Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2022-0003s

Published online 11 July 2022

Stability of Spike Solutions to the Fractional Gierer-Meinhardt System in a One-Dimensional Domain

Daniel Gomez[1,*], Jun-Cheng Wei[2] and Wen Yang[3]

[1] *Department of Mathematics, University of Pennsylvania, Philadelphia, PA USA*
[2] *Department of Mathematics, The University of British Columbia, Vancouver, BC Canada*
[3] *Wuhan Institute of Physics and Mathematics, Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, Wuhan 430071, P.R. China*

Received 7 December 2020; Accepted (in revised version) 21 November 2021

Abstract. In this paper we consider the existence and stability of multi-spike solutions to the fractional Gierer-Meinhardt model with periodic boundary conditions. In particular we rigorously prove the existence of symmetric and asymmetric two-spike solutions using a Lyapunov-Schmidt reduction. The linear stability of these two-spike solutions is then rigorously analyzed and found to be determined by the eigenvalues of a certain 2×2 matrix. Our rigorous results are complemented by formal calculations of N-spike solutions using the method of matched asymptotic expansions. In addition, we explicitly consider examples of one- and two-spike solutions for which we numerically calculate their relevant existence and stability thresholds. By considering a one-spike solution we determine that the introduction of fractional diffusion for the activator or inhibitor will respectively destabilize or stabilize a single spike solution with respect to oscillatory instabilities. Furthermore, when considering two-spike solutions we find that the range of parameter values for which asymmetric two-spike solutions exist and for which symmetric two-spike solutions are stable with respect to competition instabilities is expanded with the introduction of fractional inhibitor diffusivity. However our calculations indicate that asymmetric two-spike solutions are always linearly unstable.

AMS subject classifications: 35R11, 35B32, 60K50, 35B25

Key words: Gierer-Meinhardt system, eigenvalue, stability, fractional Laplacian, localized solution.

1. Introduction

The Gierer-Meinhardt (GM) model is a prototypical activator-inhibitor reaction-

*Corresponding author. *Email addresses:* d1gomez@sas.upenn.edu (D. Gomez), jcwei@math.ubc.ca (J. Wei), wyang@wipm.ac.cn (W. Yang)

diffusion system that has, since its introduction by Gierer and Meinhardt in 1972 [5], been the focus of numerous mathematical studies. In the singularly perturbed limit for which the activator has an asymptotically small diffusivity the GM model is known to exhibit localized solutions in which the activator concentrates at a discrete collection of points and is otherwise exponentially small. The analysis, both rigorous and formal, of the existence, structure, and linear stability of such localized solutions has been the focus of numerous studies over the last two decades (see the book [32]). The GM model in a one-dimensional domain has been particularly well studied using both rigorous PDE methods [28, 31] as well as formal asymptotic methods [13, 26]. More recent extensions to the classical one-dimensional GM model have considered the effects of precursors [15,34], bulk-membrane-coupling [7], and anomalous diffusion [20,21,33]. It is the latter of these extensions which motivates the following paper which focuses on extending the results obtained in [20, 33] for the fractional one-dimensional GM model.

The analysis of localized solutions to the GM model fits more broadly into the study of pattern formation in reaction-diffusion systems. Such reaction-diffusion systems have widespread applicability in the modelling of biological phenomena for which distinct agents diffuse while simultaneously undergoing prescribed reaction kinetics (see the classic textbook by Murray [19]). While these models have typically assumed a normal (or Brownian) diffusion process for which the mean-squared-displacement (MSD) is proportional to the elapsed time, a growing body of literature has considered the alternative of anomalous diffusion which may be better suited for biological processes in complex environments [18, 23, 24] (see also [1, Section 7.1]). In contrast to normal diffusion, for anomalous diffusion the MSD and time are related by the power law MSD $\propto$ (time)$^\alpha$ where an exponent satisfying $\alpha > 1$ or $\alpha < 1$ corresponds to superdiffusion or subdiffusion respectively. Studies of reaction-diffusion systems with subdiffusion and superdiffusion suggest that anomalous diffusion can have a pronounced impact on pattern formation (see [14] as well as [6] and the references therein). In particular studies have shown that both superdiffusion and subdiffusion can reduce the threshold for Turing instabilities when compared to the same systems with normal diffusion [6, 11]. Likewise it has been shown that the Hopf bifurcation threshold for spike solutions to the GM model with normal diffusion for the inhibitor and superdiffusion, mainly with Lévy flights, for the activator is decreased [20] whereas it is increased in the case of subdiffusion for the inhibitor and normal diffusion for the activator [21].

In this paper we consider the existence and stability of localized multi-spike solutions to the periodic one-dimensional GM model with Lévy flights for both the activator and the inhibitor. In particular we consider the fractional Gierer-Meinhardt system with periodic boundary conditions

$$\begin{cases} u_t + \varepsilon^{2s_1}(-\Delta)^{s_1}u + u - \dfrac{u^2}{v} = 0 & \text{for} \quad x \in (-1,1), \\ \tau v_t + D(-\Delta)^{s_2}v + v - u^2 = 0 & \text{for} \quad x \in (-1,1), \\ u(x) = u(x+2), \;\; v(x) = v(x+2) & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ and the parameters $0 < D < \infty$ and $\tau \geq 0$ are independent of ε. We assume the exponents satisfy $\frac{1}{4} < s_1 < 1$ and $\frac{1}{2} < s_2 < 1$. The (nonlocal) fractional Laplacian $(-\Delta)^s$ replaces the classical Laplacian as the infinitesimal generator of the underlying Lévy process for $s < 1$ and is defined for all 2-periodic functions by

$$(-\Delta)^s \phi(x) \equiv C_s \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(\bar{x})}{|x - \bar{x}|^{1+2s}} d\bar{x} = C_s \int_{-1}^{1} \big[\phi(x) - \phi(\bar{x})\big] K_s(x - \bar{x}) d\bar{x}, \quad \text{(1.2a)}$$

where

$$C_s \equiv \frac{2^{2s} s \Gamma(s + 1/2)}{\sqrt{\pi}\Gamma(1 - s)}, \quad K_s(z) \equiv \frac{1}{|z|^{1+2s}} + \sum_{j=1}^{\infty} \left(\frac{1}{|z + 2j|^{1+2s}} + \frac{1}{|z - 2j|^{1+2s}} \right), \quad \text{(1.2b)}$$

