



Localization for general Helmholtz [☆]

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Abstract

In [4], Guan, Murugan and Wei established the equivalence of the classical Helmholtz equation with a “fractional Helmholtz” equation in which the Laplacian operator is replaced by the nonlocal fractional Laplacian operator. More general equivalence results are obtained for symbols which are complete Bernstein and satisfy additional regularity conditions. In this work we introduce a novel and general set-up for this Helmholtz equivalence problem. We show that under very mild and easy-to-check conditions on the Fourier multiplier, the general Helmholtz equation can be effectively reduced to a localization statement on the support of the symbol.

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1. Introduction

A classical problem in mathematical physics is to find eigen-pairs to the linear problem $\mathcal{A}f = \lambda f$ where \mathcal{A} is a linear operator (bounded or unbounded) acting on some general function spaces. If $\mathcal{A} = -\Delta$, where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ with various boundary conditions, this becomes

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the standard Helmholtz equation. The Helmholtz equation appears ubiquitously in physical and engineering applications such as acoustic radiation, heat conduction, propagation of water waves etc. In the context of heat or wave equations, the classical Helmholtz system naturally arises as the time-independent eigen-pair form when one employs the traditional separation of variable method. Recently in [6] a scalar fractional Helmholtz type equation is derived from the Maxwell’s equations by incorporating nonlocal long range effects. In this connection a prototypical version of the fractional Helmholtz equation amounts to taking $\mathcal{A} = (-\Delta)^s$ for some $s > 0$. A first subtle mathematical issue is to understand the limit transitions $s \rightarrow 1$ (harmonic) or $s \rightarrow n \in \mathbb{N}$ (poly-harmonic) in various normed functional spaces. Another circle of questions, recently initiated in the work of Guan, Murugan and Wei [4], is concerned with the deep connection between the classical Helmholtz equation and the fractional ones, namely (taking $\lambda = 1$):

$$\{f : -\Delta f = f\} \quad \text{vs} \quad \{g : (-\Delta)^s g = g\}; \tag{1.1}$$

or more generally

$$\{f : -\Delta f = f\} \quad \text{vs} \quad \{g : \Phi(-\Delta)g = \Phi(1)g\}; \tag{1.2}$$

where $\Phi(-\Delta)$ is a suitable Dirichlet-to-Neumann operator corresponding to a suitable harmonic extension problem involving the symbol $\Phi(\xi^2)$.

In [4], by deeply exploiting the harmonic extension technique, Guan, Murugan and Wei established the following set of rigidity type results:

- (1) If $0 < s < 2, d = 1, u \in L^\infty(\mathbb{R})$ satisfies $\Lambda^s u = u$, then¹ $u(x) = c_1 \cos x + c_2 \sin x$.
- (2) If $0 < s \leq 2, d \geq 2, u \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies $\Lambda^s u = u$ and $\lim_{|x| \rightarrow \infty} u(x) = 0$, then $-\Delta u = u$.
- (3) If $m \in \mathbb{N}, d \geq 2, u \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies $(-\Delta)^m u = u$ if and only if $-\Delta u = u$.
- (4) Consider $d \geq 2$ and $\Phi(-\Delta)u = \Phi(1)u$ in \mathbb{R}^d where Φ is a complete Bernstein function. Assume $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Consider the associated extension problem: $\tilde{u} \in H_{\text{loc}}^1(\mathbb{R}_+^{d+1}, a(t))$ solves

$$\begin{cases} \nabla_{t,x} \cdot (a(t)\nabla_{t,x}\tilde{u}) = 0, & \text{on } \mathbb{R}_+^{d+1}; \\ \lim_{t \rightarrow 0} a(t)\partial_t u = -c_{n,a}\Phi(-\Delta)u, & \text{on } \mathbb{R}^d; \\ \lim_{t \rightarrow 0} \tilde{u} = u; \end{cases}$$

where one assumes the weight $a(t)$ for $\Phi(-\Delta)$ is A_2 and obeys $a(t) \sim t^\alpha$ for $t \gg 1$ and $|\alpha| < 1$. Under the above conditions, one has

$u \in L^\infty(\mathbb{R}^d)$ with $u(\infty) = 0$ solves $\Phi(-\Delta)u = \Phi(1)u$ in \mathbb{R}^d if and only if $-\Delta u = u$ in \mathbb{R}^d .

(1.3)

As was somewhat hinted earlier, the proof in [4] hinges on the machinery of harmonic extension which poses various subtle technical restrictions. For example, in dimensions $d \geq 2$, one

¹ The case $d = 1$ was first obtained by Fall and Weth in [3].

needs to impose the decay condition $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for the fractional Helmholtz problem $(-\Delta)^s u = u, 0 < s < 2$. Besides, some additional asymptotic conditions need to be imposed on the weight function $a(t)$ for the Bernstein Helmholtz problem. In recent [2], we removed the decay condition and established the equivalency of classical Helmholtz equation and the fractional Helmholtz equation corresponding to the operator $(-\Delta)^s, 0 < s < 2$. However, the proof in [2] hinges on the special form of the operator $(-\Delta)^s$ which has no obvious bearing on the general case.

In this work we shall remove all these aforementioned technical restrictions by developing a novel and general set up for this Helmholtz equivalence problem. We shall consider the general problem

$$\Phi(-\Delta)u = \Phi(1)u \quad \text{in } S'(\mathbb{R}^d), \tag{1.4}$$

where $u \in L^\infty(\mathbb{R}^d)$.

We make the following technical assumptions on the function $\Phi : [0, \infty) \rightarrow \mathbb{R}$:

- (a) Smoothness and mild growth at $z = \infty$. We assume $\Phi \in C([0, \infty)) \cap C^\infty((0, \infty))$, and all derivatives of Φ are polynomially bounded as $z \rightarrow \infty$: namely for some $z_0 \geq 2$, it holds that for all $k \geq 0$,

$$|\partial^k \Phi(z)| \leq C_{\Phi,k,z_0} z^{n_k}, \quad \forall z \geq z_0, \tag{1.5}$$

where $C_{\Phi,k,z_0} > 0$ is a constant depending only on (Φ, k, z_0) , and n_k depends on k .

