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Uniqueness of lump solution to the KP-I equation

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Abstract
The KP-I equation has family of solutions which decay to zero at space infinity. One of these solutions is the classical lump solution, which is a traveling wave, and the KP-I equation in this case reduces to the Boussinesq equation. In this paper we classify all the 'lump-type' solutions of the Boussinesq equation. Using a robust inverse scattering transform developed by Bilman–Miller for the Schrödinger equation, we show that the lump-type solutions are rational and their τ functions have to be polynomials of degree $k(k+1)$ for some integer k. In particular, this implies that the lump solution is the unique ground state of the KP-I equation (as conjectured by Klein–Saut). The problem studied in this paper was mentioned in Airault–McKean–Moser, our result can be regarded as a two-dimensional analogy of their theorem on the classification of rational solutions for the KdV equation.

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1 | INTRODUCTION

The KP equation first appeared in the 1970 paper [31] by Kadomtsev and Petviashvili, where they studied the transverse stability of the line solitons of KdV equation. It can be written as

$$\partial_x\left(\partial_t U + \partial_x^3 U + 3\partial_x\left(U^2\right)\right) - \sigma\partial_y^2 U = 0.$$

Here σ is a parameter and if $\sigma = 1$, then it is called KP-I equation and has positive dispersion, while the case of $\sigma = -1$ is of negative dispersion and called KP-II.

KP equation is an integrable system and can be regarded as a two-dimensional generalization of the classical KdV equation. It is an important PDE both in mathematics and physics. Up to now, there exists vast literature on KP equation. In the sequel, we will briefly mention some results which are most closely related to our objective.

There are various ways to study the KP equation. One of them is to use the inverse scattering transform (IST). Manakov [41] studied the IST of KP equation on a formal level. Segur in [50] then analyzed the direct scattering and rigorously obtained the solution for the direct problem under a small norm assumption, but lump solutions are not investigated in those works. Then Fokas–Ablowitz [24] obtained the lump solution in their IST framework. Their results are later extended to include higher order rational solutions in [52]. Zhou [55–57] then studied the KP-I equation and related problems in a more abstract and rigorous way. There the lump solutions correspond to poles of the associated eigenfunctions. Since the pole structure is still not well understood in the general case, the lump solutions are actually not treated. Later Boiti, etc. have also studied the IST of KP-I in [11], with initial data belonging to the Schwartz space. However, in spite of all these important progresses, in general, the IST of KP-I equation is not completely understood yet.

Observe that if U is a traveling wave of the form $u(x-t, y)$, then the KP-I equation reduces to the following Boussinesq equation:

$$\partial_x^2\left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0. \tag{1}$$

Due to the above mentioned difficulty, we would like to study the traveling wave solutions of the KP-I equation using the IST of the Boussinesq equation, which should, in principle, be easier than the KP case. The IST of the Boussinesq equation is first carried out in [20]. The first equation of the associated Lax pair turns out to be a third-order ODE, in contrast with the second-order ODE for the KdV case. In this direction, there are some related works. For instances, the IST for first-order ODE systems with generic potentials (means that the poles are all simple) has been studied in [5, 6] and the case of higher order ODEs has been treated in [7]. The case of general potentials has been studied in [21, 55] using the augmented contour approach. Recently, the hyperbolic case of the Boussinesq equation (so called 'good' Boussinesq equation) is studied in [16, 17] using Riemann–Hilbert approach. For Schwartz class initial data, long time dynamics is obtained. Note that in this case, the equation does not have lump solution.

Let us write the solution u of (1) in terms of the τ function: $u := 2\partial_x^2 \ln \tau$. Then the Boussinesq equation in bilinear form is

$$\left(\mathfrak{D}_x^4 - \mathfrak{D}_x^2 - \mathfrak{D}_y^2\right)\tau \cdot \tau = 0. \tag{2}$$

Throughout the paper, we will use the symbol $\mathfrak{D}$ to denote the bilinear derivative operator. We refer to the classical book by Hirota [28] for detailed exposition to the bilinear derivative operator and the direct method in soliton theory, including that of the KP equation. One can check that the function $\tau(x, y) = x^2 + y^2 + 3$ is a solution of the bilinear Equation (2). This function is even in both x and y variables and corresponding to the classical lump solution. The lump can be regarded as a special rogue wave extensively studied before and is a special one in the large class of lump-type solutions, whose precise definition will be given below. Note that actually the lump solution is first obtained in [42, 49] using a limiting procedure. The spectral property of lump solution is now well understood. Indeed, the first and second authors have proved in [40] using the Backlund transformation that the it is nondegenerate, in the sense that the linearized KP-I operator at this solution does not have any nontrivial kernels. This also implies that the lump is orbitally stable.

The importance of KP-I equation is also reflected by the fact that it appears in the study of many other PDEs. For instance, in [8](see also the references therein), it is shown that KP equation is related to the GP equation. The nondegeneracy result for the lump can then be used to construct traveling wave solutions of the GP equation with subsonic speed, with a perturbation argument; see [39].

It is now well known that the KP-I equation is globally well posed in the natural energy space; see, for instance [32, 44, 45] and the references therein. To fully understand the long time dynamics of the KP-I equation, a crucial step is to analyze the structure of all the lump-type solutions for the Boussinesq equation. More general rational solutions of (2) with degree $k(k+1)$ have been found in [25, 48]. Then in [26] it is proved that around the higher energy lump-type solutions, the KP-I equation has anomalous scattering with infinite phase shift. This indicates that the dynamics of the KP-I equation will be more complicated than the KdV equation. A very fascinating and in-depth description of KP equation and related dynamical, variational and other properties of its solutions, including lump, can be found in the book of Klein–Saut [34]. We refer to it and also its references for a detailed introduction on this subject.

It is worth mentioning that (1) is a special case of Boussinesq-type equation (its original form described by Boussinesq in 1870s):

$$\partial_x^2\left(pu + u^2 + \partial_x^2 u\right) + \sigma\partial_y^2 u = 0,$$

where $\sigma = \pm 1$ and p is a constant. Rational solutions of this equation has been studied in many papers, such as [3, 4, 10, 18, 25, 37]. For instance, the special case of $(\sigma, p) = (1, 0)$ is considered in [18], using the theory of Painlevé equations. Most of these works are concerned with the construction of explicit solutions and the analysis of their mathematical or physical properties.

In view of all these developments, it is desirable to have some classification on the solutions of the Boussinesq equation. In this paper, we would like to classify all the 'lump-type' solutions. Our first result is the following

Theorem 1. *Suppose u is a real-valued C^4 solution of the equation*

$$\partial_x^2\left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0 \text{ in } \mathbb{R}^2.$$

Assume that there exists $\alpha > 0$ such that

$$|u(x,y)| \leqslant \frac{C}{\left(1 + x^2 + y^2\right)^{\alpha}}. \tag{3}$$

Then $u = 2\partial_x^2 \ln \tau_k$, where τ_k is a polynomial in x, y of degree $k(k+1)$ for some $k \in \mathbb{N}$.

We remark that the assumption that u is real valued plays an essential role, since there are many complex valued solutions whose τ functions have degree $\frac{k(k+1)+j(j+1)}{2}$. The classification of complex valued solutions seems to be more difficult.

If we consider those solutions which are even, then we have

Theorem 2. *Suppose τ is a polynomial of degree $2n$ with real coefficients satisfying*

$$\tau(x, y) = \tau(x, -y) = \tau(-x, y)$$

and

$$\left(\mathfrak{D}_x^2 + \mathfrak{D}_y^2 - \mathfrak{D}_x^4\right)\tau \cdot \tau = 0.$$

Assume $n = k(k+1)/2 \leqslant 300$ for some positive integer k. Then τ is unique, up to a multiplicative constant.

We would like to emphasize that the upbound 300 is not optimal, and uniqueness of even solution is expected to be true for all $n = k(k+1)/2$. We refer to Section 5 for more details.

The KP-I equation has a variational structure and ground state solutions can be constructed using variational methods (see [12, 38]). As a corollary of Theorem 1, we see that the classical lump solution is the unique ground state of the KP-I equation, due to the fact that the energy is determined by the degree of the τ function. This answers affirmatively the uniqueness question raised in Remarks 18 and 19 by Klein–Saut in [33]. As already pointed out there, while the uniqueness of ground state of the Schrödinger equation can be proved using ODE shooting method, the uniqueness of lump is more complicated since it is not radially symmetric. We have in mind that those even traveling wave solutions of the KP-I equation should play similar role as the radially symmetric solutions of the Schrödinger equation. To our knowledge, our theorem seems to be the first classification result for solutions of semilinear elliptic equations without symmetry (also without any other assumptions such as stable or finite Morse index).

Solutions satisfying the assumptions of Theorem 1 will be called lump-type solutions. We remark that for each fixed k, there is a family of lump-type solutions, already have been found in [25]. We expect that all lump-type solutions should be included in this family. Those solutions will be recalled in the next section. However, a full classification of this type would need further detailed analysis, which will not be pursued in this paper. We expect that the moduli space of real-valued solutions is a manifold of dimension $2k$. Such a full classification would presumably yields some information of the lump-type solutions of the generalized KP equation.

Many questions remain to be answered. For instances, the classification of complex valued solutions; the computation of the Morse indices of lump-type solutions; the asymptotical stability of the lump solution; the classification of solutions to the general Boussinesq equation with zero or nonzero condition near infinity, etc. Note that by a result of [12], Pohozaev type identity tells us that the KP-II equation does not have lump-type solutions.

Let us now sketch the main ideas of our proof. We first use the robust IST developed by Bilman–Miller [9] to show that lump-type solutions have to be rational. Then we use the technique of [4], appealing to the Boussinesq hierarchy, to show that the degree of the τ function has to be $k(k+1)$. This technique is used in [4] to prove that the τ function of the rational solution of the KdV equation necessarily is a polynomial of degree $k(k+1)/2$. We hope that our method should also be applicable to other integrable systems such as 2d Toda lattice.

This paper is organized as follows. In Section 2, we recall the construction of lump-type solutions appeared in various papers of Pelinovskii and his collaborators. We emphasize that the KP-I equation is a well-studied model equation and actually there are many other constructions, using different methods. In Section 3, we use the robust IST to show that lump-type solutions are rational. We then investigate the degree of the τ functions in Section 4. The last section is devoted to analyze the even solutions. In particular, we prove that even solution is unique when the degree of its τ function is not so large.

2 | FAMILY OF LUMP-TYPE SOLUTIONS

In this section, we recall the construction of lump-type solutions. Although the materials in this section will not be used in the proof of our main results, it will be helpful to provide a rough picture of what lump solutions should be. We also feel that to fully classify and understand the moduli space of all the lump-type solutions and prove the uniqueness of even solutions, it will be a necessary step to have an explicit construction of solutions.

Real-valued rational solutions of the Boussinesq equation whose τ functions have degree $k(k+1)$ have been obtained in [48] by a limiting procedure. These solutions are even with respect to x and y variables. For instance, the function $u = 12\partial_x^2 \ln \tau$, where

$$\tau(x,y) = \left(x^2+y^2\right)^3 + 25x^4 + 90x^2y^2 + 17y^4 - 125x^2 + 475y^2 + 1875,$$

solves the equation

$$\left(-u + \frac{u^2}{2} + u_{xx}\right)_{xx} - u_{yy} = 0. \tag{4}$$

Observe that the coefficients in Equation (4) are different from that of (1). However, they can be transformed to one another simply by suitable scaling of the form $au(bx, cy)$. In the rest of the paper, we will consider the Boussinesq equation with other coefficients in different contexts. This is to make them to be consistent with the corresponding literature.

In [25], more general families of rational solutions have been derived using Wronskian representation of the solutions to the KP equation. Among these solutions, those traveling waves reduces to the Boussinesq equation. Let us recall these results in the sequel. We adopt the notations used in [25].

Consider the KP-I equation in the following form:

$$\left(-4u_{t_3} + \left(3u^2\right)_{t_1} + u_{t_1t_1t_1}\right)_{t_1} - 3u_{t_2}u_{t_2} = 0.$$

This equation has solutions expressed in terms of the τ function:

$$u(t_1,t_2,t_3) = 2\partial_{t_1}^2 \ln \tau(t_1,t_2,t_3).$$

There are different forms for the τ functions. Let us explain it now.

Let $\Psi_n^\pm$ be solutions of the system of differential equations

$$\begin{cases} \pm i\partial_{t_2}\Psi_n^\pm = \partial_{t_1}^2\Psi_n^\pm, \\ \partial_{t_3}\Psi_n^\pm = \partial_{t_1}^3\Psi_n^\pm. \end{cases}$$

We fix an integer N and define

$$\tau = \det M_N, \tag{5}$$

where M_N is the $N \times N$ matrix whose entries are given by $c_{nk} + I_{nk}, 1 \leqslant n, k \leqslant N$. Here c_{nk} are arbitrary complex parameters and

$$I_{nk} = \int_{-\infty}^{t_1} \Psi_n^+(s, t_2, t_3)\Psi_k^-(s, t_2, t_3)ds.$$

With this definition, the function $2\partial_{t_1}^2 \ln \tau$ is a solution of the KP-I equation.

The KP-I equation has another family of solutions, for which the τ function has the Wronskian form:

$$\tau = W\left(\Psi_1^\pm, \dots, \Psi_N^\pm\right) = \det\left(J_{nk}^\pm\right), \tag{6}$$

where $J_{nk}^\pm = \partial_{t_1}^{k-1}\Psi_n^\pm$.

The two forms (5) and (6) are indeed related to each other. If we choose in (5) the function

$$\Psi_k^- = \exp\left(p_k t_1 - p_k^2 t_2 + p_k^3 t_3\right) := \exp\left(\Phi_k^-\right),$$

then integration by parts yields

$$I_{nk} = \left(\frac{\Psi_n^+}{p_k} - \frac{\partial_{t_1}\Psi_n^+}{p_k^2} + \frac{\partial_{t_1}^2\Psi_n^+}{p_k^3} + \cdots\right)\exp\left(\Phi_k^-\right).$$

Assuming $p_k \gg 1$, the leading terms of τ can be written as the product of a Vandermont determinant and the Wronskian $W\left(\Psi_1^+, \dots, \Psi_N^+\right)$. Hence

$$\tau = \left[\frac{\prod_{1\leqslant m<k\leqslant N}(p_k - p_m)}{\prod_{k=1}^N p_k^N} W\left(\Psi_1^+, \dots, \Psi_N^+\right) + O\left(p^{-\left(\frac{N(N+1)}{2}+1\right)}\right)\right]\exp\left(\sum_{k=1}^N \Phi_k^-\right).$$

Dividing the right-hand side by $\exp(\sum_{k=1}^N \Phi_k^-)$ and the constant before the Wronskian W, and letting $p \to \infty$, we get (6).