and for which the equality in (1.2a) follows from the periodicity of $\phi(x)$. The system (1.1) is a prototypical model in which we can study the interplay of short range activation, long range inhibition, and intermittent periods of directed motion in the underlying stochastic processes (i.e. Lévy flights). One of the goals of this paper is to investigate the interplay of these three effects in the singularly perturbed limit for which rigorous and formal methods can be used to obtain detailed descriptions of the structure and stability of localized solutions. On a more technical note the properties of certain relevant Green's functions in this fractional case lead to a connection between localized solutions in the classical Gierer-Meinhardt model in a different numbers of spatial dimensions. We remark that the system (1.1) closely resembles the system considered in [20] with the primary difference being that we consider the effects of Lévy flights for both the activator and the inhibitor.

Before outlining the structure of this paper we outline our contributions as follows. Using a Lyapunov-Schmidt type reduction we rigorously prove the existence of symmetric and asymmetric two-spike steady solutions of (1.1) satisfying

$$\begin{cases} \varepsilon^{2s_1}(-\Delta)^{s_1} u + u - \dfrac{u^2}{v} = 0 & \text{for} \quad x \in (-1, 1), \quad \text{(1.3a)} \\ D(-\Delta)^{s_2} v + v - u^2 = 0 & \text{for} \quad x \in (-1, 1), \quad \text{(1.3b)} \\ u(x) = u(x + 2), \;\; v(x) = v(x + 2) & \text{for} \quad x \in \mathbb{R}, \quad \text{(1.3c)} \end{cases}$$

and determine their linear stability by considering the spectrum of certain 2×2 matrices. In addition we use the method of matched asymptotic expansions to formally construct N-spike quasi-equilibrium solutions and derive a system of ordinary differential equations governing their slow dynamics. We furthermore illustrate the effects of anomalous diffusion on the stability of one- and two-spike solutions by calculating thresholds for oscillatory and competition instabilities. In particular our results indicate that Lévy flights for the activator and inhibitor have, respectively, a destabilizing and stabilizing effect on the stability of single spike solutions. On the other hand we demonstrate that the stability of symmetric two-spike solutions with respect to competition instabilities is independent of s and is stabilized when the inhibitor undergoes

Lévy flights. Finally, we show that asymmetric two-spike solutions are always linearly unstable with respect to competition instabilities.

The remainder of this paper is organized as follows. In Section 2 we outline the key rigorous results established in this paper pertaining to the existence and stability of two-spike solutions. Then in Section 3 we collect preliminary results which are used in the subsequent sections. In Section 4 we use the method of matched asymptotic expansions as well as full numerical simulations to illustrate the effects of fractional diffusion on the structure and stability properties of one- and two-spike solutions. We then provide proofs of the existence and stability results in Section 5 and Section 6 respectively. Finally, in Section 7 we make some concluding remarks.

2. Main results: existence and stability

In this section we state the main results of this paper, which include the existence of two spike solutions (symmetric and asymmetric) to the steady problem of the fractional Gierer-Meinhardt system and their stability. Instead of studying the system (1.3), we replace $u(x)$ by $c_\varepsilon u(x)$ and $v(x)$ by $c_\varepsilon v(x)$, and introduce the scaling $x = \varepsilon y$ for Eq. (1.3a). Then we can write system (1.3) as

$$\begin{cases} (-\Delta)_y^s u + u - \dfrac{u^2}{v} = 0 & \text{for} \quad y \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ D(-\Delta)^s v + v - c_\varepsilon u^2 = 0 & \text{for} \quad x \in (-1,1), \\ u(\varepsilon y) = u(\varepsilon y + 2), \;\; v(x) = v(x+2) & \text{for} \quad x, y \in \mathbb{R} \end{cases} \tag{2.1}$$

with

$$c_\varepsilon = \left(\varepsilon \int_{\mathbb{R}} w^2(y) dy\right)^{-1}$$

and w being the unique solution of

$$(-\Delta)^s w + w - w^2 = 0, \quad w(x) = w(-x). \tag{2.2}$$

From now on, we shall focus on Eq. (2.1) and provide its existence and stability results.

Remark 2.1. To simplify the presentation, in the proof of Theorems 2.1 and 2.2 we have restricted our attention to the case $s_1 = s_2 = s$. The arguments can be also applied for more general cases where $s_1 \in (\frac{1}{4}, 1)$ and $s_2 \in (\frac{1}{2}, 1)$.

In order to state the main results, we introduce the Green's function associated to the steady problem with periodic boundary and make three assumptions on the Green's function that will be used for the rigorous proof and stability analysis. For $z \in (-1,1)$, let $G_D(x,z)$ be the function satisfying

$$\begin{cases} D(-\Delta)^s G_D(x,z) + G_D(x,z) = \delta(x-z) & \text{for} \quad x \in (-1,1), \\ G_D(x,z) = G_D(x+2,z) & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{2.3}$$

having the Fourier series expansion

$$G_D(x,z)=\frac{1}{2}\sum_{\ell=-\infty}^{\infty}\frac{e^{i\ell\pi(x-z)}}{1+D(\ell\pi)^{2s}}=\frac{1}{2}+\sum_{\ell=1}^{\infty}\frac{\cos(\ell\pi(x-z))}{1+D(\ell\pi)^{2s}}.$$

Let $-1<p_1^0<p_2^0<1$ be 2 points in $(-1,1)$ where the spikes concentrate. We introduce several matrices for later use. For $\mathbf{p}=(p_1,p_2)\in(-1,1)^2$ we let $\mathcal{G}_D$ be the 2×2 matrix with entries

$$(\mathcal{G}_D)_{ij}=G_D(p_i,p_j). \tag{2.4}$$

Let us denote $\frac{\partial}{\partial p_i}$ as ∇_{p_i}. When $i\neq j$, we can define $\nabla_{p_i}G_D(p_i,p_j)$ in the classical way, while if $i=j$, since $G_D(x,x)$ is a constant due to the periodic boundary condition, we have $\nabla_{p_i}G_D(p_i,p_i)=0$. Next, we define the matrix associated with the first and second derivatives of $\mathcal{G}$ as follows:

$$\nabla\mathcal{G}_D(\mathbf{p})=\big(\nabla_{p_i}G_D(p_i,p_j)\big),\quad \nabla^2\mathcal{G}_D(\mathbf{p})=\big(\nabla_{p_i}\nabla_{p_j}G_D(p_i,p_j)\big). \tag{2.5}$$

We make the following two assumptions.