- (b) Mild singularity at $z = 0$. For some $0 < \varepsilon_0 \leq \frac{1}{2}$,

$$\sum_{j=0}^{d+1} \int_0^{\varepsilon_0} |z|^j |(\partial_z^j \Phi)(z)| \cdot \frac{dz}{z} < \infty. \tag{1.6}$$

Here note that in (1.6), the condition for $j = 0$ reads as $\int_0^{\varepsilon_0} |\Phi(z)|z^{-1} dz < \infty$. This implies $\Phi(0) = 0$ by continuity of Φ .

- (c) Smooth uni-valence at $z = 1$. We assume $\Phi'(1) \neq 0$ and $\Phi(t) \neq \Phi(1)$ for any $t \in (0, 1) \cup (1, \infty)$.

Remark 1.1. A prototypical Φ satisfying conditions (a)-(c) is $\Phi(z) = z^s$ with $s > 0$. This corresponds to the usual fractional Laplacian case. Other nontrivial examples are:

$$\Phi(z) = (m^2 + z)^{\frac{s}{2}} - m, \quad s > 0;$$

$$\Phi(z) = z^{\frac{1}{2}} \tanh(z^{\frac{1}{2}});$$

$$\Phi(z) = z^{\frac{1}{2}} (\tanh(z^{\frac{1}{2}}))^{-1}.$$

The Fourier symbols $|\xi| \tanh(|\xi|), |\xi| (\tanh(|\xi|))^{-1}$ play important roles in the Dirichlet to Neumann map theory in connection with linear water waves.

One should note that for $x > 0$:

$$\frac{d}{dx} \left(\frac{\tanh x}{x} \right) = -\frac{\tanh x}{x^2} + \frac{1}{x \cosh^2 x} = \frac{1}{x^2 \cosh^2 x} \left(x - \frac{1}{2} \sinh(2x) \right) < 0.$$

Thus $h(x) = \frac{x}{\tanh x}$ is strictly increasing.

Remark 1.2. Interestingly a class of nonlocal Helmholtz problems with nonlocal “completely Bernstein” symbols are also treated in recent [5] (see also [1]).

Concerning (1.4), an immediate subtle technical issue is to ensure $\Phi(-\Delta)u \in S'(\mathbb{R}^d)$ for $u \in L^\infty(\mathbb{R}^d)$. The next proposition clarifies this point. We first examine the action of $\Phi(-\Delta)$ on test functions which are Schwartz.

Proposition 1.1. *Suppose $\Phi \in C([0, \infty)) \cap C^\infty((0, \infty))$ satisfies the conditions (a)–(c). For $\psi \in S(\mathbb{R}^d)$, define*

$$\widehat{\Phi(-\Delta)\psi}(\xi) = \Phi(|\xi|^2)\widehat{\psi}(\xi), \quad \xi \in \mathbb{R}^d.$$

Then $\Phi(-\Delta)\psi \in L^1(\mathbb{R}^d)$ and for some integers $k_1 \geq 0, k_2 \geq 0$,

$$\|\Phi(-\Delta)\psi\|_{L^1_x(\mathbb{R}^d)} \leq C_{d,k_1,k_2,\Phi} \sum_{|\alpha| \leq k_2} \|(1 + |x|^2)^{k_1} \partial^\alpha \psi\|_{L^\infty_x(\mathbb{R}^d)},$$

where $C_{d,k_1,k_2,\Phi} > 0$ is a constant depending only on (d, k_1, k_2, Φ) .

The proof of Proposition 1.1 is given in Section 2.

Thanks to Proposition 1.1, for $u \in L^\infty_x(\mathbb{R}^d)$ we can define $\Phi(-\Delta)u \in S'(\mathbb{R}^d)$ as an element in $S'(\mathbb{R}^d)$ via the following

$$\left(\Phi(-\Delta)u \right)(\varphi) = \int_{\mathbb{R}^d} u \Phi(-\Delta)\varphi dx, \quad \forall \varphi \in S(\mathbb{R}^d). \tag{1.7}$$

Alternatively and equivalently, we extend the usual L^2 -pairing of Schwartz functions (see (1.20)) to $S'(\mathbb{R}^d)$ - $S(\mathbb{R}^d)$ pairing as the following (below \bar{z} denotes the usual complex conjugate for $z \in \mathbb{C}$):

$$\langle \Phi(-\Delta)u, \psi \rangle := \int_{\mathbb{R}^d} u(x) \overline{(\Phi(-\Delta)\psi)(x)} dx, \quad \forall \psi \in S(\mathbb{R}^d). \tag{1.8}$$

In particular we have the estimate

$$|\langle \Phi(-\Delta)u, \psi \rangle| \lesssim \|u\|_{L^\infty_x(\mathbb{R}^d)} \sum_{|\alpha| \leq k_2} \|(1 + |x|^2)^{k_1} \partial^\alpha \psi\|_{L^\infty_x(\mathbb{R}^d)}.$$

Thus in its natural weak formulation, (1.4) reads as:

$$u \in L_x^\infty(\mathbb{R}^d) \text{ satisfies } \langle u, \Phi(-\Delta)\psi \rangle = \langle u, \Phi(1)\psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d) \text{ and } \Phi \text{ satisfies (a)–(c).} \tag{1.9}$$

Remark 1.3. The main difference between (1.7) and (1.8) is that the former is linear in the test function φ whereas the latter is *conjugate linear* in the test function ψ .

