19] S.-C. Chang, T.-C. Hsu, and M. S.-Y. Jong, "Integration of the peer assessment approach with a virtual reality design system for learning earth science," *Comput. Educ.*, vol. 146, 2020, Art. no. 103758, doi: 10.1016/j.compedu.2019.103758.

[20] J. Miao and K. I. Kou, "Quaternion-based bilinear factor matrix norm minimization for color image inpainting," *IEEE Trans. Signal Process.*, vol. 68, pp. 5617–5631, 2020, doi: 10.1109/TSP.2020.3025519.

[21] R. Wang, K. Wang, W. Cao, and X. Wang, "Geometric algebra in signal and image processing: A survey," *IEEE Access*, vol. 7, pp. 156,315–156,325, 2019, doi: 10.1109/ACCESS.2019.2948615.

[22] C. C. Took and D. P. Mandic, "The quaternion LMS algorithm for adaptive filtering of hypercomplex processes," *IEEE Trans. Signal Process.*, vol. 57, no. 4, pp. 1316–1327, Apr. 2009, doi: 10.1109/TSP.2008.2010600.

[23] A. N. Lasenby, "Geometric algebra as a unifying language for physics and engineering and its use in the study of gravity," *Adv. Appl. Clifford Algebr.*, vol. 27, no. 1, pp. 733–759, 2017, doi: 10.1007/s00006-016-0700-z.

[24] E. Hitzer, M. Kamarianakis, G. Papagiannakis, and P. Vašík, "Survey of new applications of geometric algebra," *Math. Methods Appl. Sci.*, early access, 2023, doi: 10.1002/mma.9575.

[25] X. Liu, M. Zhang, J. Chen, and B. Yin, "Trajectory tracking with quaternion-based attitude representation for autonomous underwater vehicle based on terminal sliding mode control," *Appl. Ocean Res.*, vol. 104, Nov. 2020, Art. no. 102342, doi: 10.1016/j.apor.2020.102342.

[26] P. Silveira, A. Teixeira, and C. G. Soares, "A method to extract the quaternion ship domain parameters from AIS data," *Ocean Eng.*, vol. 257, Aug. 2022, Art. no. 111568, doi: 10.1016/j.oceaneng.2022.111568.

[27] E. Alcaras, C. Parente, and A. Vallario, "The importance of the coordinate transformation process in using heterogeneous data in coastal and marine geographic information system," *J. Mar. Sci. Eng.*, vol. 8, no. 9, 2020, Art. no. 708, doi: 10.3390/jmse8090708.

[28] E. Hitzer and S. J. Sangwine, "Foundations of conic conformal geometric algebra and compact versors for rotation, translation and scaling," *Adv. Appl. Clifford Algebr.*, vol. 29, no. 5, 2019, Art. no. 96, doi: 10.1007/s00006-019-1016-6.

[29] E. Hitzer, W. Benger, M. Niederwieser, R. Baran, and F. Steinbacher, "Foundations for strip adjustment of airborne laserscanning data with conformal geometric algebra," *Adv. Appl. Clifford Algebr.*, vol. 32, no. 1, 2022, Art. no. 1, doi: 10.1007/s00006-021-01184-x.

[30] S. Miron, J. Flamant, N. L. Bihan, P. Chainais, and D. Brie, "Quaternions in signal and image processing: A comprehensive and objective overview," *IEEE Signal Process. Mag.*, vol. 40, no. 6, pp. 26–40, Sep. 2023, doi: 10.1109/MSP.2023.3278071.

[31] T. Xu, X. Kong, Q. Shen, Y. Chen, and Y. Zhou, "Deep and low-rank quaternion priors for color image processing," *IEEE Trans. Circuits Syst. Video Technol.*, vol. 33, no. 7, pp. 3119–3132, Jul. 2023, doi: 10.1109/TCSVT.2022.3233589.

[32] C. Wang et al., "Multi-dimensional hypercomplex continuous orthogonal moments for light-field images," *Expert Syst. Appl.*, vol. 217, 2023, Art. no. 119553, doi: 10.1016/j.eswa.2023.119553.

[33] S. Yuan, S. Zhu, D.-S. Li, W. Luo, Z.-Y. Yu, and L.-W. Yuan, "Feature preserving multiresolution subdivision and simplification of point clouds: A conformal geometric algebra approach," *Math. Methods Appl. Sci.*, vol. 41, no. 11, pp. 4074–4087, 2018, doi: 10.1002/mma.4616.

[34] A. C. O. Tanrikulu, "Least-mean kurtosis: A novel higher-order statistics based adaptive filtering algorithm," *Electron. Lett.*, vol. 30, no. 3, pp. 189–190, Feb. 1994, doi: 10.1049/el:19940129.