(H1) There exists a solution $(\hat{\xi}_1^0,\hat{\xi}_2^0)$ of the following equation:

$$\sum_{j=1}^{2}G_D\big(p_i^0,p_j^0\big)\big(\hat{\xi}_j^0\big)^2=\hat{\xi}_i^0,\quad i=1,2. \tag{2.6}$$

(H2) $\frac{1}{2}\notin\lambda(\mathcal{B})$, where $\lambda(\mathcal{B})$ is the set of eigenvalues of the 2×2 matrix $\mathcal{B}$ with entries

$$(\mathcal{B})_{ij}=G_D\big(p_i^0,p_j^0\big)\hat{\xi}_j^0. \tag{2.7}$$

By the assumption (H2) and the implicit function theorem, for $\mathbf{p}=(p_1,p_2)$ near $\mathbf{p}^0=(p_1^0,p_2^0)$, there exists a unique solution $\hat{\xi}(\mathbf{p})=(\hat{\xi}_1(\mathbf{p}),\hat{\xi}_2(\mathbf{p}))$ for the following equation:

$$\sum_{j=1}^{2}G_D(p_i,p_j)\hat{\xi}_j^2=\hat{\xi}_i,\quad i=1,2. \tag{2.8}$$

We define the following vector field:

$$F(\mathbf{p}):=\big(F_1(\mathbf{p}),F_2(\mathbf{p})\big),$$

where

$$F_i(\mathbf{p})=\sum_{j=1}^{2}\nabla_{p_i}G_D(p_i,p_j)\hat{\xi}_j^2=\sum_{j\neq i}\nabla_{p_i}G_D(p_i,p_j)\hat{\xi}_j^2,\quad i=1,2. \tag{2.9}$$

Set

$$\mathcal{M}(\mathbf{p})=\hat{\xi}_i^{-1}\nabla_{p_j}F_i(\mathbf{p}). \tag{2.10}$$

The final assumption concerns the vector field $F(\mathbf{p})$.

(H3) We assume that at $\mathbf{p}^0 = (p_1^0, p_2^0)$

$$F(\mathbf{p}^0) = 0 \quad \text{and} \quad \text{rank}\big(\mathcal{M}(\mathbf{p}^0)\big) = 1. \tag{2.11}$$

Next, let us calculate $\mathcal{M}(\mathbf{p}^0)$. Particularly, we shall show that it admits a zero eigenvalue. To compute the matrix $\mathcal{M}(\mathbf{p}^0)$, we have to derive the derivatives of $\hat{\xi}$. It is easy to see that $\hat{\xi}(\mathbf{p})$ is C^1 in $\mathbf{p}$ and from (2.8) we can calculate

$$\begin{aligned}\nabla_{p_j}\hat{\xi}_i &= 2\sum_{l=1}^{2} G_D(p_i,p_l)\hat{\xi}_l\nabla_{p_j}\hat{\xi}_l + \sum_{l=1}^{2}\nabla_{p_j}G_D(p_i,p_l)\hat{\xi}_l^2 \\ &= \begin{cases} 2\displaystyle\sum_{l=1}^{2} G_D(p_i,p_l)\hat{\xi}_l\nabla_{p_j}\hat{\xi}_l + \nabla_{p_j}G_D(p_i,p_j)\hat{\xi}_j^2, & \text{if} \quad i\neq j, \\ 2\displaystyle\sum_{l=1}^{2} G_D(p_i,p_l)\hat{\xi}_l\nabla_{p_j}\hat{\xi}_l + \sum_{l=1}^{2}\nabla_{p_j}G_D(p_i,p_l)\hat{\xi}_l^2, & \text{if} \quad i=j, \end{cases}\end{aligned} \tag{2.12}$$

where we used $\partial_{p_i}G_D(p_i,p_i) = 0$. Therefore, if we denote the matrix

$$\nabla\xi = \big(\nabla_{p_j}\hat{\xi}_i\big), \tag{2.13}$$

we have

$$\nabla\xi(\mathbf{p}) = (I - 2\mathcal{G}_D\mathcal{H})^{-1}(\nabla\mathcal{G}_D)^T\mathcal{H}^2 + \mathcal{O}\left(\sum_{j=1}^{2}|F_j(\mathbf{p})|\right), \tag{2.14}$$

where a superscript T denotes the transpose and where $\mathcal{H}$ is given by

$$\mathcal{H}(\mathbf{p}) = \big(\hat{\xi}_i(\mathbf{p})\delta_{ij}\big). \tag{2.15}$$

Let

$$\mathcal{Q} = (q_{ij}) = \left(\nabla_{p_i}\nabla_{p_j}G_D(p_1,p_2)\sum_{l\neq i}\frac{\hat{\xi}_l^2}{\hat{\xi}_i^2}\delta_{ij}\right). \tag{2.16}$$

We can compute $\mathcal{M}(\mathbf{p}^0)$ by using (2.12),

$$\mathcal{M}(\mathbf{p}^0) = \mathcal{H}^{-1}\big(\nabla^2\mathcal{G}_D + \mathcal{Q}\big)\mathcal{H}^2 + 2\mathcal{H}^{-1}\nabla\mathcal{G}_D\mathcal{H}\big(I - 2\mathcal{G}_D\mathcal{H}\big)^{-1}\big(\nabla\mathcal{G}_D\big)^T\mathcal{H}^2, \tag{2.17}$$

where A^T means the transpose of A. Using (2.11), we can further simplify the matrix $\mathcal{M}(\mathbf{p}^0)$ as the following:

$$\mathcal{M}(\mathbf{p}^0) = \begin{pmatrix} \big(\hat{\xi}_1^0\big)^{-1}\nabla_{p_1}\nabla_{p_1}G_D(p_1,p_2)\big(\hat{\xi}_2^0\big)^2 & \big(\hat{\xi}_1^0\big)^{-1}\nabla_{p_2}\nabla_{p_1}G_D(p_1,p_2)\big(\hat{\xi}_2^0\big)^2 \\ \big(\hat{\xi}_2^0\big)^{-1}\nabla_{p_1}\nabla_{p_2}G_D(p_2,p_1)\big(\hat{\xi}_1^0\big)^2 & \big(\hat{\xi}_2^0\big)^{-1}\nabla_{p_2}\nabla_{p_2}G_D(p_2,p_1)\big(\hat{\xi}_1^0\big)^2 \end{pmatrix}. \tag{2.18}$$