Now let $K \leqslant N$ be a fixed integer. If the above limiting procedure is only carried out for $\Phi_k^-, k = K+1, \dots, N$, then we obtain

$$\tau = \det(S_{nk}), \tag{7}$$

where

$$S_{nk} = \begin{cases} I_{nk}, & \text{for } k = 1, \dots, K, \\ J_{n,k-K}^+, & \text{for } k = K+1, \dots, N. \end{cases}$$

Let us now consider the function $\phi_m := \partial_p^m \exp\left(\Phi^+(t_1, t_2, t_3, p)\right)$, where

$$\Phi^+(t_1, t_2, t_3, p) = \sum_{j=1}^{\infty} \left(p^j t_j\right).$$

We have

$$\phi_m = P_m \exp\left(\Phi^+(t_1, t_2, t_3, p)\right).$$

Here P_m is a polynomial of the variables $\theta_1, \dots, \theta_m$, given by $\theta_j = \frac{1}{j!}\partial_p^j \Phi^+$. In particular,

$$\begin{aligned} \theta_1 &= t_1 + 2pt_2 + 3p^2 t_3 + \cdots, \\ \theta_2 &= t_2 + 3pt_3 + \cdots, \\ \theta_3 &= t_3 + \cdots, \end{aligned}$$

and θ_j only depends on $t_j, t_{j+1}, \dots$. We have

$$P_1 = \theta_1, P_2 = 2\theta_2 + \theta_1^2.$$

and the sequence of polynomials $\{P_j\}$ satisfies a recurrence relation of the form

$$P_{m+1} = \theta_1 P_m + \sum_{j=1}^{m} \left[(j+1)\theta_{j+1}\partial_{\theta_j} P_m\right]. \tag{8}$$

One can prove by induction that

$$\partial_{\theta_j} P_m = \partial_{\theta_1}^j P_m = \frac{m!}{(m-j)!} P_{m-j}. \tag{9}$$

Let $v = -\frac{1}{3p}$ and define the operator

$$\mathcal{S}(v) = \exp\left(-\sum_{m=1}^{+\infty} \frac{v^m}{m}\partial_{\theta_m}\right).$$

Explicitly, $\mathcal{S}(v)P_m = P_m(\theta_1 - v, \theta_2 - \frac{v^2}{2}, \dots, \theta_m - \frac{v^m}{m})$. Using the recurrence relation (8) again, we obtain

$$\mathcal{S}(v)P_m = \left(1 - v\partial_{\theta_1}\right)P_m.$$

The τ function will be a solution of the Boussinesq equation if it depends on the variables $x = t_1 + 3p^2 t_3$ and t_2. This requires

$$\partial_{\theta_2}\tau = v\partial_{\theta_3}\tau.$$

To construct solutions for the Boussinesq equation, we then define

$$\Psi_n^+ = \left(S^{N-n}(v)P_{2n-1}^+\right)\exp\left(\Phi^+(t_1,t_2,t_3,p)\right), \quad \text{for } 1 \leqslant n \leqslant N,$$
$$\Psi_k^- = \left(S^{K-k}(v)P_{2k-1}^-\right)\exp\left(\Phi^-(t_1,t_2,t_3,p)\right), \quad \text{for } 1 \leqslant k \leqslant K.$$

Then the τ function defined by (7) will correspond to a solution of the Boussinesq equation. In general, this solution is complex valued. But in the particular case of $K = N$, if we choose P_k^+, P_k^- such that $P_k^+ = \bar{P}_k^-$ and let $\mu = -\frac{1}{2p}$, then

$$\tau = \det\left(w^+(w^-)^T\right),$$

where w^+ is a matrix of size $N \times (2N-1)$, whose entries are given by

$$\left(w^+\right)_{nk} = (-\mu)^{k-1}\partial_{t_1}^{k-1}\left[S^{-k}(\mu)S^{N-n}(v)P_{2n-1}^+\right], 1 \leqslant n \leqslant N, 1 \leqslant k \leqslant 2N-1.$$

The entries of w^- are then defined to be $w_{nk}^- = \bar{w}_{nk}^+$. This implies that the determinant can be written as the sum of positive terms. Note that the condition $P_k^+ = \bar{P}_k^-$ requires t_{2k} to be imaginary and t_{2k+1} to be real. Hence there are in total $2N$ free (real) parameters, or N complex parameters. The degree of the τ functions is equal to $N(N+1)$. Let us mention that for the complex valued solutions constructed there, their degree has the form

$$N(N+1)/2 + K(K+1)/2.$$

We point out that in [53], an explicit family of rational solutions is also obtained with different methods. In the degree 6 case, the family of functions $2\partial_x^2 \ln \tau$, where

$$\begin{aligned}\tau(x,y) = {} & x^6 + y^6 + 3x^4y^2 + 3x^2y^4 + 14x^5 + 14xy^4 + 28x^3y^2 + 90x^4 \\ & + 128x^2y^2 + 22y^4 + 324x^3 + 316xy^2 + 648x^2 + 360y^2 + 648x + 324 \\ & + 2a\left(x^3 - 3xy^2 + 7x^2 - 7y^2 + 16x + 8\right) \\ & + 2by\left(y^2 - 3x^2 - 14x - 18\right) + a^2 + b^2,\end{aligned}$$

with a, b being real-valued parameters, solve the following Boussinesq equation

$$\left(-3u + 3u^2 + u_{xx}\right)_{xx} + 3u_{yy} = 0.$$

To conclude this section, we remark that there already exist many papers on the construction and analysis of solutions to the KP and related equations, for instances, [1, 14, 15, 23, 27, 29, 30, 46, 47], just to list a few of them.

3 | INVERSE SCATTERING OF THE BOUSSINESQ EQUATION AND THE RATIONALITY OF LUMP-TYPE SOLUTIONS

In this section, we will show that lump-type solutions of the Boussinesq equation have to be rational functions. The equation to be studied here reads as

$$q_{yy} = 3q_{xxxx} - 12\left(q^2\right)_{xx} - 24q_{xx}. \tag{10}$$

This can be obtained from the original Boussinesq equation (1) by a simple rescaling, that is, by setting $q(x,y) = -6u(2\sqrt{2}x, 8\sqrt{3}y)$.

Observe that every constant function solves (10). Here we will focus on the special class of solutions decaying to *zero* at infinity. At this stage, we would like to point out that the usual inverse scattering of the Boussinesq equation is developed in [20], with a 'nonzero' boundary condition near infinity. It can be seen later on that in our case, the situation is much more complicated, since the corresponding fundamental solutions have singularities in the complex plane of spectral parameter. To overcome these difficulties, we will adopt the powerful method of 'robust' IST to show that lump-type solutions of (10) have to be rational. This type of robust inverse scattering has been developed for the first time in [9], to analyze the rogue waves of the Schrödinger equation.

3.1 | Refined asymptotics of lump-type solutions

To carry out the robust inverse scattering transform, it turns out to be important to get a precise decay estimate for the lump-type solutions.

We would like to prove the following refined asymptotics estimate.

Proposition 3. *Suppose u is a real-valued C^4 solution of the Boussinesq equation*

$$\partial_x^2\left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0 \text{ in } \mathbb{R}^2.$$

Assume that for some $\alpha > 0$,

$$|u(x,y)| \leqslant \frac{C}{\left(1 + x^2 + y^2\right)^{\alpha}}. \tag{11}$$

Then there holds

$$|u(x,y)| \leqslant \frac{C}{1 + x^2 + y^2}. \tag{12}$$

Proof. The solution u satisfies

$$\partial_x^4 u - \partial_x^2 u - \partial_y^2 u = -3\partial_x^2\left(u^2\right).$$

Then

$$u(x,y) = 3\int_{\mathbb{R}^2} K(x-s, y-t)u^2(s,t)\,ds\,dt,$$

where the kernel K is defined through the Fourier transform:

$$K(x,y) = \int_{\mathbb{R}^2} \frac{\xi_1^2}{\xi_1^4 + \xi_1^2 + \xi_2^2} e^{ix\xi_1 + iy\xi_2}\,d\xi_1 d\xi_2.$$

By [13, Lemma 3.6], we have

$$\left(x^2+y^2\right)K \in L^\infty\left(\mathbb{R}^2\right). \tag{13}$$

To simplify the notation, we introduce $r=\sqrt{x^2+y^2}$ and define

$$\Omega_1 := \left\{(s,t) : (s-x)^2+(t-y)^2 \leqslant \frac{r^2}{4}\right\}, \quad \text{and} \quad \Omega_2 := \left\{(s,t) : s^2+t^2 \leqslant \frac{r^2}{4}\right\}.$$

Using (11), (13) and the integrability of the kernel $K(s,t)$ around $(0,0)$, we can estimate

$$\begin{aligned}\int_{\Omega_1} |K(x-s,y-t)|u^2(s,t)dsdt &\leqslant \int_{\Omega_1} \frac{|K(x-s,y-t)|}{\left(1+x^2+y^2\right)^{2\alpha}}dsdt\\ &\leqslant \frac{C\ln(2+r)}{\left(1+x^2+y^2\right)^{2\alpha}}.\end{aligned}$$

We also have

$$\begin{aligned}\int_{\Omega_2} |K(x-s,y-t)|u^2(s,t)dsdt &\leqslant \int_{\Omega_2} \frac{u^2(s,t)}{1+x^2+y^2}dsdt\\ &\leqslant \frac{C}{1+x^2+y^2}+\frac{C}{\left(1+x^2+y^2\right)^{2\alpha}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^2\setminus(\Omega_1\cup\Omega_2)} |K(x-s,y-t)|u^2(s,t)dsdt &\leqslant \int_{\mathbb{R}^2\setminus(\Omega_1\cup\Omega_2)} \frac{Cdsdt}{\left(1+s^2+t^2\right)^{2+2\alpha}}\\ &\leqslant \frac{C}{1+x^2+y^2}+\frac{C}{\left(1+x^2+y^2\right)^{2\alpha}}.\end{aligned}$$

Combining all these estimates, we deduce

$$|u(x,y)| \leqslant \frac{C}{1+x^2+y^2}+\frac{C}{\left(1+x^2+y^2\right)^{\frac{3\alpha}{2}}}.$$

A straightforward bootstrapping argument tells us that

$$|u(x,y)| \leqslant \frac{C}{1+x^2+y^2}.$$

This is the required decay estimate. □

Estimate (12) is optimal, as can be seen from the classical lump solution and the examples discussed in the previous section. Note that the optimal decay similar to (12) has been derived in [13] assuming that the solution is integrable in a suitable sense, meaning that it belongs to the appropriate natural energy space.

We point out that the estimate of Proposition 3 actually still holds if we only assume that u tends to zero at infinity, without any *a priori* algebraic decay assumption. However, proving this fact will be quite involved. This will appear in a future work.

3.2 | Inverse scattering

Introducing a new function p, Equation (10) can be transformed into the following system of ODEs:

$$\begin{cases} q_y = -3p_x, \\ p_y = -q_{xxx} + 8qq_x + 8q_x. \end{cases}$$

This system is corresponding to the following Lax pair equation (see [20, 54]):

$$\frac{dL}{dy} = QL - LQ = [Q, L],$$

where the operators L and Q are defined by

$$\begin{cases} L = i\frac{d^3}{dx^3} - i\left[\left(2(q+1)\frac{d}{dx} + q_x\right)\right] + p, \\ Q = i\left(3\frac{d^2}{dx^2} - 4(q+1)\right). \end{cases}$$

Here i is the imaginary unit. Let $k \in \mathbb{C}$ be a complex spectral parameter. We will consider the ODE

$$Lf = \left(k^3 + 2k\right)f. \tag{14}$$

Introducing vector $\mathbf{f} := (f_1, f_2, f_3)^T$ by $f_1 = f, f_2 = f_1', f_3 = f_2'$, we arrive at the following system of ODEs:

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q_x + pi - i\left(k^3 + 2k\right) & 2(q+1) & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{15}$$

The coefficient matrix of this system, denoted by A, will depend on the potentials q and p. As $x \to \pm\infty$, A will tend to the following trace-free constant matrix

$$T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -i\left(k^3 + 2k\right) & 2 & 0 \end{pmatrix}.$$

The eigenvalues of T can be explicitly computed. They depend on parameter k and are given by

$$\lambda_1 = ik, \quad \lambda_2 = \frac{-ik + \sqrt{3k^2 + 8}}{2}, \quad \lambda_3 = \frac{-ik - \sqrt{3k^2 + 8}}{2}.$$

It follows that T can be written as PMP^{-1}, where

$$P(k)=\begin{pmatrix} 1 & 1 & 1 \\ ik & \frac{-ik+\sqrt{3k^2+8}}{2} & \frac{-ik-\sqrt{3k^2+8}}{2} \\ -k^2 & \frac{k^2+4-ik\sqrt{3k^2+8}}{2} & \frac{k^2+4+ik\sqrt{3k^2+8}}{2} \end{pmatrix}, M(k)=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Recall that for any constant matrix B, the matrix $e^{Tx}B$ is a solution of the equation $U' = TU$. We choose $B = P$ and get the following matrix solution

$$U_{bg}(k,x) := n(k)P(k)e^{M(k)x} := E(k)e^{M(k)x}.$$

Here $n(k)$ is chosen such that $\det\left(U_{bg}\right) = 1$. Explicitly,

$$n(k) = \left(\left(3k^2+2\right)\sqrt{3k^2+8}\right)^{-1}.$$

One can see that as k tends to $\pm\frac{\sqrt{6}}{3}i$ or $\pm\frac{2\sqrt{6}}{3}i$, the function n will blow up. For $j = 1,2,3$, let us denote the jth column of $E(k)$ by ξ_j.

Let $k = s + ti$, with $s,t \in \mathbb{R}$. Direct computation tells us that the condition $\mathrm{Re}(\lambda_2) = \mathrm{Re}(\lambda_3)$ implies $\mathrm{Re}\sqrt{3k^2+8} = 0$. That is,

$$s = 0 \text{ and } t^2 > \frac{8}{3}.$$

On the other hand, $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$ or $\mathrm{Re}(\lambda_3)$ requires

$$s^2 - 3t^2 + 2 = 0.$$

Let r be a fixed large constant and B_r be the ball of radius r centered at the origin. In the region $B_r^c := \mathbb{R}^2 \backslash B_r$, we consider the curve

$$\Sigma_1 := \left\{(s,t) \in B_r^c : s^2 - 3t^2 + 2 = 0\right\} \cup \left\{(s,t) \in B_r^c : s = 0 \text{ and } t^2 > \frac{8}{3}\right\}.$$

Let us define

$$\Omega_1 = B_r^c \backslash \Sigma_1.$$

Note that Ω_1 has six connected components, which we will denote them by $\Omega_{1,1}, \dots, \Omega_{1,6}$.

In the ball B_r, we consider the curve

$$\Sigma_2 := \left\{(s,t) : s = 0, t^2 \leqslant \frac{8}{3}\right\}.$$

We also define $\Omega_2 := B_r \backslash \Sigma_2$.