[35] E. Bayro-Corrochano, "A survey on quaternion algebra and geometric algebra applications in engineering and computer science 1995–2020," *IEEE Access*, vol. 9, pp. 104,326–104,355, 2021, doi: 10.1109/ACCESS.2021.3097756.

[36] M. Buzzicotti, B. A. Storer, H. Khatri, S. M. Griffies, and H. Aluie, "Spatio-temporal coarse-graining decomposition of the global ocean geostrophic kinetic energy," *J. Adv. Model. Earth Syst.*, vol. 15, no. 6, 2023, Art. no. e2023MS003693, doi: 10.1029/2023MS003693.

[37] L. Yuan, Z. Yu, W. Luo, L. Yi, and Y. Hu, "Pattern forced geophysical vector field segmentation based on Clifford FFT," *Comput. Geosci.*, vol. 60, pp. 63–69, Oct. 2013, doi: 10.1016/j.cageo.2013.05.007.

[38] Z. Yu, W. Luo, L. Yi, Y. Hu, and L. Yuan, "Clifford algebra-based structure filtering analysis for geophysical vector fields," *Nonlinear Process. Geophys.*, vol. 20, no. 4, pp. 563–570, 2013, doi: 10.5194/npg-20-563-2013.

[39] G. Vieira and M. E. Valle, "A general framework for hypercomplex-valued extreme learning machines," *J. Comput. Math. Data Sci.*, vol. 3, Jun. 2022, Art. no. 100032, doi: 10.1016/j.jcmds.2022.100032.

[40] Y. Li, Y. Wang, Y. Wang, C. Qian, and R. Wang, "Geometric algebra based recurrent neural network for multi-dimensional time-series prediction," *Frontiers Comput. Neurosci.*, vol. 16, Dec. 2022, Art. no. 1078150, doi: 10.3389/fncom.2022.1078150.

[41] C. Lee, H. Hasegawa, and S. Gao, "Complex-valued neural networks: A comprehensive survey," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 8, pp. 1406–1426, Aug. 2022, doi: 10.1109/JAS.2022.105743.

[42] D. García-Retuerta, R. Casado-Vara, A. Martin-del Rey, F. De la Prieta, J. Prieto, and J. M. Corchado, "Quaternion neural networks: State-of-the-art and research challenges," in *Proc. Int. Conf. Intell. Data Eng. Automated Learn.*, Cham, Switzerland: Springer-Verlag, 2020, pp. 456–467, doi: 10.1007/978-3-030-62365-4_43.

[43] A. Chattopadhyay, M. Gray, T. Wu, A. B. Lowe, and R. He, "OceanNet: A principled neural operator-based digital twin for regional oceans," 2023, *arXiv:2310.00813.*

[44] J. Brandstetter, R. van de Berg, M. Welling, and J. K. Gupta, "Clifford neural layers for PDE modeling," 2022, *arXiv:2209.04934*.

[45] T. Yuan, J. Zhu, K. Ren, W. Wang, X. Wang, and X. Li, "Neural network driven by space-time partial differential equation for predicting sea surface temperature," in *Proc. IEEE Int. Conf. Data Mining (ICDM)*, Piscataway, NJ, USA: IEEE Press, 2022, pp. 656–665, doi: 10.1109/ICDM54844.2022.00076.

[46] L. Yuan, Z. Yu, and W. Luo, "Towards the next-generation GIS: A geometric algebra approach," *Ann. GIS*, vol. 25, no. 3, pp. 195–206, 2019, doi: 10.1080/19475683.2019.1612945.

[47] L. Yuan, Z. Yu, S. Chen, W. Luo, Y. Wang, and G. Lü, "CAUSTA: Clifford algebra-based unified spatio-temporal analysis," *Trans. GIS*, vol. 14, no. s1, pp. 59–83, Jul. 2010, doi: 10.1111/j.1467-9671.2010.01221.x.

[48] W. Benger and W. Dobler, "Massive geometric algebra: Visions for c++ implementations of geometric algebra to scale into the big data era," *Adv. Appl. Clifford Algebr.*, vol. 27, no. 3, pp. 2153–2174, 2017, doi: 10.1007/s00006-017-0780-4.

[49] D. Li et al., "A tensor-based approach to unify organization and operation of data for irregular spatio-temporal fields," *Int. J. Geographical Inform. Sci.*, vol. 36, no. 9, pp. 1885–1904, 2022, doi: 10.1080/13658816.2022.2092116.

[50] L. Yuan, Z. Yu, W. Luo, J. Zhang, and Y. Hu, "Clifford algebra method for network expression, computation, and algorithm construction," *Math. Methods Appl. Sci.*, vol. 37, no. 10, pp. 1428–1435, 2014, doi: 10.1002/mma.2904.