It is easy to see that the summation of both rows is zero, thus $\mathcal{M}(\mathbf{p}^0)$ is singular and admits a zero eigenvalue. While the left non-zero eigenvalue can be represented as follows:

$$\lambda_{\mathcal{M}(\mathbf{p}^0)} = \big(\hat{\xi}_1^0\big)^{-1}\nabla_{p_1}\nabla_{p_1}G_D(p_1,p_2)\big(\hat{\xi}_2^0\big)^2 + \big(\hat{\xi}_2^0\big)^{-1}\nabla_{p_2}\nabla_{p_2}G_D(p_2,p_1)\big(\hat{\xi}_1^0\big)^2. \tag{2.19}$$

Our first result is the following.

Theorem 2.1. *Assume that (H1) and (H3) are satisfied. Then for $\varepsilon \ll 1$ the problem* (1.1) *has a two-spike solution which concentrates at $p_1^\varepsilon, p_2^\varepsilon$. In addition,*

$$u_\varepsilon \sim c_\varepsilon \sum_{i=1}^{2} \hat{\xi}_i^0 w\left(\frac{x-p_i^\varepsilon}{\varepsilon}\right), \quad v_\varepsilon(p_i^\varepsilon) \sim c_\varepsilon \sum \hat{\xi}_i^0, \quad i=1,2,$$

and

$$\left(p_1^\varepsilon, p_2^\varepsilon\right) \to \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \textit{as} \quad \varepsilon \to 0.$$

Remark 2.2. In Theorem 2.1 the spike height may be the same or different yielding, respectively, symmetric and asymmetric two-spike solutions. In both cases the spike locations must satisfy $\nabla_{p_1^\varepsilon} G_D(p_1^\varepsilon, p_2^\varepsilon) = 0$ and by numerically evaluating the Green's function this implies that $|p_1^\varepsilon - p_2^\varepsilon| = 1$. As described in more detail in Section 4.4 the limiting system (2.8) can then be solved explicitly as

$$\xi_1^0 = \xi_2^0 = \frac{1}{G_D(0,0)+G_D(1,0)}, \quad \xi_1^0 = \frac{z_1}{G_D(0,0)}, \quad \xi_2^0 = \frac{z_2}{G_D(0,0)}$$

for the symmetric and asymmetric cases respectively and where z_1 and z_2 are defined in terms of $\theta = \frac{G_D(1,0)}{G_D(0,0)}$ in (4.13).

Finally, we study the stability of the 2-spikes solution constructed in Theorem 2.1.

Theorem 2.2. *Assume that $\varepsilon \ll 1$ and let $(u_\varepsilon, v_\varepsilon)$ be the solutions constructed in Theorem* 2.1 *and $\mathcal{B}$ be defined in* (2.7).

1. *If $\min_{\sigma\in\lambda(\mathcal{B})} \sigma > \frac{1}{2}$, then there exists τ_0 such that $(u_\varepsilon, v_\varepsilon)$ is linearly stable for $0 \le \tau < \tau_0$.*
2. *If $\min_{\sigma\in\lambda(\mathcal{B})} \sigma < \frac{1}{2}$, then there exists τ_0 such that $(u_\varepsilon, v_\varepsilon)$ is linearly unstable for $0 \le \tau < \tau_0$.*

Remark 2.3. We shall prove Theorem 2.2 in Section 6. Generally we have to study both large and small eigenvalue problems for the steady state. We shall see that the matrix associated with the small eigenvalues is degenerate: one eigenvalue is zero due to the translational invariance of the spike profiles. On the other hand, the other small eigenvalue is always stable. The stability of the 2-spike solution therefore depends only on the matrix $\mathcal{B}$, which naturally appears in the study of large eigenvalue problem.

Remark 2.4. The rigorous existence proof and analysis of the large eigenvalues can be extended to $N > 2$-spike solutions. However, in this case it is not clear how to determine the sign of the eigenvalues of $\mathcal{M}(\mathbf{p})$ given by (2.10) and hence the linear stability with respect to the small eigenvalues. For this reason we restrict our attention to the case of $N = 2$ spike solutions for which the linear stability can be completely rigorously determined.

Remark 2.5. The restriction $s_1 > \frac{1}{4}$ arises in the NLEP stability proof. Specifically this condition arises in the calculation following (A.15). It is not clear from our proof whether the stability of the NLEP for $\gamma > 1$ can be extended for values of $s_1 \leq \frac{1}{4}$. On the other hand the behaviour of the Green's function satisfying (2.3) as $x \to z$ is markedly different if $s_2 > \frac{1}{2}$, $s_2 = \frac{1}{2}$, or $s_2 < \frac{1}{2}$. The proceeding rigorous and formal analysis is therefore restricted to the case of $s_2 > \frac{1}{2}$.

3. Preliminaries

In this section we collect several key preliminary results needed for the existence and stability proofs in Sections 5 and 6 as well as for the formal calculations in Section 4.

Letting w be the ground state solution satisfying

$$\begin{cases} (-\Delta)^s w + w - w^2 = 0 & \text{in } \mathbb{R}, \\ w(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{3.1}$$

we have the following result [4] (also see [33, Proposition 4.1] and the references therein).