Remark 1.4. To put things into perspective, one should note that in general it is a subtle issue to define the action of fractional Laplacian on a general tempered distribution u , since the fractional Laplacian multiplier does not preserve the Schwartz space. What we basically show is that for fractional Laplacian operators and slightly more general symbols, one can start by carefully defining the action of the fractional operator on the Schwartz function, show that the corresponding norm depends only on the weighted Sobolev norm of the test function; then in a natural way one can define for (say) bounded function u , the corresponding tempered distribution $(-\Delta)^s u$. The only place where Fourier transform enter, is in the action of the symbol on the Schwartz test functions.

The main result of this paper is the following.

Theorem 1.1 (*Classification of general Helmholtz, case $\Phi'(1) \neq 0$*). *Let $d \geq 1$. Suppose $u \in L^\infty(\mathbb{R}^d)$ solves the problem (1.9). Then the following hold:*

(1) $\text{supp}(\widehat{u}) \subset K = \{\xi : \xi = 0 \text{ or } |\xi| = 1\}$. *More precisely we have*

$$\langle \widehat{u}, \phi \rangle = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \setminus K). \tag{1.10}$$

(2) *If $\Phi(1) \neq 0$, then $\text{supp}(\widehat{u}) \subset \{\xi : |\xi| = 1\}$, and*

$$\langle \widehat{u}, \phi \rangle = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \setminus \{\xi : |\xi| = 1\}). \tag{1.11}$$

Furthermore we have in this case,

$$\left\langle u, \mathcal{F}^{-1}\left((|\xi|^2 - 1)\psi(\xi)\right) \right\rangle = 0, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{1.12}$$

Remark 1.5. Since \widehat{u} is compactly supported, the function u can be identified as a $C^\infty(\mathbb{R}^d)$ function thanks to Paley-Wiener. Moreover in the case $\Phi(1) \neq 0$, we conclude that u solves the classical Helmholtz equation, namely

$$-\Delta u = u \quad \text{in } \mathbb{R}^d.$$

In Section 4 of this paper, we shall show that our theorem above already covers the general Bernstein Helmholtz case in [4]. As was already mentioned, the conditions on the symbol $\Phi(z)$ is already quite general and easy-to-check in practice.

On the other hand, as the avid reader may notice, the condition $\Phi'(1) \neq 0$ may appear a bit restrictive. As a matter of fact, one can also consider a slightly more non-degenerate condition. That is, we assume $\Phi(1) \neq 0$, and for some $j_0 \geq 1$, it holds that

$$\frac{d^j}{dz^j} \Phi(z) \Big|_{z=1} =: \Phi^{(j)}(1) = 0, \quad 1 \leq j \leq j_0 - 1, \quad \text{and} \quad \Phi^{(j_0)}(1) \neq 0.$$

If $j_0 = 1$, we simply require $\Phi'(1) \neq 0$. For $j_0 \geq 2$, $\Phi^{(j_0)}(1)$ is the first nonzero coefficient after $\Phi'(1) = 0$. As we shall show momentarily, under this general condition the corresponding localization statement takes a slightly different form, namely: the Helmholtz equation

$$\Phi(-\Delta)u = \Phi(1)u$$

can be effectively reduced to

$$(-\Delta - 1)^{j_0}u = 0.$$

For the sake of completeness, we now record the aforementioned slightly more general technical assumptions on the function $\Phi : [0, \infty) \rightarrow \mathbb{R}$ as follows

- (c1) Smoothness and mild growth at $z = \infty$. We assume $\Phi \in C([0, \infty)) \cap C^\infty((0, \infty))$, and all derivatives of Φ are polynomially bounded as $z \rightarrow \infty$: namely for some $z_0 \geq 2$, it holds that for all $k \geq 0$,

$$|\partial^k \Phi(z)| \leq C_{\Phi, k, z_0} z^{n_k}, \quad \forall z \geq z_0, \tag{1.13}$$

where $C_{\Phi, k, z_0} > 0$ is a constant depending only on (Φ, k, z_0) , and n_k depends on k .

- (c2) Mild singularity at $z = 0$. For some $0 < \varepsilon_0 \leq \frac{1}{2}$,

$$\sum_{j=0}^{d+1} \int_0^{\varepsilon_0} |z|^j |(\partial_z^j \Phi)(z)| \cdot \frac{dz}{z} < \infty. \tag{1.14}$$

- (c3) Non-degeneracy at $z = 1$. We assume $\Phi(1) \neq 0$ and $\Phi(t) \neq \Phi(1)$ for any $t \in (0, 1) \cup (1, \infty)$. Furthermore we assume for some integer $j_0 \geq 1$, it holds that

$$\Phi^{(j)}(1) = 0, \quad 1 \leq j \leq j_0 - 1, \quad \text{and} \quad \Phi^{(j_0)}(1) \neq 0. \tag{1.15}$$

If $j_0 = 1$, we simply require $\Phi'(1) \neq 0$.

Theorem 1.2 (Classification of general Helmholtz, case $\Phi^{(j_0)}(1) \neq 0$). Let $d \geq 1$. Suppose $u \in L^\infty(\mathbb{R}^d)$ solves the equation

$$\langle u, \Phi(-\Delta)\psi \rangle = \langle u, \Phi(1)\psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d); \tag{1.16}$$

where $\Phi : [0, \infty) \rightarrow \mathbb{R}$ satisfies the conditions (c1)–(c3) (as specified in (1.13)–(1.15)). Then the following hold.

We have $\text{supp}(\widehat{u}) \subset \{\xi : |\xi| = 1\}$, and

$$\langle \widehat{u}, \phi \rangle = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \setminus \{\xi : |\xi| = 1\}). \tag{1.17}$$

Furthermore we have in this case (below j_0 is the same integer as in (1.15)),

$$\left\langle u, \mathcal{F}^{-1}\left(\left(|\xi|^2 - 1\right)^{j_0} \psi(\xi)\right)\right\rangle = 0, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{1.18}$$

In yet other words, $u \in L^\infty(\mathbb{R}^d)$ can be identified as a C^∞ function and

$$(-\Delta - 1)^{j_0} u = 0. \tag{1.19}$$

In the following subsection we fix some notation used throughout this paper.