Next we define a distinguished solution matrix for (15). Note that if the matrix ϕ satisfies $\phi' = A\phi$, then $g := \phi e^{-Mx}$ will satisfy

$$g' = \phi' e^{-Mx} + \phi(-M)e^{-Mx} = Ag - gM.$$

For $k \in \mathbb{R}^2 \backslash B_r$, as will be explained below, we can choose g to be the matrix solution such that

$$\|g(x)\|_{L^\infty(\mathbb{R})} < +\infty, \text{ and } g(x) \to E(k), \text{ as } x \to -\infty. \tag{16}$$

We then define $U^{ou} = ge^{Mx}$. The solution $\phi = ge^{Mx}$ satisfies (16) is called Beals–Coifman fundamental solution. The existence of this solution is explained in [7, p, 8]. We will sketch the main steps below. As has already been pointed out there, the first step is to construct a solution with prescribed asymptotics at $-\infty$, using the arguments of [19, Problem 29, p. 104]. Since this construction will play an important role later on, we recall the precise statement of the result and its proof in the following

Lemma 4. *Assume λ_j, $j = 1, 2, 3$, are distinct. Then the equation*

$$\phi' = A\phi \tag{17}$$

has a solution ϕ_j^+ satisfying

$$\phi_j^+(x)e^{-\lambda_j x} \to \xi_j, \text{ as } x \to +\infty.$$

Similarly, (17) also has a solution ϕ_j^- with

$$\phi_j^-(x)e^{-\lambda_j x} \to \xi_j, \text{ as } x \to -\infty.$$

Proof. Let $\operatorname{Re} \lambda_j = \sigma$ and $e^{Tx} = Y_1(x) + Y_2(x)$, where the entries of Y_1 are linear combination of $e^{\lambda_k x}$ with $\operatorname{Re} \lambda_k < \sigma$, and the entries of Y_2 are linear combination of $e^{\lambda_k x}$ with $\operatorname{Re} \lambda_k \geqslant \sigma$. Thanks to the assumption that λ_j are distinct, this decomposition always exists.

We use a Picard iteration scheme and set $\eta_0(x) = e^{\lambda_j x}\xi_j$. Let a be a fixed constant. Then we can define the sequence $\{\eta_l\}$ by

$$\eta_{l+1}(x) := e^{\lambda_j x}\xi_j + \int_a^x Y_1(x-s)R(s)\eta_l(s)ds - \int_x^{+\infty} Y_2(x-s)R(s)\eta_l(s)ds.$$

The definition of Y_2 ensures that the last integral is well defined. Note that when $x \leqslant 0$, there holds $|Y_2(x)| \leqslant K_2 e^{\sigma x}$ for some constant K_2. Now if we assume $|\eta(s)| \leqslant Ce^{\sigma s}$, then there holds

$$\left|\int_x^{+\infty} Y_2(x-s)R(s)\eta(s)ds\right| \leqslant CK_2 \int_x^{+\infty} e^{\sigma(x-s)}|R(s)|e^{\sigma s}ds$$
$$= CK_2 e^{\sigma x} \int_x^{+\infty} |R(s)|ds.$$

On the other hand, there exists $\delta, K_1 > 0$ such that $\left|Y_1(x)\right| \leqslant K_1 e^{(\sigma-\delta)x}$ for $x \geqslant 0$. Hence if $|\eta(s)| \leqslant Ce^{\sigma s}$, then

$$\begin{aligned}\left|\int_a^x Y_1(x-s)R(s)\eta(s)ds\right| &\leqslant CK_1 \int_a^x e^{(\sigma-\delta)(x-s)}|R(s)|e^{\sigma s}ds \\ &\leqslant CK_1 e^{\sigma x}\int_a^x e^{-\delta(x-s)}|R(s)|ds \\ &\leqslant CK_1 e^{\sigma x}\int_a^x |R(s)|ds.\end{aligned} \tag{18}$$

It follows from these two estimates that if a is chosen such that

$$(K_1+K_2)\int_a^{+\infty}|R(s)|ds < 1,$$

then the sequence $\{\eta_l\}$ will converge to a solution ϕ_j^+ of Equation (17) satisfying

$$\left|\phi_j^+(x)\right| \leqslant Ce^{\sigma x} \text{ for } x \text{ large.}$$

Note that by the decay estimate (12) of the lump-type solution, we have

$$\int_a^{+\infty}|R(s)|ds \leqslant \frac{C}{1+|y|}\left(\frac{\pi}{2} - \arctan\frac{a}{1+|y|}\right). \tag{19}$$

Hence such a always exists.

Now since

$$\phi_j^+ = e^{\lambda_j x}\xi_j + \int_a^x Y_1(x-s)R(s)\phi_j^+(s)ds - \int_x^{+\infty} Y_2(x-s)R(s)\phi_j^+(s)ds.$$

we then can use (18) to deduce

$$\phi_j^+ e^{-\lambda_j x} - \xi_j \to 0, \text{ as } x \to +\infty.$$

Similar arguments (with straightforward modification taking care of the definition of Y_1 and Y_2) yield the solution ϕ_j^-. The choice of a for ϕ_j^+ and ϕ_j^- will be denoted by a^+ and a^- respectively. This finishes the proof. □

Now without loss of generality we assume that $\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \operatorname{Re}\lambda_3$. The matrices

$$\Phi^+ := \left[\phi_1^+, \phi_2^+, \phi_3^+\right], \Phi^- := \left[\phi_1^-, \phi_2^-, \phi_3^-\right]. \tag{20}$$

are related by a matrix M:

$$\Phi^- = \Phi^+ M.$$

The matrix M has a unique lower triangular–diagonal–upper triangular factorization $M = \mathbf{L}\delta\mathbf{U}^{-1}$, where the diagonal entries of $\mathbf{L}, \mathbf{U}$ are equal to 1. We have $\Phi^-\mathbf{U} = \Phi^+\mathbf{L}\delta$. Moreover,

$$\text{the } j\text{th column of } \ \Phi^-\mathbf{U}e^{-\lambda_j x} \to \xi_j, \ \text{ as } x \to -\infty,$$

$$\text{the } j\text{th column of } \ \Phi^+\mathbf{L}e^{-\lambda_j x} \to \xi_j, \ \text{ as } x \to +\infty.$$

Hence $\Phi^-\mathbf{U}$ is the required Beals–Coifman fundamental solution matrix.

This solution is meromorphic in each $\Omega_{1,j}$, $j = 1, \ldots, 6$. The restriction of U^{ou} to $\Omega_{1,j}$ will be denoted by U_j^{ou}. On the common boundaries of $\Omega_{1,j}$ and $\Omega_{1,j+1}$, U_j^{ou} and U_{j+1}^{ou} are related by the transfer matrix V_j. That is, for $j = 1, \ldots, 6$,

$$U_{j+1}^{ou} = U_j^{ou} V_j.$$

Here we set $U_7^{ou} = U_1^{ou}$.

Lemma 5. *The transfer matrix V_k is equal to the identity matrix I.*

Proof. In terms of the functions Φ^+ and Φ^- defined in (20), we can write

$$U_j^{ou} = \Phi^-\mathbf{U} = \Phi^+\mathbf{L}\delta.$$

We use the same notation with a tilt to denote the corresponding functions of U_{j+1}^{ou}. That is,

$$U_{j+1}^{ou} = \tilde{\Phi}^-\tilde{\mathbf{U}} = \tilde{\Phi}^+\tilde{\mathbf{L}}\tilde{\delta}.$$

The matrix V_j is independent of x, y. On the common boundaries of $\Omega_{1,j}$ and $\Omega_{1,j+1}$, by (19), we have, for $k = 1, 2, 3$,

$$\lim_{y \to +\infty} \left[\left(\phi_k^+ - \tilde{\phi}_k^+\right) e^{-\lambda_k x} \right] = 0, \text{ uniformly for } x > a^+,$$

$$\lim_{y \to +\infty} \left[\left(\phi_k^- - \tilde{\phi}_k^-\right) e^{-\lambda_k x} \right] = 0, \text{ uniformly for } x < a^-.$$

From this we deduce

$$\lim_{y \to +\infty} U_{j+1}^{ou} \left(U_j^{ou}\right)^{-1} = I.$$

Hence the transfer matrix V_j equals identity. □

In Ω_2, we define U^{in}, matrix solution of (15), such that

$$U^{in}(0) = I.$$

A key property is that U^{in} is holomorphic in Ω_2. In general, assuming the jump matrix from the interior to the outer solutions on the boundary circle ∂B_r has the form

$$G(k)E(k).$$

Then we have the following relation:

$$U^{ou} = U^{in}G(k)E(k). \tag{21}$$

Taking $(x, y) = (0, 0)$, we get

$$U^{ou}(0, 0) = G(k)E(k).$$

It turns out that the Beals–Coifman fundamental solution $U^{ou}(k; x, y)$ has the form

$$U^{ou}(k; x, y) = \left[I + \sum_{k_j^*} \sum_{s=1}^{n_j} \left(\frac{A_{j,s}(x, y)}{\left(k - k_j^*\right)^s} \right) \right] E(k)e^{M(k)x}, \tag{22}$$

for certain complex numbers k_j^*. Here $A_{j,s}$ are 3×3 matrices. We then obtain

$$\left[I + \sum_{k_j^*} \sum_{s=1}^{n_j} \left(\frac{A_{j,s}(x, y)}{\left(k - k_j^*\right)^s} \right) \right] E(k)e^{Mx}E(k)^{-1}G(k)^{-1} = U^{in}. \tag{23}$$

As we already mentioned, U^{in} is a holomorphic function in the radius r disk. This will yield a system of equations for the entries of $A_{j,s}$. Next we would like to show that the system has a unique solution.

Lemma 6. *For fixed G, the system (21) has a unique solution.*

Proof. We have

$$U^{ou}(k; x, y) = U^{in}(k; x, y)G(k)E(k).$$

Suppose there is another pair $\left(\tilde{U}^{ou}, \tilde{U}^{in}\right)$ such that

$$\tilde{U}^{ou}(k; x, y) = \tilde{U}^{in}(k; x, y)G(k)E(k).$$

We claim that $\tilde{U}^{ou} = U^{ou}$.

Indeed, since $\tilde{U}^{ou}$ is invertible, the matrix $U^{ou}\left(\tilde{U}^{ou}\right)^{-1}$ is holomorphic outside B_r, while $U^{in}\left(\tilde{U}^{in}\right)^{-1}$ is holomorphic inside B_r. Moreover, they are equal to each other on ∂B_r. Hence they patch up to an entire holomorphic function which is also bounded. Hence in view of their asymptotics at infinity, we obtain

$$U^{ou} = \tilde{U}^{ou}, \text{ and } U^{in} = \tilde{U}^{in}.$$

This finishes the proof. □

With this result at hand, next we show that the solution q has to be rational. Let us define

$$\Phi := E(k)e^{Mx}E(k)^{-1}.$$

Lemma 7. *The matrix Φ is holomorphic in k with removable singularities at*

$$k_{1,\pm} = \pm\frac{\sqrt{6}}{3}i, k_{2,\pm} = \pm\frac{2\sqrt{6}}{3}i.$$

Proof. For instance, if $\Phi_{i,j}$ represents the entry of Φ on the ith row and jth column, then

$$\Phi_{12} = \frac{ki\sqrt{3k^2+8} - 3k^2 - 4}{\left(6k^2+4\right)\sqrt{3k^2+8}} e^{-\frac{x}{2}\left(ki+\sqrt{3k^2+8}\right)} - \frac{ki}{3k^2+2}e^{kix}$$
$$+ \frac{ki\sqrt{3k^2+8} + 3k^2 + 4}{\left(6k^2+4\right)\sqrt{3k^2+8}} e^{-\frac{x}{2}\left(ki-\sqrt{3k^2+8}\right)}.$$

Letting $t = \frac{x}{2}\sqrt{3k^2+8}\, i$, we see that

$$\Phi_{12} = \frac{ki}{3k^2+2}e^{-\frac{kxi}{2}}\cos t + \frac{3k^2+4}{6k^2+4}xe^{-\frac{kxi}{2}}\frac{\sin t}{t} - \frac{ki}{3k^2+2}e^{kix}.$$

Note that $\cos t$ and $\sin t$ are holomorphic in k, and the derivatives of them with respect to k contains polynomials of x as coefficients. Similarly, for Φ_{22} :

$$\Phi_{22} = \frac{\left(k^2+1\right)\sqrt{3k^2+8} - ki}{\left(3k^2+2\right)\sqrt{3k^2+8}} e^{-\frac{x}{2}\left(ki+\sqrt{3k^2+8}\right)} + \frac{k^2}{3k^2+2}e^{kix}$$
$$+ \frac{\left(k^2+1\right)\sqrt{3k^2+8} + ki}{\left(3k^2+2\right)\sqrt{3k^2+8}} e^{-\frac{x}{2}\left(ki-\sqrt{3k^2+8}\right)}$$
$$= \frac{2k^2+2}{3k^2+2}e^{-\frac{kxi}{2}}\cos t + \frac{ki}{3k^2+2}xe^{-\frac{kxi}{2}}\frac{\sin t}{t} + \frac{k^2}{3k^2+2}e^{kix}.$$

The other entries can be treated in a similar way.

Note that potentially $\Phi_{i,j}$ also has singularities when $k = \pm\frac{\sqrt{6}}{3}i$. However, one can also show that they are removable. □

Note that the second equation in the Lax pair reads as

$$\partial_y f = i\left(3\frac{d^2}{dx^2} - 4(q+1)\right)f. \tag{24}$$

Recall that we have defined $f_1 = f$, and $\mathbf{f} := (f_1, f_2, f_3)^T$, which solves (15), the ODE system corresponding to the first equation of the Lax pair. In view of the asymptotic behavior imposed on the Beals–Coifman solution, we have $\mathbf{f}e^{-\lambda_j x} \to \xi_j$ as $x \to +\infty$. We then see that the function

$$e^{i\left(3\lambda_j^2-4\right)y} f_1(x)$$

will solve Equation (24). Let us set $\sigma_j = i(3\lambda_j^2 - 4)$, and $\Lambda_j = \lambda_j x + \sigma_j y$.