Proposition 3.1. *Eq.* (3.1) *admits a positive, radially symmetric solution satisfying the following properties:*

(a) There exists a positive constant $\mathfrak{b}_s$ depending only on s such that

$$w(x) = \frac{\mathfrak{b}_s}{|x|^{1+2s}}\big(1 + o(1)\big) \quad \textit{as} \quad |x| \to \infty.$$

Moreover $w'(x) < 0$ for $x > 0$ and

$$w'(x) = -\frac{(1+2s)\mathfrak{b}_s}{x^{2+2s}}\big(1 + o(1)\big) \quad \textit{as} \quad x \to \infty.$$

(b) Let $L_0 = (-\Delta)^s + 1 - 2w$ be the linearized operator. Then we have

$$\mathrm{Ker}(L_0) = \mathrm{span}\left\{\frac{\partial w}{\partial x}\right\}.$$

(c) The eigenvalue problem

$$(-\Delta)^s \phi + \phi - 2w\phi + \alpha\phi = 0$$

has a unique positive eigenvalue $\alpha > 0$.

Next we consider the stability of a system of nonlocal eigenvalue problems (NLEPs). We first establish the following result which we prove in Appendix A.

Theorem 3.1. *Consider the following nonlocal eigenvalue problem:*

$$(-\Delta)^s\phi+\phi-2w\phi+\gamma\int_{\mathbb{R}}w\phi dx\left(\int_{\mathbb{R}}w^2dx\right)^{-1}w^2+\alpha\phi=0. \tag{3.2}$$

(1) If $\gamma<1$, then there is an eigenvalue α to (3.2) *such that $\Re(\alpha)>0$.*

(2) If $\gamma>1$ and $s>\frac{1}{4}$, then for any nonzero eigenvalue α of (3.2)*, we have*

$$\Re(\alpha)\le -c_0<0.$$

(3) If $\gamma\neq 1$ and $\alpha=0$, then $\phi=c_0\partial_x w$ for some constant c_0.

In our application to the case when $\tau>0$, we have to deal with the situation when the coefficient γ is a function of $\tau\alpha$. Letting $\gamma=\gamma(\tau\alpha)$ be a complex function of $\tau\alpha$, we suppose that

$$\gamma(0)\in\mathbb{R},\quad |\gamma(\tau\alpha)|\le C\quad \text{for}\quad \alpha_R\ge 0,\quad \tau\ge 0, \tag{3.3}$$

where C is a generic constant independent of τ,α. Then we have the following result.

Theorem 3.2. *Consider the following nonlocal eigenvalue problem:*

$$(-\Delta)^s\phi+\phi-2w\phi+\gamma(\tau\alpha)\int_{\mathbb{R}}w\phi dx\left(\int_{\mathbb{R}}w^2dx\right)^{-1}w^2+\alpha\phi=0, \tag{3.4}$$

where $\gamma(\tau\alpha)$ satisfies (3.3)*. Then there is a small number $\tau_0>0$ such that for $\tau<\tau_0$,*

(1) if $\gamma(0)<1$, then there is a positive eigenvalue to (3.4)*;*

(2) if $\gamma(0)>1$ and $s>\frac{1}{4}$, then for any nonzero eigenvalue α of (3.4)*, we have*

$$\Re(\alpha)\le -c_0<0.$$

Proof. The above theorem follows from Theorem 3.1 by a perturbation argument. To make sure that the perturbation works, we have to show that if $\alpha_R\ge 0$ and $0<\tau<1$, then $|\alpha|\le C$, where C is a generic constant (independent of τ). In fact, multiplying (3.4) by $\bar{\phi}$ - the conjugate of ϕ - and integrating by parts, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\left|(-\Delta)^{\frac{s}{2}}\phi\right|^2+|\phi|^2-2w|\phi|^2\right)dx\\ &=-\alpha\int_{\mathbb{R}}|\phi|^2-\gamma(\tau\alpha)\int_{\mathbb{R}}w\phi dx\left(\int_{\mathbb{R}}w^2dx\right)^{-1}\int_{\mathbb{R}}w^2\overline{\phi}dx.\end{aligned} \tag{3.5}$$

From the imaginary part of (3.5), we obtain that

$$|\alpha_I|\le C_1|\gamma(\tau\alpha)|,$$

where $\alpha = \alpha_R + \sqrt{-1}\alpha_I$ and C_1 is a positive constant (independent of τ). By assumption (3.3), $|\gamma(\tau\alpha)| \leq C$ and so $|\alpha_I| \leq C$. Taking the real part of (3.5), we note that the left-hand side of (3.5) is not smaller than $C\int_{\mathbb{R}} |\phi|^2 dx$ for some $C \in \mathbb{R}$, then we obtain that $\alpha_R \leq C_2$ where C_2 is a positive constant independent of $\tau > 0$. Therefore, $|\alpha|$ is uniformly bounded and hence a perturbation argument gives the desired conclusion. □

We now consider the following system of linear operators:

$$L\Phi := (-\Delta)^s \Phi + \Phi - 2w\Phi + 2\mathcal{B}\left(\int_{\mathbb{R}} w\Phi dx\right)\left(\int_{\mathbb{R}} w^2 dx\right)^{-1} w^2, \tag{3.6}$$

where $\mathcal{B}$ is given by (2.7) and

$$\Phi := (\phi_1, \phi_2)^T \in \left(H^{2s}(\mathbb{R})\right)^2.$$

The conjugate operator of L under the scalar product in $L^2(\mathbb{R})$ is

$$L^*\Psi := (-\Delta)^s \Psi + \Psi - 2w\Psi + 2\mathcal{B}^T\left(\int_{\mathbb{R}} w^2\Psi dx\right)\left(\int_{\mathbb{R}} w^2 dx\right)^{-1} w, \tag{3.7}$$

where

$$\Psi := (\psi_1, \psi_2)^T \in \left(H^{2s}(\mathbb{R})\right)^2.$$

We then have the following result.

Lemma 3.1. *Assume that (H2) holds. Then*

$$\mathrm{Ker}(L) = \mathrm{Ker}(L^*) = X_0 \oplus X_0, \tag{3.8}$$

where $X_0 = \mathrm{Span}\{w'(x)\}$.