Notation

For any two nonnegative quantities X and Y , we write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some harmless constant $C > 0$. We write $X \ll Y$ or $Y \gg X$ if $X \leq cY$ for some sufficiently small constant $c > 0$. The needed smallness of the constant c is usually clear from the context.

We denote by $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d \rightarrow \mathbb{C})$ the usual space of complex-valued Schwartz functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

For $u \in \mathcal{S}(\mathbb{R}^d)$, we adopt the following convention for Fourier transform:

$$(\mathcal{F}u)(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^d} u(y)e^{-iy \cdot \xi} dy \quad \text{and} \quad u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{u}(\xi)e^{i\xi \cdot x} d\xi =: (\mathcal{F}^{-1}\widehat{u})(x).$$

Let $s > 0$. For $u \in \mathcal{S}(\mathbb{R}^d)$, $d \geq 1$, the fractional Laplacian $\Lambda^s u = (-\Delta)^{\frac{s}{2}} u$ is defined via Fourier transform as

$$\widehat{\Lambda^s u}(\xi) = |\xi|^s \widehat{u}(\xi), \quad \xi \in \mathbb{R}^d.$$

For $f_1 : \mathbb{R}^d \rightarrow \mathbb{C}$, $f_2 : \mathbb{R}^d \rightarrow \mathbb{C}$, f_1, f_2 Schwartz, we denote the usual L^2 pairing:

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1(x) \overline{f_2(x)} dx, \tag{1.20}$$

where \bar{z} denotes the usual complex conjugate of $z \in \mathbb{C}$. The usual Plancherel formula reads

$$\langle \widehat{f}_1, \widehat{f}_2 \rangle = (2\pi)^d \langle f_1, f_2 \rangle.$$

If we denote $f_3 = \widehat{f}_2$, then $f_2 = \mathcal{F}^{-1}(f_3)$. Thus for $f_1, f_3 \in \mathcal{S}(\mathbb{R}^d)$, it holds that

$$\langle \widehat{f}_1, f_3 \rangle = (2\pi)^d \langle f_1, \mathcal{F}^{-1}(f_3) \rangle.$$

More generally for tempered distribution u , we have

$$\langle \widehat{u}, \phi \rangle = (2\pi)^d \langle u, \mathcal{F}^{-1}(\phi) \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

2. Proof of Proposition 1.1

Proof of Proposition 1.1. We begin by noting that in the regime $t \gtrsim 1$, the function $\Phi(t)$ along with its derivatives grow at most polynomially. Let $\chi_{|\xi| \gtrsim 1}$ be a smooth cut-off function localized to the regime $|\xi| \gtrsim 1$. Since $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^d)$, it is not difficult to check that $\mathcal{F}^{-1}(\Phi(|\xi|^2)\chi_{|\xi| \gtrsim 1}\widehat{\psi}) \in L_x^1(\mathbb{R}^d)$ and

$$\|\mathcal{F}^{-1}(\Phi(|\xi|^2)\chi_{|\xi| \gtrsim 1}\widehat{\psi})\|_{L_x^1(\mathbb{R}^d)} \lesssim \sum_{|\alpha| \leq k_2} \|(1 + |x|^2)^{k_1} \partial^\alpha \psi\|_{L^\infty(\mathbb{R}^d)}, \tag{2.1}$$

where $k_1 \geq 0, k_2 \geq 0$ are integers.

It suffices for us to examine the piece

$$\beta(x) = \int_{\mathbb{R}^d} \chi(\xi) \Phi(|\xi|^2) \widehat{\psi}(\xi) e^{i\xi \cdot x} d\xi, \tag{2.2}$$

where $\chi \in C_c^\infty(\mathbb{R}^d)$ is a radial bump function localized to $\{\xi : |\xi| \ll 1\}$. Clearly

$$\|\beta\|_{L_x^1(\mathbb{R}^d)} \lesssim \|\beta_1\|_{L_x^1(\mathbb{R}^d)} \|\psi\|_{L_x^1(\mathbb{R}^d)}, \tag{2.3}$$

where

$$\beta_1(x) = \int_{\mathbb{R}^d} \chi(\xi) \Phi(|\xi|^2) e^{i\xi \cdot x} d\xi. \tag{2.4}$$

Thus to finish the proof of Proposition 1.1, we only need to prove the next proposition. \square

Proposition 2.1. *Let ε_0 be a small positive number such that χ is supported in $\{x \in \mathbb{R}^d \mid |x| \leq \varepsilon_0\}$. Then we have*

$$\left\| \mathcal{F}^{-1}(\Phi(|\xi|^2)\chi(\xi)) \right\|_{L_x^1(\mathbb{R}^d)} \lesssim \sum_{j=0}^{d+1} \int_0^{\varepsilon_0} |z|^j |(\partial_z^j \Phi)(z)| \cdot \frac{dz}{z}. \tag{2.5}$$

Proof of Proposition 2.1. The case for dimension $d = 1$ is left to the reader as an exercise.