Lemma 8. *There holds*

$$\Lambda_1\left(k_{1,+}\right) = -\frac{\sqrt{6}}{3}x - 2iy, \Lambda_2\left(k_{1,+}\right) = \frac{2\sqrt{6}}{3}x + 4iy, \Lambda_3\left(k_{1,+}\right) = -\frac{\sqrt{6}}{3}x - 2iy,$$
$$\Lambda_1\left(k_{1,-}\right) = \frac{\sqrt{6}}{3}x - 2iy, \Lambda_2\left(k_{1,-}\right) = \frac{\sqrt{6}}{3}x - 2iy, \Lambda_3\left(k_{1,-}\right) = -\frac{2\sqrt{6}}{3}x + 4iy.$$

Moreover,

$$\Lambda_1\left(k_{2,+}\right) = -\frac{2\sqrt{6}}{3}x + 4iy, \Lambda_2\left(k_{1,+}\right) = \frac{\sqrt{6}}{3}x - 2iy, \Lambda_3\left(k_{1,+}\right) = \frac{\sqrt{6}}{3}x - 2iy,$$
$$\Lambda_1\left(k_{2,-}\right) = \frac{2\sqrt{6}}{3}x + 4iy, \Lambda_2\left(k_{1,-}\right) = -\frac{\sqrt{6}}{3}x - 2iy, \Lambda_3\left(k_{1,-}\right) = -\frac{\sqrt{6}}{3}x - 2iy.$$

Proof. This follows from direct computation. □

Lemma 9. *Suppose Q is a rational function of the x, y variables. Then for each fixed y,*

$$\lim_{x\to\infty} \frac{\partial_x Q}{Q} = 0.$$

Proof. Q can be written as $\frac{V}{W}$, where W, V are polynomials. For fixed y, without loss of generality, we assume $V, W > 0$ for x large. Then

$$\frac{\partial_x Q}{Q} = \partial_x \ln Q = \partial_x \ln V - \partial_x \ln W = \frac{\partial_x V}{V} - \frac{\partial_x W}{W}.$$

Since V, W are polynomials, we conclude

$$\lim_{x\to\infty} \frac{\partial_x Q}{Q} = 0.$$

The proof is then completed. □

Now we are ready to prove the main result of this section.

Theorem 10. *Suppose q is a solution of the Boussinesq equation satisfying the assumption of Theorem 1, then q is rational.*

Proof. Once the solution q is given, the matrix G is determined. On the other hand, the solution q, which appears as a potential in the Lax pair equation, is determined by the Beals–Coifman function U^{ou} and U^{in}. To do this, we insert (22) into the equation

$$(U^{ou})' = AU^{ou}.$$

Comparing the $(3,1)$ entry on both sides, we can see that q, p are determined by $A_{j,s}$. Hence we need to determine the matrices $A_{j,s}$ in (23).

We first show that the possible poles k_j^* in (23) have to be $k_{1,\pm}$ and $k_{2,\pm}$. To see this, we use the fact that the Beals–Coifman fundamental solution is unique (this follows from condition (16)). Then for y large, the constants $a^\pm$ appeared in the construction of Beals–Coifman fundamental solution can both be chosen to be zero. Note that this construction works provided that λ_j are distinct. Hence again using estimate (19), we then see that as y tends to ∞, the asymptotic behavior of the solution ϕ_j^+ is 'close' to the asymptotic behavior of ϕ_j^- for x large. Hence if a complex number k^* is not equal to $k_{1,\pm}$ or $k_{2,\pm}$, then it cannot appear in the set of poles.

Since the right-hand side of (23) is holomorphic in k, we see that the matrices $A_{j,s}$ satisfy a system of linear equation whose entries are polynomial in x. Now by Lemma 6, the solution has to be unique. Hence the linear system does not have kernel and q contains rational functions and exponential functions in its expression.

We claim that q is rational. Indeed, supposes q also have exponential functions, then we can write

$$q(x,y) = \sum_{k,j=0}^{+\infty} \left[Q_{j,k}(x,y) e^{-\sqrt{\frac{2}{3}} jx - 2kyi} \right].$$

Inserting it into the equation

$$KP(q) := 3\partial_x^2\left(\partial_x^2 q - 4q^2 - 8q\right) - q_{yy} = 0,$$

we see that $KP\left(Q_{0,0}\right) = 0$, and

$$3\partial_x^2\left[\partial_x^2\left(Q_{1,0}e^{-\sqrt{\frac{2}{3}}x}\right) - 8Q_{0,0}Q_{1,0}e^{-\sqrt{\frac{2}{3}}x} - 8\left(Q_{1,0}e^{-\sqrt{\frac{2}{3}}x}\right)\right] - \partial_y^2\left(Q_{1,0}e^{-\sqrt{\frac{2}{3}}x}\right) = 0.$$

Let us set $a = -\sqrt{\frac{2}{3}}$. Then the left-hand side can be written as

$$\left(3a^4 - 8a^2 + P(x,y)\right)Q_{1,0} = 0,$$

where P is determined by $Q_{0,0}$ and derivatives of $Q_{1,0}$. In particular, applying Lemma 9, we have

$$P(x,y) \to 0, \text{ as } x \to +\infty.$$

It follows that

$$Q_{1,0} = 0.$$

Now for general $Q_{j,k}$, it satisfies an equation of the form

$$\left[3\left((aj)^4 - 8(aj)^2\right) - (2ki)^2 + P_{j,k}\right]Q_{j,k} = 0.$$

Observe that

$$3\left((aj)^4 - 8(aj)^2\right) - (2ki)^2 = \frac{4}{3}\left(j^4 - 12j^2 + 3k^2\right).$$

This is nonzero for all integers j, k. Hence same arguments as above imply that $Q_{j,k} = 0$ for $k + j \geqslant 1$. We then conclude that q is a rational solution.

It is worth pointing out that by the Krichever theorem (see [35, 36]), if the solution is rational in x, then it will also be rational in the y variable. □

4 | THE BOUSSINESQ HIERARCHY AND THE STRUCTURE OF RATIONAL SOLUTIONS

In this section, we will extend the elegant techniques developed by Airault–McKean–Moser [4] for the complex valued rational solutions of KdV equation to the Boussinesq equation, and classify its rational solutions. This problem is originally raised in [4, pp. 123–124], and is much more complicated than the KdV case. In the KdV case, it has been shown in [4] that the space of rational solutions with degree $d(d+1)/2$ is a manifold of complex dimension d. The polynomials corresponding to these rational solutions are the famous Adler–Moser polynomials, studied in [2]. However, in our case, a proof of similar result of this type turns out to be more delicate, due to the facts that various differential operators involved in the Boussinesq hierarchy is of different orders, and the locus M defined by (30) is not scaling invariant.

4.1 | The Boussinesq hierarchy

In [43], Mckean found the Boussinesq hierarchy associated to the Boussinesq equation. The equation he studied is the following:

$$\partial_y^2 q = 3\partial_x^2\left(\partial_x^2 q + 4q^2\right). \tag{25}$$

Related works on the Boussinesq hierarchy can be found in [22].

We use D to denote the differentiation with respect to the x variable (this D is not the bilinear derivative operator). Define the operator

$$\mathcal{D} = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}.$$

Let

$$L_0 := D^5 + 5\left(qD^3 + D^3 q\right) - 3\left(q''D + Dq''\right) + 16qDq,$$

and define

$$\mathcal{K}_0 = \begin{bmatrix} D^3 + qD + Dq & 3pD + 2p' \\ 3pD + p' & \frac{1}{3}L_0 \end{bmatrix}.$$

Mckean used $\mathcal{K}_0$ to define recursively a sequence of vector fields, which generate the Boussinesq hierarchy.

In our case, we are actually considering those solutions of the Boussinesq equation (25) with nonzero boundary condition, say $\tilde{q} \to -\frac{1}{8}$ as $x^2 + y^2 \to +\infty$. Indeed, introducing new variable $\tilde{q}$ by $q = \tilde{q} - \frac{1}{8}$ in (25), we obtain

$$\partial_y^2 \tilde{q} = 3\partial_x^2\left(\partial_x^2 \tilde{q} + 4\tilde{q}^2 - \tilde{q}\right). \tag{26}$$

This is the equation we will study in this section. Note that if we set $\tilde{q}\,(x, y) = \frac{3}{4}u(x, \sqrt{3}y)$, then u satisfies the version of the Boussinesq equation appeared in Section 1, that is,

$$\partial_x^2\left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0.$$

We are thus lead to consider the shifted operator L defined by

$$\begin{aligned} L := D^5 + 5\left[\left(q - \frac{1}{8}\right)D^3 + D^3\left(q - \frac{1}{8}\right)\right] \\ - 3\left(q''D + Dq''\right) + 16\left(q - \frac{1}{8}\right)D\left(q - \frac{1}{8}\right). \end{aligned}$$

Note that

$$L = L_0 - \frac{5}{4}D^3 - 2(qD + Dq) + \frac{1}{4}D.$$

We then define

$$\mathcal{K}_j = \begin{bmatrix} D^3 + \left(q - \frac{1}{8}\right)D + D\left(q - \frac{1}{8}\right) & 3\left(p + (-1)^j a\right)D + 2p' \\ 3\left(p + (-1)^j a\right)D + p' & \frac{1}{3}L \end{bmatrix}.$$

Here the constant a is chosen such that

$$(3a)^2 + \frac{1}{48} = 0.$$

We will explain later on why a should be chosen in this way. Let $H_0 = \int \frac{3}{2}p$. Then a series of vector fields can be defined recursively by

$$X_{j+1} = \mathcal{K}_j \nabla H_j, \text{ and } \mathcal{D}\nabla H_j = X_j.$$

More precisely, once we obtained X_j, we can find ∇H_j by using the relation $\mathcal{D}\nabla H_j = X_j$. Then we can find X_{j+1} by $X_{j+1} = \mathcal{K}\nabla H_j$.

In particular,

$$X_0 = \mathcal{D}\nabla H_0 = 0, X_1 = \mathcal{K}\nabla H_0 = \left(3p', q''' + 8qq' - q'\right)^T.$$

Hence X_1 is the Boussinesq flow. Here $'$ represents the derivative with respect to the x variable. As a matter of fact, there is another family of the Boussinesq hierarchy, starting from $\nabla H = (1, 0)$. But we will not use them.

Suppose u is a rational solution of the KP-I equation. Then from [35, 36], we know that u can be written in the form

$$u = -\frac{3}{2}\sum_{j=1}^{n} \frac{1}{\left(x - \xi_j(y, t)\right)^2}.$$

In this case, $u = \frac{3}{2}\partial_x^2 \ln \tau$, where τ is a polynomial in the x variable.

For rational solutions q of the Boussinesq equation, we have

$$q = -\frac{3}{2}\sum_{j=1}^{n} \frac{1}{\left(x - \eta_j(y)\right)^2}. \tag{27}$$

For real-valued solutions, as we will see, the main-order term of the τ function is $(x^2 + 3y^2)^n$. Inserting (27) into Equation (26), we find that for each fixed index $j = 1, \ldots, n$, there holds

$$\begin{cases} \partial_y^2 \eta_j - \sum_{k \neq j} \frac{72}{\left(\eta_j - \eta_k\right)^3} = 0, \\ \eta_j'^2 + 36 \sum_{k \neq j} \left(\eta_j - \eta_k\right)^{-2} + 3 = 0. \end{cases} \tag{28}$$

Recall that (28) is the famous Caloger–Moser system. More precisely, let $\partial_y \eta_j = \beta_j$. Then the CM flow can be written as

$$\begin{cases} \partial_y \eta_j = \beta_j, \\ \partial_y \beta_j = \sum_{k \neq j} \frac{72}{\left(\eta_j - \eta_k\right)^3}. \end{cases} \tag{29}$$

Now one can show that the function (27) solves the Boussinesq equation if and only (η, β) satisfies the CM system (29) restricted to the set

$$M := \left\{(\eta, \beta) \in \mathbb{C}^{2n} \, : \, \nabla(F_1 + F_3) = 0\right\},$$

where

$$F_1 = 3\sum_{j=1}^{n} \beta_j, \quad F_3 = \frac{1}{3}\sum_{j=1}^{n} \beta_j^3 + 36 \sum_{j=1}^{n}\sum_{k \neq j} \frac{\beta_j}{\left(\eta_j - \eta_k\right)^2}.$$

The proof of this fact follows from similar lines as that of [4], although in that paper the case of hyperbolic Boussinesq equation is treated, instead of the elliptic case we are studying now. Therefore the details of the computation will be omitted.

Explicitly, a point $(\eta_1, \ldots, \eta_n, \beta_1, \ldots, \beta_n) \in M \subset \mathbb{C}^{2n}$ if and only if for each fixed $j = 1, \ldots, n$, the following identities hold:

$$\begin{cases} \sum_{k \neq j} \dfrac{\beta_j + \beta_k}{\left(\eta_j - \eta_k\right)^3} = 0, \\ \beta_j^2 + \sum_{k \neq j} \frac{36}{\left(\eta_j - \eta_k\right)^2} + 3 = 0. \end{cases} \tag{30}$$

In the case of lump solution, we have $n = 2$ and

$$\eta_1 = i\sqrt{3y^2 + 3}, \;\; \eta_2 = -i\sqrt{3y^2 + 3},$$

$$\beta_1 = \frac{3yi}{\sqrt{3y^2 + 3}}, \;\; \beta_2 = -\frac{3yi}{\sqrt{3y^2 + 3}}.$$

As a consequence of (30), a vector $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in TM$, the tangent space of M at $(\eta_1, \ldots, \eta_n, \beta_1, \ldots, \beta_n)$, if and only if for each fixed $j = 1, \ldots, n$,

$$\begin{cases} \sum_{k \neq j} \left(\dfrac{b_j + b_k}{\left(\eta_j - \eta_k\right)^3} - 3 \dfrac{\left(\beta_j + \beta_k\right)\left(a_j - a_k\right)}{\left(\eta_j - \eta_k\right)^4} \right) = 0, \\ \beta_j b_j - \sum_{k \neq j} \frac{36\left(a_j - a_k\right)}{\left(\eta_j - \eta_k\right)^3} = 0. \end{cases} \tag{31}$$

Recall that the Boussinesq equation reads as

$$\begin{cases} q_y = 3p', \\ p_y = q''' + 8qq' - q'. \end{cases}$$

The rational solution q of the Boussinesq equation can be written as

$$q = -\frac{3}{2} \sum_{j=1}^{n} \frac{1}{\left(x - \eta_j\right)^2}.$$

Therefore, for this q, we have

$$p = \frac{1}{2} \sum_{j=1}^{n} \frac{\beta_j}{\left(x - \eta_j\right)^2}.$$

where $\beta_j = \partial_y \eta_j$. Now for initial condition (q_0, p_0) of this form, for each k, the vector field X_k corresponds the kth Boussinesq flow can be defined by

$$(q_y, p_y)^T = X_k((q,p)^T).$$

Denote the flow induced by this equation as $e(yX_k)$.

Proposition 11. *The* kth *Boussinesq flow* $e(yX_k)$ *induces a flow on* M. *More precisely,*

$$X_k((q,p)^T) = \left(6\sum_{j=1}^{n}\frac{a_j}{(x-\eta_j)^3}, -\sum_{j=1}^{n}\left(\frac{2\beta_j a_j}{(x-\eta_j)^3} + \frac{b_j}{(x-\eta_j)^2}\right)\right)^T, \tag{32}$$

where $(a_1, \dots, a_n, b_1, \dots, b_n) \in TM$.

Proof. The proof is quite tedious and long, and it may be omitted at the first reading.