Proof. We first prove $\mathrm{Ker}(L) \subset X_0 \oplus X_0$. Suppose $L\Phi = 0$. By the fact that $\mathcal{G}_D$ is symmetric and $\mathcal{H}(\mathbf{p})$ is a diagonal matrix, we can diagonalize $\mathcal{B}$. Let

$$P^{-1}\mathcal{B}P = \mathcal{J},$$

where P is an orthogonal matrix and $\mathcal{J}$ is a diagonal matrix, i.e.,

$$\mathcal{J} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

with suitable real numbers $\sigma_i, i = 1, 2$. Defining $\Phi = P\tilde{\Phi}$ we have

$$(-\Delta)^s \tilde{\Phi} + \tilde{\Phi} - 2w\tilde{\Phi} + 2\left(\int_{\mathbb{R}} w^2 dx\right)^{-1}\left(\int_{\mathbb{R}} w\mathcal{J}\tilde{\Phi} dx\right) w^2 = 0. \tag{3.9}$$

For $i = 1, 2$ we look at the i-th equation of system (3.9)

$$(-\Delta)^s \tilde{\Phi}_i + \tilde{\Phi}_i - 2w\tilde{\Phi}_i + 2\sigma_i \left(\int_{\mathbb{R}} w^2 dx \right)^{-1} \left(\int_{\mathbb{R}} w\tilde{\Phi}_i dx \right) w^2 = 0. \tag{3.10}$$

By Theorem 3.1(3), Eq. (3.10) implies $\tilde{\Phi}_i \in X_0$ since by condition (H2) we know that $2\sigma_i \neq 1$.

We proceed similarly to prove $\mathrm{Ker}(L^*) \subset X_0 \oplus X_0$. Using $\sigma(\mathcal{B}) = \sigma(\mathcal{B}^T)$ the i-th equation of the diagonalized system is as follows:

$$(-\Delta)^s \tilde{\Psi}_i + \tilde{\Psi}_i - 2w\tilde{\Psi}_i + 2\sigma_i \left(\int_{\mathbb{R}} w^2 dx \right)^{-1} \left(\int_{\mathbb{R}} w^2\tilde{\Psi}_i dx \right) w = 0. \tag{3.11}$$

Multiplying the above equation by w and integrating over the real line, we obtain

$$(1 - 2\sigma_i) \int_{\mathbb{R}} w^2 \tilde{\Psi}_i = 0, \tag{3.12}$$

which together with the fact that $2\sigma_i \neq 1$ implies that

$$\int_{\mathbb{R}} w^2 \tilde{\Psi}_i = 0, \quad i = 1, 2.$$

Thus all the nonlocal terms vanish and we have $L_0\tilde{\Psi}_i = 0$ for $i = 1, 2$, which in turn implies that $\Psi_i \in X_0$ for $i = 1, 2$. On the other hand, it is obvious that $X_0 \oplus X_0 \subset \mathrm{Ker}(L)$ and $X_0 \oplus X_0 \subset \mathrm{Ker}(L^*)$. Therefore, we conclude that (3.8) holds. □

Lemma 3.2. *The operator $L : (H^{2s}(\mathbb{R}))^2 \to (L^2(\mathbb{R}))^2$ is invertible if it is restricted as follows:*

$$L : (X_0 \oplus X_0)^\perp \cap \left(H^{2s}(\mathbb{R})\right)^2 \to (X_0 \oplus X_0)^\perp \cap \left(L^2(\mathbb{R})\right)^2.$$

Moreover, L^{-1} is bounded.

Proof. This follows from the Fredholm Alternatives Theorem and Lemma 3.1. □

Finally we study the eigenvalue problem (see (3.6) for the definition of L)

$$L\Phi + \alpha\Phi = 0, \tag{3.13}$$

for which we have the following lemma.

Lemma 3.3. *Assume that all the eigenvalues of $\mathcal{B}$ are real. Then we have*

(1) If $2\min_{\sigma\in\sigma(\mathcal{B})} \sigma > 1$ then for any nonzero eigenvalue of (3.13) *we have $\Re(\alpha) \leq -c_0 < 0$.*

(2) If there exists $\sigma \in \sigma(\mathcal{B})$ such that $2\sigma < 1$, then there exists a positive eigenvalue of (3.13).

Proof. We first prove (1). Let (Φ, α) satisfy (3.13) and $2\min_{\sigma\in\sigma(\mathcal{B})}\sigma > 1$. Suppose $\alpha_R \geq 0$ and $\alpha \neq 0$. Similar to Lemma 3.2 we diagonalize (3.13)

$$(-\Delta)^s\Phi + \Phi - 2w\Phi + 2\left(\int_{\mathbb{R}} w^2 dx\right)^{-1}\left(\int_{\mathbb{R}} w\mathcal{J}\Phi\right)w^2 + \alpha\Phi = 0, \tag{3.14}$$

and the i-th equation of system (3.14) becomes

$$(-\Delta)^s\Phi_i + \Phi_i - 2w\Phi_i + 2\sigma_i\left(\int_{\mathbb{R}} w^2 dx\right)^{-1}\left(\int_{\mathbb{R}} w\Phi_i\right)w^2 + \alpha\Phi_i = 0. \tag{3.15}$$

The first conclusion follows by Theorem 3.1(2) and the fact that $2\sigma_i > 1$. We conclude that either $\Phi_1 = \Phi_2 = 0$ or $\alpha \leq -c_0 < 0$. Since Φ does not vanish and $\alpha < 0$, thus (1) is proved.

Next we prove (2) and assume that $2\sigma_i < 1$ for some $\sigma_i \in \sigma(\mathcal{B})$. Then the equation corresponding to σ_i becomes

$$(-\Delta)^s\Phi_i + \Phi_i - 2w\Phi_i + 2\sigma_i\left(\int_{\mathbb{R}} w^2\right)^{-1}\left(\int_{\mathbb{R}} w\Phi_i dx\right)w^2 + \alpha\Phi_i = 0.$$

By Theorem 3.1(1) we know that there exists an eigenvalue $\alpha_0 > 0$ and an eigenfunction Φ_0 such that

$$L_0\Phi_0 + 2\sigma_i\left(\int_{\mathbb{R}} w^2 dx\right)^{-1}\left(\int_{\mathbb{R}} w\Phi_0 dx\right)w^2 + \alpha_0\Phi_0 = 0. \tag{3.16}$$

Let us take $\Phi_i = \Phi_0$ and $\Phi_j = 0$ for $j \neq i$. Then (Φ, α_0) satisfies (3.13) which establishes (2). □