We now consider for example for $d = 3$, denoting $r = |x| \geq r_0 \gg 1$ and $\rho = |\xi|$, we have (below we slightly abuse the notation and still denote $\chi(\rho) = \chi(\xi)$)

$$\begin{aligned}
 |\beta_1(x)| &\lesssim \left| \int_0^{\varepsilon_0} \chi(\rho)\Phi(\rho^2) \frac{\sin \rho r}{\rho r} \rho^2 d\rho \right| \\
 &\lesssim \underbrace{\left| \int_0^{\varepsilon_0} \chi(\rho)\chi_1(r\rho)\Phi(\rho^2) \frac{\sin \rho r}{\rho r} \rho^2 d\rho \right|}_{=:I_1(r)} + \underbrace{\left| \int_0^{\varepsilon_0} \chi(\rho)(1 - \chi_1(r\rho))\Phi(\rho^2) \frac{\sin \rho r}{\rho r} \rho^2 d\rho \right|}_{=:I_2(r)}.
 \end{aligned} \tag{2.6}$$

In the above $\chi_1 \in C_c^\infty(\mathbb{R})$ is an even function such that $\chi_1(z) = 1$ for $|z| \leq 0.9$ and $\chi_1(z) = 0$ for $|z| \geq 1$. Clearly

$$\int_{r_0}^\infty |I_1(r)|r^2 dr \lesssim \int_0^{\varepsilon_0} \chi(\rho)|\Phi(\rho^2)|\rho^2 \left(\int_{r \lesssim \rho^{-1}} r^2 dr \right) d\rho \lesssim \int_0^{\varepsilon_0} |\Phi(\rho^2)| \frac{d\rho}{\rho} \lesssim \int_0^{\varepsilon_0^2} |\Phi(z)|z^{-1} dz. \tag{2.7}$$

On the other hand,

$$I_2(r) = \frac{1}{r} \int_0^{\varepsilon_0} \rho \chi(\rho)(1 - \chi_1(r\rho))\Phi(\rho^2) \sin(\rho r) d\rho. \tag{2.8}$$

By using successive integration by parts, we have

$$|I_2(r)| \lesssim \frac{1}{r^4} \int_0^{\varepsilon_0} \left| \frac{d^3}{d\rho^3} \left(\rho \chi(\rho)(1 - \chi_1(r\rho))\Phi(\rho^2) \right) \right| d\rho. \tag{2.9}$$

Thus

$$\int_{r_0}^\infty |I_2(r)|r^2 dr \lesssim \text{R.H.S. of (2.5)}. \tag{2.10}$$

Adding the estimates for $I_1(r)$ and $I_2(r)$, we obtain the proof for the case $d = 3$.

For the general case $d \geq 2$, we only need to work with the expression

$$\int_{r_0}^\infty \left| \int_0^{\varepsilon_0} \chi(\rho)\Phi(\rho^2)F_d(\rho r)\rho^{d-1} d\rho \right| r^{d-1} dr, \tag{2.11}$$

where

$$F_d(\lambda) = \int_{\mathbb{S}^{d-1}} e^{i\lambda e_1 \cdot \omega} d\sigma(\omega).$$

In Subsection 2.1, we collect some standard material on the Bessel functions and some needed auxiliary estimates on the function $F_d(\lambda)$.

The regime $\rho r \lesssim 1$ is clearly under control, i.e.

$$\int_{r_0}^{\infty} \left| \int_0^{\varepsilon_0} \chi(\rho) \Phi(\rho^2) F_d(\rho r) \chi_{r\rho \lesssim 1} \rho^{d-1} d\rho \right| r^{d-1} dr \lesssim \int_0^{\varepsilon_0} \rho^{-1} \chi(\rho) |\Phi(\rho^2)| d\rho \lesssim \int_0^{\varepsilon_0^2} |\Phi(z)| z^{-1} dz. \tag{2.12}$$

On the other hand, for $\lambda = \rho r \gg 1$, we note that by (2.18) and taking K large

$$\left| F_d(\lambda) - \text{finitely many terms of the form } \lambda^{-\alpha} e^{i\lambda} \right| \lesssim \lambda^{-K}.$$

Clearly the error term is under control:

$$\int_{r_0}^{\infty} \left| \int_0^{\varepsilon_0} \chi(\rho) |\Phi(\rho^2)| (r\rho)^{-K} \chi_{r\rho \gg 1} \rho^{d-1} d\rho \right| r^{d-1} dr \lesssim \int_0^{\varepsilon_0} \chi(\rho) |\Phi(\rho^2)| \frac{d\rho}{\rho} \lesssim \int_0^{\varepsilon_0^2} |\Phi(z)| z^{-1} dz. \tag{2.13}$$

It remains to treat the terms

$$\int_{r_0}^{\infty} \left| \int_0^{\varepsilon_0} \chi(\rho) \Phi(\rho^2) (r\rho)^{-\alpha} e^{ir\rho} \chi_{r\rho \gg 1} \rho^{d-1} d\rho \right| r^{d-1} dr. \tag{2.14}$$

One can perform successive integration by parts in much the same way as in (2.9). We omit the details. \square

2.1. Bessel functions and auxiliary estimates for the function $F_d(\lambda)$

For $\nu > -\frac{1}{2}$, we recall the following formula for the standard Bessel function J_ν

$$J_\nu(\lambda) = \frac{1}{2^\nu \Gamma(\nu + \frac{1}{2}) \sqrt{\pi}} \lambda^\nu \int_{-1}^1 e^{i\lambda t} (1 - t^2)^{\nu - \frac{1}{2}} dt, \quad \lambda > 0. \tag{2.15}$$

We shall need the well-known asymptotic formula for $J_\nu(\lambda)$ (see [7, Section 17.5] or [9]): here we assume $\nu \geq 0, \lambda \gg 1$, then

$$J_\nu(\lambda) \sim \left(\frac{2}{\pi\lambda} \right)^{\frac{1}{2}} \left(\cos \omega_\lambda \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k}(\nu)}{\lambda^{2k}} - \sin \omega_\lambda \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k+1}(\nu)}{\lambda^{2k+1}} \right), \tag{2.16}$$

where

$$\omega_\lambda = \lambda - \frac{1}{2}v\pi - \frac{1}{4}\pi, \quad \text{and} \quad a_k(v) = \frac{(4v^2 - 1)(4v^2 - 3^2) \dots (4v^2 - (2k - 1)^2)}{k!8^k}.$$

In particular, for any integer $K \geq 1$, we have

$$\left| J_\nu(\lambda) - \lambda^{-\frac{1}{2}} \cos \lambda \sum_{j=0}^K \alpha_{j,\nu} \lambda^{-j} - \lambda^{-\frac{1}{2}} \sin \lambda \sum_{j=0}^K \beta_{j,\nu} \lambda^{-j} \right| \leq C_{K,\nu} \lambda^{-K-\frac{3}{2}}, \quad \forall \lambda \geq 10, \tag{2.17}$$

where $C_{K,\nu} > 0$ depends on (K, ν) , and $\alpha_{j,\nu}, \beta_{j,\nu}$ are computable coefficients.