We would like to compute the explicit form of each vector field X_k acting on $(q,p)^T$, in an inductive way. For this purpose, we will use the recursive formula of X_k defined through the operator $\mathcal{K}_k$.

To simplify notations, let us set

$$m_j(x) := \frac{1}{(x-\eta_j)^2} \quad \text{and} \quad n_j(x) := \frac{1}{x-\eta_j}.$$

Adopting the notation $\Sigma_j = \Sigma_{j=1}^n$, we have

$$q = -\frac{3}{2}\Sigma_j m_j \quad \text{and} \quad p = \frac{1}{2}\Sigma_j(\beta_j m_j).$$

We split the proof into two steps.

Step 1. Suppose the vector X_k already has the form (32), which implies

$$\begin{aligned}\nabla H_k &= \begin{bmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{bmatrix}\left(6\Sigma_j\frac{a_j}{(x-\eta_j)^3}, -\Sigma_j\left(\frac{2\beta_j a_j}{(x-\eta_j)^3} + \frac{b_j}{(x-\eta_j)^2}\right)\right)^T \\ &= \left(\left(\Sigma_j(\beta_j a_j m_j + b_j n_j)\right), -3\Sigma_j(a_j m_j)\right)^T,\end{aligned}$$

we would like to show that X_{k+1} also has the form (32).

Recall that X_{k+1} is defined to be $\mathcal{K}\nabla H_k$. Let us denote the first component of $\mathcal{K}\nabla H_k$ by $(\mathcal{K}\nabla H_k)^{(1)}$. Assuming $X_k = \mathcal{K}\nabla H_{k-1}$ already has the form (32), we find that $(\mathcal{K}\nabla H_k)^{(1)}$ equals

$$\begin{aligned}&\left(D^3 + \left(q - \frac{1}{8}\right)D + D\left(q - \frac{1}{8}\right)\right)\left(\Sigma_j(\beta_j a_j m_j + b_j n_j)\right) \\ &\qquad + \left(3(p+a)D + 2p'\right)\left(-3\Sigma_j(a_j m_j)\right).\end{aligned}$$

The points η_j, $j = 1, \ldots$, are possible poles. To analyze the pole structure of function, we would like to expand it around each η_j.

Let us fix an index j. The coefficient before $\frac{1}{(x-\eta_j)^5}$ is

$$\left(-24 + 2\left(-\frac{3}{2}\right)(-2) + \left(-\frac{3}{2}\right)(-2) - 9\left(\frac{1}{2}\right)(-2) - 6\left(\frac{1}{2}\right)(-2)\right)\beta_j a_j = 0.$$

Therefore, $(\mathcal{K}\nabla H_k)^{(1)}$ does not have pole of order 5.

Next we consider the term $\frac{1}{(x-\eta_j)^4}$. We see that it only comes from

$$\left(D^3 + qD + Dq\right)\left(b_j n_j\right).$$

The coefficient vanishes, since it equals

$$b_j\left((-1)(-2)(-3) + \left(-\frac{3}{2}\right)(-1) + \left(-\frac{3}{2}\right)(-3)\right) = 0.$$

For pole of order 3, that is, the term $\frac{1}{(x-\eta_j)^3}$, it comes from

$$\begin{aligned}&\left(2q - \frac{1}{4}\right)\Sigma_j\left(\beta_j a_j m_j'\right) + q'\Sigma_j\left(\beta_j a_j m_j\right)\\&+ q'\Sigma_j\left(b_j n_j\right) - 9(p+a)\Sigma_j\left(a_j m_j'\right) - 6p'\Sigma_j\left(a_j m_j\right).\end{aligned}$$

We will use the notation Σ_k' to denote the summation over the index k which is not equal to j. The coefficient I_3 of $\frac{1}{(x-\eta_j)^3}$ equals

$$\begin{aligned}&2\left(-\frac{3}{2}\right)\Sigma_k'\left((-2)\beta_j a_j m_k\left(\eta_j\right)\right) - \frac{1}{4}(-2)\beta_j a_j\\&+\left(-\frac{3}{2}\right)(-2)\Sigma_k'\left(\beta_k a_k m_k\left(\eta_j\right) + b_k n_k\left(\eta_j\right)\right)\\&- 9\left(\frac{1}{2}\right)\Sigma_k'\left(-2\beta_k a_j m_k\left(\eta_j\right)\right) - 9\alpha(-2)a_j - 6\left(\frac{1}{2}\right)(-2)\Sigma_k'\left(\beta_j a_k m_k\left(\eta_j\right)\right).\end{aligned}$$

That is,

$$\begin{aligned}I_3 = {}& 6\Sigma_k'\left(\beta_j a_j m_k\left(\eta_j\right)\right) + \frac{1}{2}\beta_j a_j\\&+ 3\Sigma_k'\left(\beta_k a_k m_k\left(\eta_j\right)\right) + 3\Sigma_k'\left(b_k n_k\left(\eta_j\right)\right)\\&+ 9\Sigma_k'\left(\beta_k a_j m_k\left(\eta_j\right)\right) + 18\alpha a_j + 6\Sigma_k'\left(\beta_j a_k m_k\left(\eta_j\right)\right).\end{aligned}$$

Next, to analyze the $\frac{1}{(x-\eta_j)^2}$ term, we need to compute

$$\begin{aligned}&2q\Sigma_j\left(\beta_j a_j m_j' + b_j n_j'\right) + q'\Sigma_j\left(\beta_j a_j m_j + b_j n_j\right) - \frac{1}{4}\Sigma_j\left(b_j n_j'\right)\\&+ 3p\Sigma_j\left(-3a_j m_j'\right) + 2p'\Sigma_j\left(-3a_j m_j\right).\end{aligned}$$

Using the formula of q and p, we can compute the corresponding coefficient I_2:

$$\begin{aligned}
&2\left(-\frac{3}{2}\right)\left[(-2)\Sigma_k'\left(\beta_j a_j m_k'(\eta_j)\right)+\Sigma_k'\left(\beta_k a_k m_k'(\eta_j)\right)\right]\\
&+\left(-\frac{3}{2}\right)\left[\Sigma_k'\left(\beta_j a_j m_k'(\eta_j)\right)+(-2)\Sigma_k\left(\beta_k a_k m_k'(\eta_j)\right)\right]\\
&+2\left(-\frac{3}{2}\right)(-1)b_j\Sigma_k'\left(m_k(\eta_j)\right)+2\left(-\frac{3}{2}\right)\Sigma_k'\left(b_k n_k'(\eta_j)\right)\\
&+\left(-\frac{3}{2}\right)(-2)\Sigma_k'\left(b_k n_k'(\eta_j)\right)-\frac{1}{4}(-1)b_j\\
&-9\left(\frac{1}{2}\right)\left[(-2)\Sigma_k'\left(\beta_k a_j m_k'(\eta_j)\right)+\Sigma_k'\left(\beta_j a_k m_k'(\eta_j)\right)\right]\\
&-6\left(\frac{1}{2}\right)\left[\Sigma_k'\left(\beta_k a_j m_k'(\eta_j)\right)+(-2)\Sigma_k'\left(\beta_j a_k m_k'(\eta_j)\right)\right].
\end{aligned}$$

Simplifying this expression, we find that I_2 equals

$$\frac{9}{2}\Sigma_k'\left(\beta_j a_j m_k'(\eta_j)\right)+6\Sigma_k'\left(\beta_k a_j m_k'(\eta_j)\right)+\frac{3}{2}\Sigma_k'\left(\beta_j a_k m_k'(\eta_j)\right)+3b_j\Sigma_k'\left(m_k(\eta_j)\right)+\frac{1}{4}b_j.$$

Note that on the locus M we have the following two identities:

$$12\Sigma_k'\left(m_k(\eta_j)\right)+1=-\frac{1}{3}\beta_j^2,\quad \text{and}\quad \Sigma_k'\left[\beta_k m_k'(\eta_j)\right]=-\Sigma_k'\left[\beta_j m_k'(\eta_j)\right].$$

It follows that

$$I_2=\frac{3}{2}\Sigma_k'\left[\beta_j(a_k-a_j)m_k'(\eta_j)\right]-\frac{1}{12}b_j\beta_j^2.$$

Since $(a_1,\dots,a_N,b_1,\dots,b_N)$ belongs to the tangent space of M, we obtain $I_2=0$.

We proceed to compute the coefficient of $\frac{1}{x-\eta_j}$. It comes from

$$2q\Sigma_j\left(\beta_j a_j m_j'+b_j n_j'\right)+q'\Sigma_j\left(\beta_j a_j m_j+b_j n_j\right)-9p\Sigma_j\left(a_j m_j'\right)-6p'\Sigma_j\left(a_j m_j\right).$$

The corresponding coefficient I_1 is

$$\begin{aligned}
&2\left(-\frac{3}{2}\right)\Sigma_k'\left(\beta_k a_k m_k''(\eta_j)+b_k n_k''(\eta_j)\right)+2\left(-\frac{3}{2}\right)\Sigma_k'\left((-2)\frac{1}{2}\beta_j a_j m_k''(\eta_j)\right)\\
&+2\left(-\frac{3}{2}\right)\Sigma_k'\left((-1)b_j m_k'(\eta_j)\right)\\
&+\left(-\frac{3}{2}\right)(-2)\Sigma_k'\left(\frac{1}{2}\beta_k a_k m_k''(\eta_j)+\frac{1}{2}b_k n_k''(\eta_j)\right)\\
&+\left(-\frac{3}{2}\right)\Sigma_k'\left(\beta_j a_j m_k''(\eta_j)\right)+\left(-\frac{3}{2}\right)\Sigma_k'\left(b_j m_k'\right)\\
&-9\left(\frac{1}{2}\right)\left(\Sigma_k'\left(\beta_j a_k m_k''(\eta_j)\right)+\Sigma_k'\left((-2)\frac{1}{2}\beta_k a_j m_k''\right)\right)\\
&-6\left(\frac{1}{2}\right)\left((-2)\Sigma_k'\left(\frac{1}{2}\beta_j a_k m_k''(\eta_j)\right)+\Sigma_k'\beta_k a_j m_k''\right).
\end{aligned}$$

It follows that

$$
\begin{aligned}
I_1 = {} & \frac{3}{2}\Sigma_k'\left(\beta_j a_j m_k''(\eta_j)\right) + \frac{3}{2}\Sigma_k'\left(\beta_k a_j m_k''(\eta_j)\right) \\
& - \frac{3}{2}\Sigma_k'\left(\beta_j a_k m_k''(\eta_j)\right) - \frac{3}{2}\Sigma_k'\left(\beta_k a_k m_k''(\eta_j)\right) \\
& + \frac{3}{2}\Sigma_k'\left(b_j m_k'(\eta_j)\right) + \frac{3}{2}\Sigma_k'\left(b_k m_k'(\eta_j)\right).
\end{aligned} \tag{33}
$$

Using the first identity of (31), we then deduce that $I_1 = 0$.

Now we consider the second component $(\mathcal{K}\nabla H_k)^{(2)}$ of the vector field $\mathcal{K}\nabla H_k$. We have

$$
(\mathcal{K}\nabla H_k)^{(2)} = -L\Sigma_j\left(a_j m_j\right) + \left(3(p+a)D + p'\right)\Sigma_j\left(\beta_j a_j m_j + b_j n_j\right).
$$

Similar (but more tedious, and the most complicated term is $16qDq$) computation as above shows that the coefficient of the term $\frac{1}{(x-\eta_j)^l}$ vanishes for $l = 1, 4, 5, 6, 7$.

Let us compute the coefficient J_3 of $\frac{1}{(x-\eta_j)^3}$. Recall that

$$
L = L_0 - \frac{5}{4}D^3 - 2(qD + Dq) + \frac{1}{4}D,
$$

where

$$
L_0 := D^5 + 5\left(qD^3 + D^3 q\right) - 3\left(q''D + Dq''\right) + 16qDq.
$$

Observe that $D^3\left(q\Sigma_j\left(a_j m_j\right)\right)$ does not contain $\frac{1}{(x-\eta_j)^3}$ term. From the operator L_0, the contribution to the coefficient is

$$
\begin{aligned}
& 5\left(-\frac{3}{2}\right)(-24)\Sigma_k'\left(\frac{1}{2}a_j m_k''(\eta_j)\right) \\
& - 3\left(-\frac{3}{2}\right)\left(2(6)\Sigma_k' a_k m_k''(\eta_j) + 2(-2)\Sigma_k'\left(a_j m_k''(\eta_j)\right) + (-24)\Sigma_k'\left(\frac{1}{2}a_k m_k''(\eta_j)\right)\right) \\
& + 16\left(\frac{9}{4}\right)\left[\Sigma_k'\left((-2)a_j m_k''(\eta_j)\right) - 2\Sigma_k' m_k(\eta_j)\Sigma_k'\left(\left(a_k + a_j\right)m_k(\eta_j)\right)\right].
\end{aligned}
$$

From the operator $-\frac{5}{4}D^3 - 2\left(qD + Dq\right) + \frac{1}{4}D$, we get

$$
-2\left(-\frac{3}{2}\right)\left[2(-2)\Sigma_k'\left(a_j m_k(\eta_j)\right) - 2\Sigma_k'\left(a_k m_k(\eta_j)\right)\right] + \frac{1}{4}\left(-2a_j\right)
$$

Finally, from

$$
\left(3(p+a)D + p'\right)\Sigma_j\left(\beta_j a_j m_j + b_j n_j\right),
$$

we obtain

$$
\begin{aligned}
& 3\left(\frac{1}{2}\right)\Sigma_k'\left((-2)\beta_k\beta_j a_j m_k(\eta_j)\right) + 3a(-2)\beta_j a_j \\
& + \left(\frac{1}{2}\right)(-2)\Sigma_k'\left(\beta_j\beta_k a_k m_k(\eta_j)\right) + \frac{1}{2}(-2)\Sigma_k'\beta_j b_k n_k(\eta_j).
\end{aligned}
$$

Combining these, we find that

$$\begin{aligned} J_3 = {} & 72\Sigma_k'\big(m_k(\eta_j)\big)\Sigma_k'\big((a_k + a_j)m_k(\eta_j)\big) \\ & + 12\Sigma_k'\big(a_j m_k(\eta_j)\big) + 6\Sigma_k'\big(a_k m_k(\eta_j)\big) + \frac{1}{2}a_j \\ & - 3\Sigma_k'\big(\beta_k\beta_j a_j m_k(\eta_j)\big) - \Sigma_k'\big(\beta_j\beta_k a_k m_k(\eta_j)\big) \\ & - \Sigma_k'\big(\beta_j b_k n_k(\eta_j)\big) - 6\alpha\beta_j a_j. \end{aligned}$$

Now using the identity

$$\beta_j^2 + 36\Sigma_k' m_k(\eta_j) + 3 = 0,$$

we see that $J_3 = -\frac{1}{3}\beta_j I_3$.