4. Formal analysis of N-spike equilibrium solutions and their linear stability

Although the fractional Laplacian $(-\Delta)^s$ is nonlocal, the method of matched asymptotic expansions can nevertheless be used to construct leading order asymptotic approximations to equilibrium solutions of (1.1). Indeed, assuming $-1 < p_1 < \cdots < p_N < 1$ ($N \geq 1$) are well separated in the sense that $p_1 + 1 = \mathcal{O}(1)$, $1 - p_N = \mathcal{O}(1)$, and $|p_{i+1} - p_i| = \mathcal{O}(1)$ for all $i = 1, \ldots, N-1$ then it is clear from the definition (1.2b) that

$$K_s(p_i + \varepsilon y - p_j - \varepsilon\bar{y}) = \begin{cases} \mathcal{O}(1), & j \neq i, \\ \dfrac{1}{\varepsilon^{1+2s}}\dfrac{1}{|y-\bar{y}|^{1+2s}} + \mathcal{O}(1), & j = i, \end{cases} \qquad y, \bar{y} = \mathcal{O}(1).$$

Moreover, for any bounded and periodic function $\phi(x)$ such that $\phi(x) \sim \Phi(y)$ for $x = p_i + \varepsilon y$ and $y = \mathcal{O}(1)$

$$(-\Delta)^s\phi(x) \sim \varepsilon^{-2s}(-\Delta)^s\Phi + \mathcal{O}(1),$$
$$(-\Delta)^s\Phi \equiv C_s\int_{-\infty}^{\infty}\frac{\Phi(y) - \Phi(\bar{y})}{|y-\bar{y}|^{1+2s}}d\bar{y},$$

which effectively separates the inner region problems in the method of matched asymptotic expansions. In the remainder of this section we use the method of matched asymptotic expansions to formally construct multi-spike equilibrium solutions to (1.1) and determine their linear stability.

4.1. Multi-spike solutions and their slow dynamics

With the separation of inner region problems as outlined above, the construction of quasi-equilibrium solutions follows closely that for the classical case when $s_1 = s_2 = 1$ as detailed in [13]. In particular letting $-1 < p_1 < \cdots < p_N < 1$ be given as above, then we obtain the inner expansions

$$u \sim \varepsilon^{-1}\big(\xi_i w_{s_1}(y) + o(1)\big), \quad v \sim \varepsilon^{-1}\big(\xi_i + o(1)\big) \quad \text{for} \quad x = p_i + \varepsilon y, \quad y = \mathcal{O}(1)$$

for each $i = 1, \ldots, N$ where w_{s_1} satisfies the core problem (3.1) with $s = s_1$ and $\xi_i > 0$ is an undetermined constant. Therefore for all $-1 < x < 1$

$$u(x) \sim \varepsilon^{-1} \sum_{i=1}^{N} \xi_i w_{s_1}\big(\varepsilon^{-1}|x - p_i|\big) + o(\varepsilon^{-1}), \tag{4.1a}$$

where the corrections due to the algebraic decay of the core solution do not contribute until $\mathcal{O}(\varepsilon^{2s_1})$. Moreover, in the sense of distributions we calculate the limit $u^2 \to \varepsilon^{-1}\omega_{s_1}\sum_{j=1}^{N}\xi_j^2\delta(x-p_j)$ as $\varepsilon \to 0^+$ from which it follows that for all x such that $|x - p_i| \gg \varepsilon$ for all $i = 1, \ldots, N$ the inhibitor is given by

$$v \sim \varepsilon^{-1}\omega_{s_1} \sum_{j=1}^{N} \xi_j^2 G_D(x, p_j) + o(\varepsilon^{-1}), \quad \omega_{s_1} \equiv \int_0^\infty w_{s_1}(y)^2 dy, \tag{4.1b}$$

where $G_D(\cdot,\cdot)$ is the Green's function satisfying (2.3) with $s = s_2$. Since $v \to \varepsilon^{-1}(\xi_i + o(1))$ as $x \to p_i$ we obtain the nonlinear algebraic system

$$\boldsymbol{\xi} - \omega_{s_1}\mathcal{G}_D\boldsymbol{\xi}^2 = 0, \tag{4.2a}$$

where

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)^T, \quad \mathcal{G}_D = \big(G_D(p_i, p_j)\big)_{i,j=1}^{N}. \tag{4.2b}$$

If $N = 2$ we recover the system (2.6), and if $N = 1$ we obtain $\xi_1 = [\omega_{s_1} G_D(p_1, p_1)]^{-1}$.

Given a fixed configuration $-1 < p_1 < \cdots < p_N < 1$, the algebraic system (4.2) can be solved for the unknown constants $\xi_1, \ldots, \xi_N$ yielding quasi-equilibrium solution to (1.1) given by (4.1). We emphasize that the resulting solutions is not, for arbitrary spike locations, a stationary solution of (1.1). Indeed, while the solution (4.1) is stationary over an $\mathcal{O}(1)$ timescale the spike locations drift slowly over an $\mathcal{O}(\varepsilon^{-2})$ timescale according to the system of differential equations (see Appendix D for details)

$$\frac{dp_i}{dt} = -\varepsilon^2 \kappa_{s_1} \xi_i^{-1} \sum_{j \neq i} \xi_j^2 \nabla_{p_i} G_D(p_i, p_j), \quad \kappa_{s_1} \equiv \frac{\int_{-\infty}^{\infty} w_{s_1}^2 dy \int_{-\infty}^{\infty} w_{s_1}^3 dy}{3\int_{-\infty}^{\infty} |dw_{s_1}/dy|^2 dy}, \tag{4.3}$$

where ∇_{p_i} denotes the derivative with respect to the first argument and we remark that this is to be solved concurrently with the algebraic system (4.2). In particular, if $-1 < p_1 < \cdots < p_N < 1$ are chosen so that

$$\sum_{j\neq i} \xi_j^2 \nabla_{p_i} G_D(p_i, p_j) = 0 \tag{4.4}$$

for all $i = 1, \ldots, N$, then (4.1) is an equilibrium solution of (1.1). Theorem 2.1 and the proof found in Section 5 rigorously establish the existence of the equilibrium solution constructed in this section for $N = 2$.