Consider dimension $d \geq 2$ and denote by $d\sigma = d\sigma(\omega)$ the standard spherical measure on the unit-sphere $\mathbb{S}^{d-1} = \{\omega \in \mathbb{R}^d : |\omega| = 1\}$. Denote $e_1 = (1, 0, \dots, 0)^T$. Then for $\lambda > 0$,

$$\begin{aligned} F_d(\lambda) &= \int_{\mathbb{S}^{d-1}} e^{i\lambda\omega \cdot e_1} d\sigma(\omega) = c_d^{(1)} \int_0^\pi e^{i\lambda \cos \phi_1} \sin^{d-2} \phi_1 d\phi_1 \\ &= c_d^{(2)} \int_{-1}^1 e^{i\lambda t} (1 - t^2)^{\frac{d-3}{2}} dt = c_d^{(3)} \lambda^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\lambda), \end{aligned}$$

where $c_d^{(1)} > 0, c_d^{(2)} > 0, c_d^{(3)} > 0$ are constants depending only on the dimension d .

By (2.17), we obtain (below $a_{j,d}, b_{j,d}$ are coefficients) for any integer $K \geq 1$

$$\left| F_d(\lambda) - \lambda^{-\frac{d-1}{2}} \cos \lambda \sum_{j=0}^K a_{j,d} \lambda^{-j} - \lambda^{-\frac{d-1}{2}} \sin \lambda \sum_{j=0}^K b_{j,d} \lambda^{-j} \right| \leq \tilde{C}_{K,d} \lambda^{-K-\frac{d+1}{2}}, \quad \forall \lambda \geq 10, \tag{2.18}$$

where $\tilde{C}_{K,d} > 0$ depends only on (K, d) .

3. Proof of Theorem 1.1 and Theorem 1.2

To prove Theorem 1.1, we only need to prove the following theorem.

Theorem 3.1. *Let Φ satisfy the conditions (a)–(c) (see (1.5)–(1.6)). Suppose $u \in L^\infty(\mathbb{R}^d)$ and satisfy*

$$\left\langle u, \mathcal{F}^{-1} \left((\Phi(|\xi|^2) - \Phi(1))\phi(\xi) \right) \right\rangle = 0, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d). \tag{3.1}$$

Then the following hold:

(1) $\text{supp}(\widehat{u}) \subset K = \{\xi : \xi = 0 \text{ or } |\xi| = 1\}$. More precisely we have

$$\langle \widehat{u}, \phi \rangle = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \setminus K). \tag{3.2}$$

(2) If $\Phi(1) \neq 0$, then $\text{supp}(\widehat{u}) \subset \{\xi : |\xi| = 1\}$, and

$$\langle \widehat{u}, \phi \rangle = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \setminus \{\xi : |\xi| = 1\}). \tag{3.3}$$

Furthermore we have in this case,

$$\left\langle u, \mathcal{F}^{-1}\left((|\xi|^2 - 1)\psi(\xi)\right) \right\rangle = 0, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{3.4}$$

Proof. We sketch the details.

(1) Consider $\phi \in C_c^\infty(\mathbb{R}^d \setminus \{\xi = 0 \text{ or } |\xi| = 1\})$. Clearly we have the decomposition

$$\phi = \phi_1 + \phi_2,$$

where $\phi_1 \in C_c^\infty(\{\xi : 0 < |\xi| < 1\})$ and $\phi_2 \in C_c^\infty(\{\xi : |\xi| > 1\})$. With no loss we may assume that $\phi_1 \in C_c^\infty(\{\xi : \delta_1 < |\xi| < 1 - \delta_1\})$ and $\phi_2 \in C_c^\infty(\{\xi : 1 + \delta_1 < |\xi| < \frac{1}{\delta_1}\})$ for some $\delta_1 > 0$ sufficiently small. This assumption is harmless since ϕ_1 and ϕ_2 are both compactly supported.

By our assumption (c) on the function Φ , we have

$$\sup_{\substack{\delta_1 < |\xi| < 1 - \delta_1 \\ \text{or } 1 + \delta_1 < |\xi| < \frac{1}{\delta_1}}} \frac{1}{|\Phi(|\xi|^2) - \Phi(1)|} \lesssim 1. \tag{3.5}$$

This is because Φ is smooth and $\Phi(t) \neq \Phi(1)$ for any $t \in (0, 1) \cup (1, \infty)$.

It is then not difficult to check that

$$\frac{\phi_1(\xi)}{\Phi(|\xi|^2) - \Phi(1)} \in C_c^\infty(\{\xi : 0 < |\xi| < 1\}), \quad \frac{\phi_2(\xi)}{\Phi(|\xi|^2) - \Phi(1)} \in C_c^\infty(\{\xi : |\xi| > 1\}).$$

Then

$$\begin{aligned} \langle \widehat{u}, \phi_1 \rangle &= \left\langle \widehat{u}, (\Phi(|\xi|^2) - \Phi(1)) \cdot \frac{\phi_1(\xi)}{\Phi(|\xi|^2) - \Phi(1)} \right\rangle = 0; \\ \langle \widehat{u}, \phi_2 \rangle &= \left\langle \widehat{u}, (\Phi(|\xi|^2) - \Phi(1)) \cdot \frac{\phi_2(\xi)}{\Phi(|\xi|^2) - \Phi(1)} \right\rangle = 0. \end{aligned}$$

Thus (3.2) holds.