The coefficient of $\frac{1}{(x-\eta_j)^2}$ is in general nonzero and can be computed in a similar way. The result is

$$\begin{aligned} J_2 = {} & 9a_j\Sigma_k' m_k'(\eta_j) + \frac{3}{2}\Sigma_k' a_j m_k'''(\eta_j) - \frac{15}{2}\Sigma_k' a_k m_k'''(\eta_j) + 108\Sigma_k' m_k(\eta_j)\Sigma_k' a_j m_k'(\eta_j) \\ & + 36\Sigma_k' m_k'(\eta_j)\Sigma_k' a_k m_k(\eta_j) + \frac{1}{2}\beta_j\Sigma_k'(a_k\beta_k m_k'(\eta_j) + b_k n_k'(\eta_j)) - \frac{3}{2}b_j\Sigma_k'\beta_k m_k(\eta_j) \\ & - \frac{5}{2}\beta_j a_j\Sigma_k'\beta_k m_k'(\eta_j) - 36ab_j. \end{aligned}$$

In the sequel, for $j = 1, \dots, n$, we use $\tilde{I}_j$ to denote the coefficient of degree -3 term for the pole η_j. Up to now, we have proved that $X_{k+1} = \mathcal{K}\nabla H_k$ has the form

$$\left(6\Sigma_j\frac{\tilde{I}_j}{(x-\eta_j)^3}, -\Sigma_j\left(\frac{2\beta_j\tilde{I}_j}{(x-\eta_j)^3} + \frac{B_j}{(x-\eta_j)^2}\right)\right)^T = \mathcal{D}\nabla H_{k+1}.$$

Step 2. We show that the vector

$$(\tilde{I}_1, \dots, \tilde{I}_n, B_1, \dots, B_n)$$

lies in the tangent space of M at the point $(\eta_1, \dots, \eta_n, \beta_1, \dots, \beta_n)$.

To see this, it will be suffice to show that $\mathcal{K}\nabla H_{k+1}$ is residue free at each pole due to our previous computation, this means exactly it is in the tangent space of the locus M.

Let us write the operator $\mathcal{K}$ as

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}.$$

We also write $\nabla H_{k+1} = (\phi_1, \phi_2)^T$. That is,

$$(\phi_1, \phi_2) = \big(\Sigma_j\big(\tilde{I}_j\beta_j m_j + B_j n_j\big), -3\Sigma_j\big(\tilde{I}_j m_j\big)\big).$$

Introducing

$$\sigma = \Sigma_j \left(a_j \beta_j m_j + b_j n_j \right), \quad \tau = -3\Sigma_j \left(a_j m_j \right), \tag{34}$$

we get

$$\phi_1' = K_{21}\sigma + K_{22}\tau, \quad \phi_2' = K_{11}\sigma + K_{12}\tau. \tag{35}$$

Let l be a closed path around the pole η_j in the complex x plane. To see that the residue is zero(that is, does not have $\frac{1}{x-\eta_j}$ term in the Laurent expansion around η_j), we compute the integral

$$Q := \int_l \left(\mathcal{K} \nabla H_{k+1} \right)^T dx = \int_l [K_{11}\phi_1 + K_{12}\phi_2, K_{21}\phi_1 + K_{22}\phi_2] dx.$$

It is important to observe that each operator K_{11}, K_{22} is skew-symmetric, and moreover the adjoint of K_{12} is $-K_{21}$, that is,

$$\int (gK_{12}h) = -\int (hK_{21}g).$$

This is to say that the matrix operator $\mathcal{K}$ is skew-symmetric. Integrating by parts tells us that Q equals

$$-\int_l [\phi_1 K_{11}(1) + \phi_2 K_{21}(1), \phi_1 K_{12}(1) + \phi_2 K_{22}(1)] dx.$$

Let us define

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} := \mathcal{K} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \mathcal{K} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then for some functions w, z, we have

$$\mu = \mathcal{D}(w_1, w_2)^T \text{ and } v = \mathcal{D}(z_1, z_2)^T.$$

Explicitly,

$$\mu = \left(q', p'\right)^T, \ v = \left(2p', \frac{1}{3}\left(2q''' + 16qq' - 2q'\right)\right)^T.$$

With these notations,

$$\begin{aligned} Q &= -\int_l [\phi_1 \mu_1 + \phi_2 \mu_2, \phi_1 v_1 + \phi_2 v_2] dx \\ &= -\int_l \left[\phi_1 w_2' + \phi_2 w_1', \phi_1 z_2' + \phi_2 z_1'\right] dx = \int_l \left[\phi_1' w_2 + \phi_2' w_1, \phi_1' z_2 + \phi_2' z_1\right] dx. \end{aligned}$$

Using (35), we find that Q is equal to

$$\int_l [(K_{21}\sigma + K_{22}\tau)w_2 + (K_{11}\sigma + K_{12}\tau)w_1, (K_{21}\sigma + K_{22}\tau)z_2 + (K_{11}\sigma + K_{12}\tau)z_1]dx$$
$$= -\int_l [(K_{11}w_1 + K_{12}w_2)\sigma + (K_{21}w_1 + K_{22}w_2)\tau, (K_{11}z_1 + K_{12}z_2)\sigma + (K_{21}z_1 + K_{22}z_2)\tau]dx.$$

On the other hand, using integration by parts,

$$\int_l [(K_{11}w_1 + K_{12}w_2)w_1 + (K_{21}w_1 + K_{22}w_2)w_2]dx$$
$$= -\int_l [(K_{11}w_1 + K_{12}w_2)w_1 + (K_{21}w_1 + K_{22}w_2)w_2]dx.$$

This implies

$$\int_l [(K_{11}w_1 + K_{12}w_2)w_1 + (K_{21}w_1 + K_{22}w_2)w_2]dx = 0. \tag{36}$$

Therefore, if we write

$$\mathcal{K}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(-3\Sigma_j\left(s_j m_j'\right), \Sigma_j\left(s_j\beta_j m_j' + t_j n_j'\right)\right)^T,$$

then in view of (33), $(s_1, \ldots, s_n, t_1, \ldots, t_n)$ satisfies the first equation of (31). That is,

$$\Sigma_k'\left(\frac{t_j + t_k}{\left(\eta_j - \eta_k\right)^3} - \frac{3\left(\beta_j + \beta_k\right)\left(s_j - s_k\right)}{\left(\eta_j - \eta_k\right)^4}\right) = 0, j = 1, \ldots, n. \tag{37}$$

We would like to show that $(s_1, \ldots, s_n, t_1, \ldots, t_n)$ also satisfies the second equation of (31). To do this, we first compute $\mathcal{K}(w_1, w_2)^T$ and derive that

$$s_j = \Sigma_k'(\beta_j + \beta_k)m_k(\eta_j) + \frac{1}{24}\beta_j + \frac{3}{2}a,$$

and

$$t_j = -\frac{9}{2}\Sigma_k' m_k'(\eta_j) + 3\Sigma_k' m_k'''(\eta_j) - 72\Sigma_k' m_k(\eta_j)\Sigma_k' m_k'(\eta_j) + \beta_j\Sigma_k'(\beta_k m_k'(\eta_j)).$$

Then, we get that

$$\beta_j t_j - 36\Sigma_k'\frac{s_j - s_k}{(\eta_j - \eta_k)^3}$$
$$= 72\beta_j\Sigma_k'\frac{1}{(\eta_j - \eta_k)^3}\Sigma_k'\frac{1}{(\eta_j - \eta_k)^2} - 72\beta_j\Sigma_k'\frac{1}{(\eta_j - \eta_k)^5} \tag{38}$$
$$- 36\Sigma_k'\Sigma_{l\neq j}\frac{\beta_j + \beta_l}{(\eta_l - \eta_j)^2(\eta_j - \eta_k)^3} + 36\Sigma_k'\Sigma_{l\neq k}\frac{\beta_k + \beta_l}{(\eta_j - \eta_k)^3(\eta_l - \eta_j)^2}.$$

Now we claim the right-hand side of the above equation is 0. Without loss of generality, we consider the case $j = 1$ and rewrite the right-hand side of (38) as

$$
\begin{aligned}
&72\beta_1 \sum_{1<l<k\leqslant n} \left(\frac{1}{(\eta_1-\eta_l)^3(\eta_1-\eta_k)^2} + \frac{1}{(\eta_1-\eta_l)^2(\eta_1-\eta_k)^3} \right) \\
&- 36 \sum_{1<l<k\leqslant n} \left(\frac{\beta_1+\beta_k}{(\eta_1-\eta_l)^3(\eta_1-\eta_k)^2} + \frac{\beta_1+\beta_l}{(\eta_1-\eta_l)^2(\eta_1-\eta_k)^3} \right) \\
&+ 36 \sum_{1<l<k\leqslant n} \left(\frac{\beta_l+\beta_k}{(\eta_1-\eta_l)^3(\eta_l-\eta_k)^2} + \frac{\beta_k+\beta_l}{(\eta_l-\eta_k)^2(\eta_1-\eta_k)^3} \right).
\end{aligned} \tag{39}
$$

Using the identity $\det \begin{vmatrix} 1 & \frac{1}{(\eta_1-\eta_l)^2} & \frac{1}{(\eta_1-\eta_l)^3} \\ 1 & \frac{1}{(\eta_l-\eta_k)^2} & \frac{1}{(\eta_l-\eta_k)^3} \\ 1 & \frac{1}{(\eta_k-\eta_1)^2} & \frac{1}{(\eta_k-\eta_1)^3} \end{vmatrix} \equiv 0$. We can further rewrite the third term of (39) as

$$
\begin{aligned}
&36 \sum_{1<l<k\leqslant n} \left(\frac{\beta_l+\beta_k}{(\eta_1-\eta_l)^3(\eta_1-\eta_k)^2} + \frac{\beta_l+\beta_k}{(\eta_1-\eta_l)^2(\eta_l-\eta_k)^3} \right) \\
&- 36 \sum_{1<l<k\leqslant n} \left(\frac{\beta_l+\beta_k}{(\eta_l-\eta_k)^3(\eta_1-\eta_k)^2} + \frac{\beta_l+\beta_k}{(\eta_1-\eta_l)^2(\eta_k-\eta_1)^3} \right).
\end{aligned}
$$

Substituting it into (39) and using the second equation of (30) we get that the right-hand side of (38) can be written as

$$
\begin{aligned}
&72\beta_1 \sum_{l\neq 1}\sum_{k\neq 1} \frac{1}{(\eta_1-\eta_l)^2(\eta_1-\eta_k)^3} - 72\beta_1 \sum_{l\neq 1} \frac{1}{(\eta_1-\eta_l)^5} \\
&+ 36 \sum_{l\neq 1} \frac{\beta_1+\beta_l}{(\eta_1-\eta_l)^5} + 36 \sum_{l\neq 1} \frac{1}{(\eta_1-\eta_l)^2} \left(-\sum_{k\neq 1} \frac{2\beta_1}{(\eta_1-\eta_k)^3} + \frac{\beta_1-\beta_l}{(\eta_1-\eta_l)^3} \right) = 0.
\end{aligned}
$$

Hence we can conclude that

$$
\beta_j t_j - \sum_{k\neq j} \frac{36\left(s_j - s_k\right)}{\left(\eta_j-\eta_k\right)^3} = 0. \tag{40}
$$

Next, for σ, τ with the form (34), we compute the first component of the integral Q along the closed circle l which surrounds the jth pole x_j. We have

$$
\begin{aligned}
I_{Q,1} &:= \int_l ((K_{11}w_1 + K_{12}w_2)\sigma + (K_{21}w_1 + K_{22}w_2)\tau)dx \\
&= -3 \int_l \left[\Sigma_k\left(s_k m_k'\right)\Sigma_\mu\left(a_\mu\beta_\mu m_\mu + b_\mu n_\mu\right) + \Sigma_\mu\left(s_\mu\beta_\mu m_\mu' + t_\mu n_\mu'\right)\Sigma_k(a_k m_k)\right] dx \\
&= -3\Sigma_k'\left[(-1)s_j a_k\beta_k m_k''(\eta_j)\right] - 3\Sigma_k'\left[(-1)s_j b_k n_k''\right]
\end{aligned}
$$

$$- 3\Sigma_k' \left[s_k m_k''(\eta_j) a_j \beta_j \right] - 3\Sigma_k' \left[s_k m_k'(\eta_j) b_j \right]$$
$$- 3\Sigma_k' \left[(-1) s_j \beta_j a_k m_k''(\eta_j) \right] - 3\Sigma_k' \left[(-1) t_j a_k m_k'(\eta_j) \right]$$
$$- 3\Sigma_k' \left[a_j s_k \beta_k m_k''(\eta_j) \right] - 3\Sigma_k' \left[a_j t_k n_k'' \right].$$

Using (37), we find that it equals

$$\begin{aligned} &3\Sigma_k' \left[(\beta_j + \beta_k)(s_j - s_k) a_j m_k''(\eta_j) \right] - 3\Sigma_k' \left[(\beta_j + \beta_k)(a_j - a_k) s_j m_k''(\eta_j) \right] \\ &\quad - 3\Sigma_k' \left[(s_j b_k + s_k b_j - t_j a_k - a_j t_k) m_k'(\eta_j) \right] \\ &= -3\Sigma_k' \left[(a_j (t_j + t_k) - s_j (b_j + b_k) + s_j b_k + s_k b_j - t_j a_k - a_j t_k) m_k'(\eta_j) \right] \\ &= -3\Sigma_k' \left[(a_j - a_k) t_j m_k'(\eta_j) \right] + 3\Sigma_k' \left[(s_j - s_k) b_j m_k'(\eta_j) \right]. \\ &= \frac{\beta_j b_j t_j}{6} + 3\Sigma_k' \left[(s_j - s_k) b_j m_k'(\eta_j) \right]. \end{aligned}$$

By (40), this is equal to 0.

Now applying the similar computation to $\sigma = z_1, \tau = z_2$, we find that

$$\mathcal{K} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left(-3\Sigma_j \left(\bar{s}_j m_j' \right), \Sigma_j \left(\bar{s}_j \beta_j m_j' + \bar{t}_j n_j' \right) \right)^T,$$

where $(\bar{s}_1, \dots, \bar{s}_n, \bar{t}_1, \dots, \bar{t}_n)$ also lies in the tangent space of M. With this at hand, we can then compute the integral Q using similar residue computation as that for $I_{Q,1}$, and show that the second component of Q also equals zero, which readily implies that the vector

$$(\tilde{I}_1, \dots, \tilde{I}_n, B_1, \dots, B_n)$$

lies in the tangent space of M at the point $(\eta_1, \dots, \eta_n, \beta_1, \dots, \beta_n)$. The proof is thus completed. □

It will be interesting if one can find another simpler proof of the above result. We can also compare our result with that of [51], where the relation between CM hierarchy and KP hierarchy has been studied from different point of view.