[51] M. C. L. Belon and D. Hildenbrand, "Practical geometric modeling using geometric algebra motors," *Adv. Appl. Clifford Algebr.*, vol. 27, no. 3, pp. 2019–2033, Sep. 2017, doi: 10.1007/s00006-017-0777-z.

[52] L. Ren, X. Ran, J. Peng, and C. Shi, "Saliency detection for small maritime target using singular value decomposition of amplitude spectrum," *IETE Tech. Rev.*, vol. 34, no. 6, pp. 631–641, 2017, doi: 10.1080/02564602.2016.1231023.

[53] M. E. Valle and R. A. Lobo, "Hypercomplex-valued recurrent correlation neural networks," *Neurocomputing*, vol. 432, pp. 111–123, Apr. 2021, doi: 10.1016/j.neucom.2020.12.034.

[54] A. Aristidou and J. Lasenby, "Inverse kinematics solutions using conformal geometric algebra," in *Guide to Geometric Algebra in Practice*. London, U.K.: Springer-Verlag, 2011, pp. 47–62.

[55] A. Schmitt, A. Wendleder, R. Kleynmans, M. Hell, A. Roth, and S. Hinz, "Multi-source and multi-temporal image fusion on hypercomplex bases," *Remote Sens.*, vol. 12, no. 6, 2020, Art. no. 943, doi: 10.3390/rs12060943.

[56] J. He, Y. Guo, and H. Yuan, "Ship target automatic detection based on hypercomplex flourier transform saliency model in high spatial resolution remote-sensing images," *Sensors*, vol. 20, no. 9, 2020, Art. no. 2536, doi: 10.3390/s20092536.

[57] A. Stanton and M. D. Sacchi, "Vector reconstruction of multicomponent seismic data," *Geophysics*, vol. 78, no. 4, pp. V131–V145, 2013, doi: 10.1190/geo2012-0448.1.

[58] X. Li et al., "Deep-learning-based information mining from ocean remote-sensing imagery," *Nat. Sci. Rev.*, vol. 7, no. 10, pp. 1584–1605, 2020, doi: 10.1093/nsr/nwaa047.

[59] M. Reichstein et al., "Deep learning and process understanding for data-driven earth system science," *Nature*, vol. 566, no. 7743, pp. 195–204, 2019, doi: 10.1038/s41586-019-0912-1.

[60] Z. Chen, Y. Liu, and H. Sun, "Physics-informed learning of governing equations from scarce data," *Nature Commun.*, vol. 12, no. 1, 2021, Art. no. 6136, doi: 10.1038/s41467-021-26434-1.

[61] L. Zanna and T. Bolton, "Data-driven equation discovery of ocean mesoscale closures," *Geophys. Res. Lett.*, vol. 47, no. 17, 2020, Art. no. e2020GL088376, doi: 10.1029/2020GL088376.

[62] Y. Li, J. Yang, Z. Yuan, and Y. Zhang, "Marine oil spills detection and classification from polsar images based on complex-valued convolutional neural network," in *Proc. IEEE Int. Geosci. Remote Sens. Symp. (IGARSS)*, 2022, pp. 7085–7088, doi: 10.1109/IGARSS46834.2022.9883991.

[63] S. Thiruvengadam, J. S. Tan, and K. Miller, "Time series, hidden variables and spatio-temporal ordinality networks," *Adv. Appl. Clifford Algebr.*, vol. 30, no. 3, 2020, Art. no. 37, doi: 10.1007/s00006-020-01061-z.

[64] D. Ruhe, J. K. Gupta, S. De Keninck, M. Welling, and J. Brandstetter, "Geometric clifford algebra networks," 2023, *arXiv:2302.06594*.

[65] L. Dorst, "3D oriented projective geometry through versors of $R^{3,3}$," *Adv. Appl. Clifford Algebr.*, vol. 26, no. 4, pp. 1137–1172, 2016, doi: 10.1007/s00006-015-0625-y.

[66] S. Sane, R. Bujack, C. Garth, and H. Childs, "A survey of seed placement and streamline selection techniques," *Comput. Graph. Forum*, vol. 39, no. 3, pp. 785–809, 2020, doi: 10.1111/cgf.14036.

[67] C. O. Ausoni and P. Frey, "Geometric algebra for vector field analysis and visualization: Mathematical settings, overview and applications," in *Topological and Statistical Methods for Complex Data: Tackling Large-Scale, High-Dimensional, and Multivariate Data Spaces*. Berlin, Germany: Springer-Verlag, 2014, pp. 183–203.

[68] K. O. Klausen, "Visualizing stokes' theorem with geometric algebra," 2022, *arXiv:2206.07177.*