(2). The case $\Phi(1) \neq 0$. Choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(z) = 1$ for $|z| \leq \frac{1}{2}$ and $\chi(z) = 0$ for $|z| \geq 1$. Clearly

$$\begin{aligned} \left\langle u, \mathcal{F}^{-1}\left((\Phi(|\xi|^2) - \Phi(1))\chi\left(\frac{\xi}{\varepsilon}\right)\psi(\xi)\right) \right\rangle &= 0; \quad (\text{by (3.1)}) \\ \lim_{\varepsilon \rightarrow 0} \left\langle u, \mathcal{F}^{-1}\left(\Phi(|\xi|^2)\chi\left(\frac{\xi}{\varepsilon}\right)\psi(\xi)\right) \right\rangle &= 0; \quad (\text{by Proposition 2.1}) \\ \lim_{\varepsilon \rightarrow 0} \left\langle u, \mathcal{F}^{-1}\left(|\xi|^2\chi\left(\frac{\xi}{\varepsilon}\right)\psi(\xi)\right) \right\rangle &= 0. \quad (\text{obvious}) \end{aligned}$$

A suitable linear combination of the above yields (here we use $\Phi(1) \neq 0$)

$$\lim_{\varepsilon \rightarrow 0} \left\langle u, \mathcal{F}^{-1} \left((|\xi|^2 - 1) \chi \left(\frac{\xi}{\varepsilon} \right) \psi(\xi) \right) \right\rangle = 0.$$

On the other hand, it is not difficult to check that

$$\lim_{\varepsilon \rightarrow 0} \left\langle u, \mathcal{F}^{-1} \left((|\xi|^2 - 1) (1 - \chi(\varepsilon\xi)) \psi(\xi) \right) \right\rangle = 0.$$

Note that $\left(1 - \chi \left(\frac{\xi}{\varepsilon} \right)\right) (1 - \chi(\varepsilon\xi)) = 1 - \chi(\varepsilon\xi)$. Thus

$$\lim_{\varepsilon \rightarrow 0} \left\langle u, \mathcal{F}^{-1} \left((|\xi|^2 - 1) \left(1 - \chi \left(\frac{\xi}{\varepsilon} \right)\right) (1 - \chi(\varepsilon\xi)) \psi(\xi) \right) \right\rangle = 0. \tag{3.6}$$

We now only need to check for each small $\varepsilon > 0$ the identity

$$\left\langle u, \mathcal{F}^{-1} \left((|\xi|^2 - 1) \underbrace{\left(1 - \chi \left(\frac{\xi}{\varepsilon} \right)\right) \chi(\varepsilon\xi)}_{\psi_\varepsilon} \psi(\xi) \right) \right\rangle = 0. \tag{3.7}$$

Observe that $\psi_\varepsilon \in C_c^\infty$ and

$$(|\xi|^2 - 1) \psi_\varepsilon(\xi) = (\Phi(|\xi|^2) - \Phi(1)) \cdot \underbrace{\frac{|\xi|^2 - 1}{\Phi(|\xi|^2) - \Phi(1)}}_{\in C_c^\infty(\mathbb{R}^d)} \psi_\varepsilon(\xi). \tag{3.8}$$

Here we use the crucial assumption (c) on Φ , namely: 1) near $|\xi| = 1$, $\Phi'(1) \neq 0$; 2) away from $|\xi| = 1$ (and in a compact neighborhood of $|\xi| = 1$), $|\Phi(|\xi|^2) - \Phi(1)| \gtrsim 1$. These two facts yield that $\frac{|\xi|^2 - 1}{\Phi(|\xi|^2) - \Phi(1)}$ can be defined as a smooth function in the whole neighborhood of $|\xi| = 1$.

Thus (3.7) holds and we have

$$\left\langle u, \mathcal{F}^{-1} \left((|\xi|^2 - 1) \psi(\xi) \right) \right\rangle = 0, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{3.9}$$

The statement (3.3) can be proved along similar lines. We omit the details. \square

Proof of Theorem 1.2. The main modification is in (3.8):

$$(|\xi|^2 - 1)^{j_0} \psi_\varepsilon(\xi) = (\Phi(|\xi|^2) - \Phi(1)) \cdot \underbrace{\frac{(|\xi|^2 - 1)^{j_0}}{\Phi(|\xi|^2) - \Phi(1)}}_{\in C_c^\infty(\mathbb{R}^d)} \psi_\varepsilon(\xi). \tag{3.10}$$

Clearly the result follows. \square

4. Connection with the Bernstein Helmholtz case in [4]

We now show that in the more general Bernstein Helmholtz case introduced in [4], the conditions on $\Phi(\lambda)$ in [4] are stronger than our conditions on the function $\Phi(\lambda)$.

Recall that in [4], one assumes that $\Phi(\lambda)$ is a complete Bernstein function, and in the corresponding harmonic extension problem, the weight function $a(t) \in A_2$ (in particular a is locally integrable) and satisfies $a(t) \sim t^\alpha$ for $t \gg 1$ where $|\alpha| < 1$.

We shall show that in [4], as long as Φ is complete Bernstein, a is weakly integrable, $a(t) \sim t^\alpha$ for $t \gg 1$ where $|\alpha| < 1$, then such Φ will satisfy our conditions (a)–(c) (see (1.5)–(1.6)) with the property $\Phi(1) \neq 0$.

1) Since $\Phi(\cdot) : [0, \infty) \rightarrow [0, \infty)$ is complete Bernstein, we have

$$\Phi(\lambda) = c_1 + c_2\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + s} \frac{m(ds)}{s}, \tag{4.1}$$

for some constants $c_1 \geq 0, c_2 \geq 0$, and the nonnegative measure m satisfies

$$\int_{(0,\infty)} \frac{1}{1 + s} \frac{m(ds)}{s} < \infty. \tag{4.2}$$

Clearly $\Phi \in C^\infty((0, \infty)) \cap C([0, \infty))$. It is easy to check that (1.5) holds.