Next we show that the kth Boussinesq flow will be trivial in suitable sense, provided that k is large.

Lemma 12. *Let n be fixed. Then for k large, $X_k^{(1)} = 0$.*

Proof. Since parameter a in the recursive operator $\mathcal{K}$ is chosen such that $(3a)^2 + \frac{1}{48} = 0$, we can compute

$$\begin{bmatrix} -\frac{D}{4} & -3aD \\ -3aD & \frac{D}{12} \end{bmatrix} \begin{bmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{bmatrix} \begin{bmatrix} -\frac{D}{4} & 3aD \\ 3aD & \frac{D}{12} \end{bmatrix} = 0.$$

This identity guarantees that if the main-order term of the X_k is x^{-k}, then the main-order term of X_{k+2} will be at the order x^{-k-2}. Therefore, if the index k is an odd number, then the main-order term of X_k is $O(\frac{1}{x^{k+2}})$.

We define

$$\pi_s := \Sigma_j \eta_j^s, \quad \Pi_s := \Sigma_j \left(\beta_j \eta_j^s\right).$$

Since

$$\frac{1}{(1-t)^2} = \sum_{s=0}^{+\infty} [(s+1)t^s],$$

we can write, for x large,

$$q = -\frac{3}{2}\sum_{s=0}^{\infty}\left[\frac{(s+1)\Sigma_j \eta_j^s}{x^{s+2}}\right], \quad p = \frac{1}{2}\sum_{s=0}^{\infty}\left[\frac{(s+1)\Sigma_j\left(\beta_j \eta_j^s\right)}{x^{s+2}}\right].$$

Proposition 11 tells us that X_k induces a flow on M. We can then write

$$X_k\left((q,p)^T\right) = \sum_{s=0}^{\infty}\left(-\frac{3(s+1)X_k^{(1)}\pi_s}{2x^{s+2}}, \frac{(s+1)X_k^{(2)}\Pi_s}{2x^{s+2}}\right)^T.$$

Since the main order of $X_k\,(q,p)$ is x^{-k-2}, we see that if $s < k$, then

$$X_k^{(1)}\pi_s = 0 \quad \text{and} \quad X_k^{(2)}\Pi_s = 0.$$

On the other hand, on M, those η_j have to be distinct to each other (if some of them are equal, then the τ function reduces to a lower degree 1). It follows that if $k_0 \geqslant n$, then $\pi_1, \ldots, \pi_{k_0}$ form a basis for the first n coordinate components of the locus M. From this we deduce that if $k > n$, then the first component of the flow X_k is trivial. It is also worth to be pointed out that if $y = 0$, then it is possible that all the β_j in Π_s vanish, making the analysis of $X_k^{(2)}$ to be more delicate. □

4.2 | Degree of the τ function

We need some information on the high degree terms of the τ functions. For this purpose, we first prove the following:

Lemma 13. *Suppose η is a homogeneous polynomial in x, y of degree m and*

$$\left(\mathfrak{D}_x^2 + \mathfrak{D}_y^2\right)\eta \cdot \eta = 0.$$

Then

$$\eta(x,y) = a\left(x^2+y^2\right)^j (x+yi)^k,$$

where a is a constant and $2j+k=m$. In particular, if η is real valued, then $\eta = a(x^2+y^2)^{m/2}$ for some real number a.

Proof. In the polar coordinate (r,θ), where $r=\sqrt{x^2+y^2}$, we can write $\eta = r^m g(\theta)$. Then

$$\begin{aligned}\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)\eta\cdot\eta &= 2\left(\eta\Delta\eta - |\nabla\eta|^2\right)\\ &= 2r^m g\left(m^2 r^{m-2} g + r^{m-2} g''\right) - 2\left(m^2 r^{2m-2} g^2 + r^{2m-2} g'^2\right).\end{aligned}$$

From this we obtain

$$gg'' - g'^2 = 0,$$

which implies $g(\theta) = ae^{b\theta}$ for some constants a and b. Since g has to be 2π-periodic in θ, we have $b = ki$ for some integer k. It follows that

$$\eta = ar^m\left(e^{i\theta}\right)^k = ar^{m-k}(x+yi)^k.$$

Setting $j=\frac{m-k}{2}$, we arrive at the desired result. □

Let τ be a polynomial solution of the bilinear equation

$$\left(\mathfrak{D}_x^4 - \mathfrak{D}_x^2 - \mathfrak{D}_y^2\right)\tau\cdot\tau = 0, \tag{41}$$

with $\deg(\tau) = m$. Note that the bilinear equation corresponding to (26) differs from (41) only by a scaling of the variables x, y.

By Lemma 13, we can assume without loss of generality that the highest degree terms of τ are of the form

$$\left(x^2+y^2\right)^j (x+yi)^k = z^{j+k}\bar{z}^j := \tau_m,$$

where $z = x+yi$ and $\bar{z} = x-yi$ are complex variables. Let us denote those terms of τ with degree $m-1$ by τ_{m-1}. The previous lemma can also be proved using the $(z,\bar{z})$ coordinate. In the sequel, we study τ_{m-1} in this new coordinate.

Lemma 14. *$\tau_{m-1} = a_1 z^{j+k-1}\bar{z}^j + a_2 z^{j+k}\bar{z}^{j-1}$ for some constants a_1, a_2. In particular, if τ is real valued, then for some constant a,*

$$\tau_{m-1} = az^{j-1}\bar{z}^j + \bar{a}z^j\bar{z}^{j-1}.$$

Proof. The sum of degree $2m-3$ terms in the left-hand side of (41) will be of the form

$$-\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)\tau_m \cdot \tau_{m-1}.$$

Suppose $z^r\bar{z}^s$ is a term appearing in τ_{m-1}, then there holds

$$\mathfrak{D}_z\mathfrak{D}_{\bar{z}}\left(z^{j+k}\bar{z}^j\right)\cdot\left(z^r\bar{z}^s\right)=0.$$

Direct computation of the left-hand side tells us that

$$(j+k-r)(j-s)z^{j+k+r-1}\bar{z}^{j+s-1}=0.$$

In the case of $k=0$, we have $r=j$ or $s=j$.

If in addition τ is real valued, then $k=0$ and $\tau_m=z^j\bar{z}^j$. Hence

$$\tau_{m-1}=az^{j-1}\bar{z}^j+\bar{a}z^j\bar{z}^{j-1}.$$

This completes the proof. □

By this lemma, in the real-valued case, if we introduce new variables $Z=z+\frac{a}{j}$ and $\bar{Z}=\bar{z}+\frac{\bar{a}}{j}$, then we see that

$$z^j\bar{z}^j+az^{j-1}\bar{z}^j+\bar{a}z^j\bar{z}^{j-1}=Z^j\bar{Z}^j+P,$$

where P is a polynomial of $Z,\bar{Z}$ with degree less than $j-1$. This means that we can find real numbers b_1,b_2 such that in the new variables $\tilde{x}=x+b_1, \tilde{y}=y+b_2$, the highest degree term of τ is $\left(\tilde{x}^2+\tilde{y}^2\right)^j$ and τ does not have terms with degree $2j-1$.

Lemma 15. *Suppose $q=\frac{3}{2}\partial_x^2\ln\tau$ is a real-valued rational solution of the Boussinesq equation (26), where τ is a polynomial of degree $2n$. Let $p=\int_{-\infty}^x\partial_y q\,dx$. Then for x large, at $y=0$,*

$$q=-\frac{3n}{x^2}+O\left(x^{-3}\right),\quad p=O\left(x^{-5}\right),\quad p'=O\left(x^{-6}\right).$$

Proof. The fact that $q=\frac{3n}{x^2}+O\left(x^{-3}\right)$ is relatively easy to check. We focus on the estimate of p.

We use τ_j to denote the sum of those degree j terms in τ. Since τ is real valued, after a possible translation of the coordinate (and a scaling of the y variable), we can assume

$$\tau_{2n}=z^n\bar{z}^n \text{ and } \tau_{2n-1}=0.$$

By Lemma 18, we have

$$\begin{aligned}\tau_{2n-2}=&\frac{1}{2}\left(n-n^2\right)z^{n+1}\bar{z}^{n-3}+3n^2z^{n-1}\bar{z}^{n-1}+\frac{1}{2}\left(n-n^2\right)z^{n-3}\bar{z}^{n+1}\\&+cz^n\bar{z}^{n-2}+cz^{n-2}\bar{z}^n.\end{aligned}\tag{42}$$

On the other hand, we have

$$p = -\frac{3}{2}\partial_y\partial_x \ln\tau = -\frac{3}{2}\frac{\tau\partial_y\partial_x\tau - \partial_x\tau\partial_y\tau}{\tau^2}.$$

By (42), in terms of the x, y coordinates, τ_{2n-2} does not have the term $x^{2n-3}y$. Hence when $y = 0$,

$$p = O\left(x^{-5}\right),\ p' = O\left(x^{-6}\right).$$

This is the desired estimate for p. We emphasize that the estimate is not true if $y \neq 0$. □

Now we are at a position to prove the main result of this section.

Theorem 16. *Assume that q is a real-valued rational solution of the Boussinesq equation (26) satisfies the decay assumption (3). Then* $q = \frac{3}{2}\partial_x^2 \ln\tau$, *where* τ *is a polynomial in* x, y *with degree* $k(k+1)$ *for some* $k \in \mathbb{N}$.

Proof. For x large, the main-order term of q in its Laurent expansion is $\frac{m}{x^2}$. Since the degree of the polynomial is expected to be $k(k+1)$, we expect m to be $-\frac{3}{2}k(k+1)$.

We compute the third-order derivatives in the $(1,1)$ entry of $\mathcal{K}$:

$$\begin{aligned}&\left(D^3 + qD + Dq\right)\left(\frac{1}{x^j}\right)\\ &= [-j(j+1)(j+2) - m(j+j+2)]\frac{1}{x^{j+3}}\\ &= -(j+1)(j(j+2)+2m)\frac{1}{x^{j+3}} \ := b_m(j)\frac{1}{x^{j+3}}.\end{aligned}$$

Similarly, the third-order derivatives in the $(2,2)$ entry of $\mathcal{K}$:

$$\begin{aligned}&\left(-\frac{5}{4}D^3 - 2(qD+Dq)\right)\left(\frac{1}{x^j}\right)\\ &= \frac{5}{4}j(j+1)(j+2) + 2m(j+j+2)\frac{1}{x^{j+3}}\\ &= \frac{1}{8}(j+1)(10j(j+2)+32m)\frac{1}{x^{j+3}} =:\ B_m(j)\frac{1}{x^{j+3}}.\end{aligned}$$

By Lemma 15, the off-diagonal operators in $\mathcal{K}$ do not enter the computation of main-order terms. On the other hand, Lemma 12 tells us that the first component of the kth flow is trivial for k large. Hence vanishing of terms requires

$$\frac{1}{4}b_m(j) - \frac{1}{4}B_m(j) = 0.$$

That is,

$$m = -\frac{3}{8}j(j+2).$$

Note that j should be even since $q = O(\frac{m}{x^2})$. Taking $j = 2k$, we find that

$$m = -\frac{3}{2}k(k+1).$$

This completes the proof. □

Summarizing all the previous discussion, we conclude that Theorem 1 is proved.

Now suppose q is a solution of the equation

$$\partial_y^2 q = 3\partial_x^2\left(\partial_x^2 q + 4q^2 - q\right). \tag{43}$$

The energy of q is

$$E(q) := \int_{\mathbb{R}^2}\left[\frac{3}{2}\left|\partial_x q\right|^2 - 4q^3 + \frac{3}{2}q^2 + \left|\partial_x^{-1}\partial_y q\right|^2\right].$$

We now know that q has the form $\frac{3}{2}\partial_x^2 \ln \tau$, where τ is a polynomial with degree $k(k+1)$. The classical lump solution for (43) is

$$u_0(x, y) = \frac{3}{2}\partial_x^2 \ln\left(x^2 + 3y^2 + 3\right).$$

Note that up to a translation in the x and y variables, the τ function with degree 2 is unique.

Following the same proof as that of the appendix of Gorshkov–Pelinovskii–Stepanyants [26] (see Equation (A6) there), we obtain

$$E(q) = \frac{k(k+1)}{2}H(u_0). \tag{44}$$

On the other hand, we also know from [12] that Equation (43) has variational structure and possesses a ground state. From the energy quantization identity (44), we infer immediately that the classical lump solution is the unique ground state, up to translation in the plane.

5 | THE ANALYSIS OF EVEN SOLUTIONS

In this section, we would like to analyze the uniqueness of the (nontrivial) even solutions of the Boussinesq equation. Combining our classification result obtained in the previous section with the existence result of [48] mentioned in Section 2, we can show that even solutions exist if and only if their τ functions are polynomials of degree $2n = k(k+1)$.

The importance of even solutions comes from the fact that in principle, using them, we can construct traveling wave solutions of the GP or generalized KP equation. This construction relies on certain nondegeneracy properties of the solutions, which is expected to be true for the even solutions in the space of even perturbations. Note that without evenness assumption, we will not have nondegeneracy, since the space of solutions will be a manifold. From the semilinear elliptic PDE point of view, even solutions should play similar role as the radially symmetric solutions of the Schrödinger equation.

Now suppose q is an even solution. From Lemma 13, we can assume that the sum of the degree $2n$ terms of τ is $T_{n,0} = \left(x^2+y^2\right)^n$. We also denote the sum of all the degree $2n-2j$ terms in τ by $T_{n,j}$.

We will first analyze the even solution in the x-y coordinate. Then later on we will adopt a slightly different approach using the z-$\bar{z}$ coordinate.

Let us define functions (throughout this section, i is set to be an index number)

$$g_j := \left(x^2+y^2\right)^{n-3j} x^{2j} y^{2j}, \text{ and } \xi_{i,j} := \left(\mathfrak{D}_x^2 + \mathfrak{D}_y^2\right) g_i \cdot g_j.$$

Observe that actually $\xi_{i,j}$ can be divided by $\left(x^2+y^2\right)^{2n-3i-3j-1}$. We then introduce the constants

$$d_{i,j} := \left.\frac{\left(\mathfrak{D}_x^2 + \mathfrak{D}_y^2\right) g_i \cdot g_j}{\left(x^2+y^2\right)^{2n-3i-3j-1}}\right|_{\left(x^2=-1, y^2=1\right)}.$$

Direct computation shows that

$$d_{i,j} = -12(i-j)^2(-1)^{i+j}.$$

We also need the function

$$\mathfrak{D}_x^4 \left(x^2+y^2\right)^{n-3i} \cdot \left(x^2+y^2\right)^{n-3j}.$$

Since we have taken the fourth-order derivatives, this function is dividable by $(x^2+y^2)^{2n-3i-3j-4}$. Define

$$p_{i,j} = \left.\frac{\mathfrak{D}_x^4 g_i \cdot g_j}{\left(x^2+y^2\right)^{2n-3i-3j-4}}\right|_{\left(x^2=-1, y^2=1\right)}.$$

Explicitly, $(-1)^{i+j} p_{i,j}$ is equal to

$$\begin{aligned}
&1296i^4 - 5184i^3 j + 7776i^2 j^2 - 5184ij^3 + 1296j^4 + 2592i^3 - 2592i^2 j \\
&- 1728i^2 n - 2592ij^2 + 3456ijn + 2592j^3 - 1728j^2 n + 1584i^2 - 1440ij \\
&- 576in + 1584j^2 - 576jn + 192n^2 + 288i + 288j - 192n.
\end{aligned}$$

With all these constants $d_{i,j}, p_{i,j}$ at hand, we would like to define, in a recursive way, a sequence of numbers a_m, $m = 0, 1, \ldots$, depending on n, in the following way.

First take $a_0 = 1$. Then a_m is determined by $a_1, \ldots, a_{m-1}$ through the following recursive relation:

$$\sum_{i,j \leqslant m, i+j=m} \left(a_i a_j d_{i,j}\right) = \sum_{i,j \leqslant m, i+j=m-1} \left(a_i a_j p_{i,j}\right).$$

These a_j can also be regarded as polynomials of the variable n. To state our result, the following constant J_n will be important:

$$J_n := \sum_{i,j \leqslant [n/3], i+j=[n/3]+1} \left(a_i a_j d_{i,j}\right) - \sum_{i,j \leqslant [n/3], i+j=[n/3]} \left(a_i a_j p_{i,j}\right).$$

Here $[m]$ denotes the largest integer which does not exceed m. The following result provides a necessary condition for the existence of even solutions.

Proposition 17. *Let n be a fixed integer. If $J_n \neq 0$, then the Boussinesq equation has no rational even solution with degree $2n$.*

Proof. First of all, observe that $T_{n,j}$ has the form

$$a_j\left(x^2+y^2\right)^{n-3j}x^{2j}y^{2j}+\left(x^2+y^2\right)^{n-3j+1}\Gamma(x,y),$$

where Γ is a homogeneous polynomial in x, y with degree $4j-2$.

Let us denote the function $(\mathfrak{D}_x^4-\mathfrak{D}_x^2-\mathfrak{D}_y^2)f\cdot f$ by K_f. Since we have chosen a_0 to be 1, K_f is a polynomial of degree at most $4n-4$. The terms with degree $4n-4$ are given by

$$\mathfrak{D}_x^4T_{n,0}\cdot T_{n,0}-2\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)T_{n,0}\cdot T_{n,1}.$$

This function is dividable by $(x^2+y^2)^{2n-4}$. We write is as

$$b_1\left(x^2+y^2\right)^{2n-4}x^2y^2+\left(x^2+y^2\right)^{2n-3}M(x,y).$$

Inserting $x^2=-1, y^2=1$ into this function, we find that necessary $b_1=0$. Therefore, we get

$$a_0^2p_{0,0}-a_0a_1d_{0,1}=0.$$

Similarly, consider the terms with degree $4n-6$, we get

$$\mathfrak{D}_x^4T_{n,0}\cdot T_{n,1}-\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)T_{n,1}\cdot T_{n,1}-\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)T_{n,0}\cdot T_{n,2}=0.$$

Then

$$a_0a_1p_{0,1}-a_1^2d_{1,1}-a_0a_2d_{0,2}=0.$$

Similarly, for $m\leqslant[n/3]$,

$$\sum_{i,j\leqslant m,i+j=m}\left(a_ia_jd_{i,j}\right)=\sum_{i,j\leqslant m,i+j=m-1}\left(a_ia_jp_{i,j}\right).$$

Since we require that the solution is a polynomial, the function

$$\sum_{i,j\leqslant[n/3],i+j=[n/3]+1}a_ia_j\left(\mathfrak{D}_x^2+\mathfrak{D}_y^2\right)T_{n,i}\cdot T_{n,j}-\sum_{i,j\leqslant[n/3],i+j=[n/3]}a_ia_j\mathfrak{D}_x^4T_{n,i}\cdot T_{n,j}$$

should be dividable by $(x^2+y^2)^{n-1}$, this implies that $J_n=0$. □

We have computed the constants a_j and J_n, using software like *Mathematica*. It turns out that at least for $n \leqslant 300$, J_n is equal to zero if and only if $n = \frac{k(k+1)}{2}$ for some integer k.

The previous analysis can also be viewed in a slightly different way, in terms of the z and $\bar{z}$ variables. Let us explain this in more details.

From the proof of Lemma 14, we know that if η satisfies

$$\mathfrak{D}_z \mathfrak{D}_{\bar{z}} T_{n,0} \cdot \eta = 0,$$

then for some constants c_1, c_2,

$$\eta = c_1 z^n \bar{z}^s + c_2 z^m \bar{z}^n.$$

This also tells us that the equation

$$\mathfrak{D}_z \mathfrak{D}_{\bar{z}} T_{n,0} \cdot \eta = z^\alpha \bar{z}^\beta,$$

is not solvable if either α or β equal $2n - 1$. Since $T_{n,0} = z^n \bar{z}^n$, another necessary condition is that

$$\alpha \geqslant n - 1 \text{ and } \beta \geqslant n - 1.$$

Lemma 18. *The $T_{n,1}$ term has the following form:*

$$T_{n,1} = \frac{1}{2}\left(n - n^2\right) z^{n+1} \bar{z}^{n-3} + 3n^2 z^{n-1} \bar{z}^{n-1} + \frac{1}{2}\left(n - n^2\right) z^{n-3} \bar{z}^{n+1} + c z^n \bar{z}^{n-2} + c z^{n-2} \bar{z}^n,$$

where c is a real-valued constant.

Proof. We compute

$$\mathfrak{D}_x^4 T_{n,0} \cdot T_{n,0} = \left(12n^2 - 12n\right) z^{2n} \bar{z}^{2n-4} + 24n^2 z^{2n-2} \bar{z}^{2n-2} + \left(12n^2 - 12n\right) z^{2n-4} \bar{z}^{2n}.$$

Since our solution is even, the conclusion then follows from the fact that $T_{n,1}$ solves the equation (note that the constant is 8, rather than 4)

$$8 \mathfrak{D}_z \mathfrak{D}_{\bar{z}} T_{n,0} \cdot T_{n,1} = \mathfrak{D}_x^4 T_{n,0} \cdot T_{n,0}.$$

The fact the our solution is real and even forces the coefficients before $z^n \bar{z}^{n-2}$ and $z^{n-2} \bar{z}^n$ to be a same real constant. This completes the proof. □

We emphasize that in general the constant c will not be zero. For instance, the degree 12 solution obtained in [48] is

$$(x^2 + y^2)^6 + 2(x^2 + y^2)^3 (49x^4 + 198x^2 y^2 + 29y^4) + 5(147x^8 + 3724x^6 y^2 + 7490x^4 y^4 + 7084x^2 y^6 + 867y^8)$$

$$+\frac{140}{3}(539x^6+4725x^4y^2-315x^2y^4+5707y^6)$$
$$+\frac{1225}{9}(391\,314x^2-12\,705x^4+4158x^2y^2+40\,143y^4+736\,890x^2+717\,409).$$

It can also be written as

$$z^6\bar{z}^6-15z^7\bar{z}^3+10z^6\bar{z}^4+108z^5\bar{z}^5+10z^4\bar{z}^6-15z^3\bar{z}^7$$
$$-45z^8+150z^7\bar{z}-875z^6\bar{z}^2-1050z^5\bar{z}^3+4375z^4\bar{z}^4-1050z^3\bar{z}^5-875z^2\bar{z}^6+150z\bar{z}^7-45\bar{z}^8$$
$$-22\,330z^6/3+20\,895z^5\bar{z}-52850z^4\bar{z}^2+103\,950z^3\bar{z}^3-52\,850z^2\bar{z}^4+20\;895z\bar{z}^5-22\,330\bar{z}^6/3$$
$$+594\,125z^4/3-1\,798\,300z^3\bar{z}+1\,471\,225z^2\bar{z}^2-1\,798\,300z\bar{z}^3+594\,125z^4/3$$
$$+38\,390\,275z^2+76\,780\,550z\bar{z}+38\,390\,275\bar{z}^2+878\,826\,025/9.$$

As a polynomial of variables $z, \bar{z}$, the total degree of the homogeneous polynomial $T_{n,j}$ is equal to $2n-2j$. For each fixed j, inspecting the term in $T_{n,j}$ with lowest degree in $\bar{z}$, we find that it has to be of the form $\sigma_j z^{n+j}\bar{z}^{n-3j}$. Indeed, the constants σ_j can be defined recursively and uniquely by the following equation: For $j = 1, \ldots,$

$$4\mathfrak{D}_z\mathfrak{D}_{\bar{z}}T_{n,0}\cdot\left(\sigma_j z^{n+j}\bar{z}^{n-3j}\right)$$
$$=\sum_{k+m=j-1}\left[\mathfrak{D}_x^4\left(\left(\sigma_k z^{n+k}\bar{z}^{n-3k}\right)\right)\cdot\left(\sigma_m z^{n+m}\bar{z}^{n-3m}\right)\right]$$
$$-4\sum_{k,m\leqslant j-1,k+m=j}\left[\mathfrak{D}_z\mathfrak{D}_{\bar{z}}\left(\left(\sigma_k z^{n+k}\bar{z}^{n-3k}\right)\right)\cdot\left(\sigma_m z^{n+m}\bar{z}^{n-3m}\right)\right].$$

Observe that the degree of $\mathfrak{D}_z\mathfrak{D}_{\bar{z}}T_{n,0}\cdot\left(z^{n+j}\bar{z}^{n-3j}\right)$ is equal to $z^{2n+j-1}\bar{z}^{2n-3j-1}$. However, as discussed above, the equation

$$\mathfrak{D}_z\mathfrak{D}_{\bar{z}}T_{n,0}\cdot\eta = z^{2n+j-1}\bar{z}^{2n-3j-1} \tag{45}$$

will not be solvable if

$$2n-3j-1<n-1.$$

That is, $n<3j$. This means that it necessary condition for an even solution to exist is

$$\sigma_{j_0}=0,\ \text{for}\ j_0=\left[\frac{n}{3}\right]+1.$$

We have also verified that for $0\leqslant n\leqslant 300$, σ_n equals zero if and only if $n=k(k+1)/2$ for some integer k.

This algorithm inspires us to study the uniqueness of even solution. The possible nonuniqueness arises from the fact that Equation (45) has kernels of the form $z^n\bar{z}^{n-2q}+z^{n-2q}\bar{z}^n$. Note that for each fixed $q=1,\ldots,[n/2]$, the lowest possible degree term generated by the function $z^n\bar{z}^{n-2q}$ in $T_{n,q+j}$ is of the form $\beta_j z^{n+j}\bar{z}^{n-2q-3j}$. Here $\beta_0=1$, and similar to σ_j, for $j\geqslant 1$, the sequence β_j

is determined by the following recursive formula:

$$
\begin{aligned}
&4\mathfrak{D}_z\mathfrak{D}_{\bar z}(z^n\bar z^n)\cdot\left(\beta_j z^{n+j}\bar z^{n-2q-3j}\right)\\
&=\sum_{k+m=j-1}\left[\mathfrak{D}_x^4\left(\left(\sigma_k z^{n+k}\bar z^{n-3k}\right)\right)\cdot\left(\beta_m z^{n+m}\bar z^{n-2q-3m}\right)\right]\\
&\quad-4\sum_{k,m\leqslant j-1,k+m=j}\left[\mathfrak{D}_z\mathfrak{D}_{\bar z}\left(\left(\sigma_k z^{n+k}\bar z^{n-3k}\right)\right)\cdot\left(\beta_m z^{n+m}\bar z^{n-2q-3m}\right)\right].
\end{aligned}
$$

Note that β_j are also depending on q. The degree of $\bar z$ in $\mathfrak{D}_z\mathfrak{D}_{\bar z}\,(z^n\bar z^n)\cdot\left(\beta_j z^{n+j}\bar z^{n-2q-3j}\right)$ is $2n-2q-3j-1$. For

$$j=\bar j\ :=[(n-2q)/3]+1,$$

there holds

$$2n-2q-3j-1<n-1.$$

We then define, for $q=1,\dots,[n/2]$,

$$\gamma_q\ :=\beta(\bar j).$$

We have the following:

Lemma 19. *For given $n=k\,(k+1)\,/2$, if $\gamma_q\neq 0$ for all $q=1,\dots,[n/2]$, then the even solution is unique.*

Proof. Note that the kernel terms $z^n\bar z^{n-2q}+z^{n-2q}\bar z^n$ are the only possible sources of nonuniqueness. We consider them for each q, starting from $q=1$.

Since $\gamma_1\neq 0$, we see that the coefficient of $z^n\bar z^{n-2}+z^{n-2}\bar z^n$ is uniquely determined in $T_{n,1}$, otherwise one of the equations for the terms in $T_{n,[(n-2)/3]+2}$ will not be solvable. Once $z^n\bar z^{n-2}+z^{n-2}\bar z^n$ is determined, we use the assumption that $\gamma_2\neq 0$ to conclude that we do not have the freedom to choose the kernel $z^n\bar z^{n-4}+z^{n-4}\bar z^n$ in $T_{n,2}$. Proceeding with this argument, we see that all the kernel terms $z^n\bar z^{n-2q}+z^{n-2q}\bar z^n$ are uniquely determined. This finishes the proof. □

We can compute the precise value of the constant γ_q explicitly for each n (we used *Mathematica*). It turns out that for $n=k\,(k+1)\,/2\leqslant 300$, all the constants γ_q are nonzero. For example, when $n=15$, we have

$$
\begin{aligned}
\gamma_1&=\frac{3\,219\,950\,475}{374},\gamma_2=-\frac{800\,391\,375}{416},\gamma_3=24\,045\,525/4,\\
\gamma_4&=\frac{34\,505\,100}{187},\gamma_5=-\frac{74\,025}{52},\gamma_6=\frac{55\,335}{2},\gamma_7=-\frac{5460}{17}.
\end{aligned}
$$

It should be pointed out that all these computations are actually rigorous. We are therefore arriving at the following:

Theorem 20. *Suppose τ is a polynomial of degree $2n$ with real coefficients satisfying*

$$\tau(x,y) = \tau(x,-y) = \tau(-x,y)$$

and

$$\left(\mathfrak{D}_x^2 + \mathfrak{D}_y^2 - \mathfrak{D}_x^4\right)\tau \cdot \tau = 0.$$

Assume $n = k\,(k+1)/2 \leqslant 300$ for some positive integer k. Then τ is unique, up to a multiplicative constant.

The upper bound 300 appeared in this theorem can be significantly improved. We actually expect the uniqueness of even solution to be true for all fixed $n \in \mathbb{N}$ (obviously, by our result, n has to be $k\,(k+1)/2$). We believe this is true and leave it for another work.

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