2) We check (1.6). Although in general the exact profile of the Krein correspondence $\Phi \leftrightarrow a$ is hard to determine, we can work out the asymptotic information via the quadratic form inequality (cf. Theorem II of [8], note that the convention of Fourier transform therein differs from ours by a constant): namely

$$\int_0^\infty \int_{\mathbb{R}} a(t) (|\partial_t u|^2 + |\partial_x u|^2) dx dt \geq \text{const} \cdot \int_{\mathbb{R}} \Phi(|\xi|^2) |\widehat{f}(\xi)|^2 d\xi, \tag{4.3}$$

where $u(0, x) = f(x)$. Note that the equality (with sharp constants) is achieved when u is a suitable harmonic extension of f .

Setting $\widehat{u}(t, \xi) = e^{-t|\xi|} \widehat{f}(\xi)$, we obtain

$$\int_{\mathbb{R}} \left(\int_0^\infty a(t) e^{-2t|\xi|} dt \right) |\xi|^2 |\widehat{f}(\xi)|^2 d\xi \gtrsim \int_{\mathbb{R}} \Phi(|\xi|^2) |\widehat{f}(\xi)|^2 d\xi. \tag{4.4}$$

Since by assumption $a(t) \sim t^\alpha$ for $t \gg 1$ and $|\alpha| < 1$, we have (below $R_0 \gg 1$ is the constant for which $a(t) \sim t^\alpha$ when $t \geq R_0$)

$$\int_0^{R_0} a(t) dt \int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi + \int_{\mathbb{R}} |\xi|^{1-\alpha} |\widehat{f}(\xi)|^2 d\xi \gtrsim \int_{\mathbb{R}} \Phi(|\xi|^2) |\widehat{f}(\xi)|^2 d\xi. \tag{4.5}$$

Since by assumption a is locally integrable, we have $\int_0^{R_0} a(t)dt \lesssim 1$. Since $|\alpha| < 1$, we have for all \widehat{f} with support in $\{\xi : |\xi| < 1\}$,

$$\int_{|\xi|<1} |\xi|^{1-\alpha} |\widehat{f}(\xi)|^2 d\xi \gtrsim \int_{|\xi|<1} \Phi(|\xi|^2) |\widehat{f}(\xi)|^2 d\xi. \tag{4.6}$$

By choosing suitable $\widehat{f}(\xi) \sim |\xi|^{-1+\delta}$ ($2\delta > \alpha$) when $|\xi| \ll 1$, we obtain

$$\int_{|\xi|\ll 1} \Phi(|\xi|^2) |\xi|^{-2+2\delta} d\xi \lesssim 1.$$

Here we note that $\alpha < 1$, and we can choose $2\delta = 1 - \eta$ for some $\eta > 0$ sufficiently small. This easily implies

$$\int_0^{\varepsilon_0} |\Phi(z)||z|^{-1} dz \lesssim 1.$$

Note that here we actually proved $c_1 = 0$ in (4.1). By (4.1), we have for $\lambda > 0$,

$$\Phi(\lambda) = c_1 + c_2\lambda + \int_{(0,\infty)} \left(1 - \frac{s}{\lambda + s}\right) \frac{m(ds)}{s}; \tag{4.7}$$

$$\Phi'(\lambda) = c_2 + \int_{(0,\infty)} \frac{s}{(\lambda + s)^2} \frac{m(ds)}{s}; \tag{4.8}$$

$$\lambda|\Phi'(\lambda)| \leq c_2\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + s} \frac{s}{\lambda + s} \frac{m(ds)}{s} \leq \Phi(\lambda). \tag{4.9}$$

Similar estimates hold for higher derivatives. Thus (1.6) holds.

3). We check the uni-valence of Φ at $\Phi(1)$. First we show $\Phi(1) \neq 0$. By (4.1), we have

$$\Phi(1) = c_1 + c_2 + \int_{(0,\infty)} \frac{1}{1 + s} \frac{m(ds)}{s}. \tag{4.10}$$

If $\Phi(1) = 0$, then $c_1 = c_2 = 0$, and $\int_{(0,\infty)} \frac{1}{1+s} \frac{m(ds)}{s} = 0$. It follows that $\Phi \equiv 0$ which contradicts to the assumption that $a(t) \sim t^\alpha$ for $t \gg 1$.

Next we show $\Phi'(1) \neq 0$. Suppose $\Phi'(1) = 0$. By (4.8), we obtain

$$c_2 = 0, \quad \int_{(0,\infty)} \frac{1}{(1 + s)^2} m(ds) = 0. \tag{4.11}$$

By using Lebesgue monotone convergence, we have

$$\int_{(0,\infty)} \frac{1}{(1+s)} \frac{m(ds)}{s} = \lim_{\varepsilon \rightarrow 0^+} \int_{(\varepsilon,\infty)} \frac{1}{1+s} \frac{m(ds)}{s} = 0.$$

This implies that $\Phi \equiv c_1 = \Phi(0) = 0$. Thus we rule out this possibility and conclude $\Phi'(1) \neq 0$.

Finally we observe that $\Phi'(1) \neq 0$ and $\Phi \in C^\infty((0, \infty))$. Clearly Φ is strictly monotone near $\lambda = 1$. By monotonicity we have for $\delta_1 > 0$ sufficiently small,

$$\max_{0 \leq \lambda \leq 1 - \delta_1} \Phi(\lambda) \leq \Phi(1 - \delta_1) < \Phi(1), \quad \Phi(1) < \Phi(1 + \delta_1) \leq \Phi(\lambda), \quad \forall \lambda \geq 1 + \delta_1. \tag{4.12}$$

Thus Φ satisfies our condition (c).

Data availability

No data was used for the research described in the article.

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