4.2. Linear stability of multi-spike solutions

We now consider the linear stability of the N-spike equilibrium solutions constructed above which we denote by u_e and v_e. Substituting $u = u_e + e^{\lambda t}\phi$ and $v = v_e + e^{\lambda t}\psi$ where $|\phi|, |\psi| \ll 1$ into (1.1) and linearizing we obtain

$$\varepsilon^{2s_1}(-\Delta)^{s_1}\phi + \phi - 2v_e^{-1}u_e\phi + v_e^{-2}u_e^2\psi + \lambda\phi = 0, \quad -1 < x < 1, \tag{4.5a}$$

$$D(-\Delta)^{s_2}\psi + \psi - 2u_e\phi + \tau\lambda\psi = 0, \quad -1 < x < 1, \tag{4.5b}$$

where we assume in addition that both ϕ and ψ are 2-periodic. We focus first on the case where $\lambda = \mathcal{O}(1)$, the so-called large eigenvalues, and make a brief comment on the case of small eigenvalues for which $\lambda = \mathcal{O}(\varepsilon^2)$ at the end of this section. Proceeding with the method of matched asymptotic expansions as in the previous section we deduce that $\phi \sim \phi_i(y) + o(1)$ when $x = p_i + \varepsilon y$ and $y = \mathcal{O}(1)$ for each $i = 1, \ldots, N$. It follows that $\phi \sim \sum_{j=1}^N \phi_j(\varepsilon^{-1}(x - p_j)) + o(1)$ for all $-1 < x < 1$ and furthermore $u_e\phi \to \sum_{j=1}^N \xi_j \int_{-\infty}^{\infty} w_{s_1}(y)\phi_j(y)dy\delta(x - p_j)$ as $\varepsilon \to 0^+$ in the sense of distributions. Substituting this into (4.5b) we deduce that

$$\psi(x) = 2\sum_{j=1}^N \xi_j \int_{-\infty}^{\infty} w_{s_1}(y)\phi_j(y)dyG_D^{\lambda}(x, p_j),$$

where $G_D^{\lambda}(x, z)$ is the eigenvalue dependent Green's function satisfying

$$D(-\Delta)^{s_2}G_D^{\lambda} + (1 + \tau\lambda)G_D^{\lambda} = \delta(x - z), \quad -1 < x, z < 1 \tag{4.6}$$

with periodic boundary conditions. It follows that for $x = p_i + \varepsilon y$ Eq. (4.5a) becomes

$$L_0\phi_i + 2w_{s_1}^2 \sum_{j=1}^N \xi_j \int_{-\infty}^{\infty} w_{s_1}(y)\phi_j(y)dyG_D^{\lambda}(p_i, p_j) + \lambda\phi_i = 0$$

for each $i = 1, \ldots, N$ where L_0 is the linear operator of Proposition 3.1 with $s = s_1$. This system of equations is conveniently rewritten as the system of NLEPs

$$L_0\Phi + 2w_{s_1}^2 \frac{\int_{-\infty}^{\infty} w_{s_1}\mathcal{E}^{\lambda}\Phi dy}{\int_{-\infty}^{\infty} w_{s_1}^2 dy} + \lambda\Phi = 0, \tag{4.7a}$$

where

$$\Phi \equiv \begin{pmatrix} \phi_1(y) \\ \vdots \\ \phi_N(y) \end{pmatrix}, \quad \mathcal{E}^\lambda = \begin{pmatrix} \hat{\xi}_1 G_D^\lambda(p_1,p_1) & \cdots & \hat{\xi}_N G_D^\lambda(p_1,p_N) \\ \vdots & \ddots & \vdots \\ \hat{\xi}_1 G_D^\lambda(p_N,p_1) & \cdots & \hat{\xi}_N G_D^\lambda(p_N,p_N) \end{pmatrix}, \quad \hat{\xi}_i = \omega_{s_1}\xi_i. \quad (4.7b)$$

Let $\boldsymbol{p}_k^\lambda$ and χ_k^λ be the eigenpairs of $\mathcal{E}^\lambda$ satisfying $\mathcal{E}^\lambda \boldsymbol{p}_k^\lambda = \chi_k^\lambda \boldsymbol{p}_k^\lambda$ for each $k = 0, 1, \ldots, N-1$. We can further diagonalize (4.7a) by setting $\Phi = \Phi_k \boldsymbol{p}_k^\lambda$ to get the decoupled system of NLEPs

$$L_0\Phi_k + 2\chi_k^\lambda w_{s_1}^2 \frac{\int_{-\infty}^{\infty} w_{s_1}\Phi_k dy}{\int_{-\infty}^{\infty} w_{s_1}^2 dy} + \lambda\Phi_k = 0. \quad (4.8)$$

An N-spike equilibrium solution is linearly stable with respect to the large eigenvalues provided that all eigenvalues of (4.8) satisfy $\Re(\lambda) < 0$ for all $k = 0, \ldots, N-1$. Finally, we remark that the NLEP (4.8) can be further reduced to the algebraic equation

$$\mathcal{A}_k(\lambda) \equiv \frac{1}{\chi_k^\lambda} + \mathcal{F}_{s_1}(\lambda) = 0, \quad \mathcal{F}_{s_1}(\lambda) \equiv 2\frac{\int_{-\infty}^{\infty} w_{s_1}(L_0+\lambda)^{-1} w_{s_1}^2 dy}{\int_{-\infty}^{\infty} w_{s_1}^2 dy}, \quad (4.9)$$

which will in general require the numerical evaluation of $\mathcal{F}_{s_1}(\lambda)$.

The stability of a multi-spike equilibrium solution with respect to the small eigenvalues is closely related to the slow dynamics given by (4.3). In particular, whereas the large eigenvalues correspond to amplitude instabilities occurring on an $\mathcal{O}(1)$ timescale, the small eigenvalues are linked to the linear stability of the spike pattern with respect to the slow dynamics (4.3) and therefore occur on an $\mathcal{O}(\varepsilon^{-2})$ timescale. In the case of two-spike equilibrium solutions Theorem 2.2 rigorously establishes the linear stability with respect to the large eigenvalues. On the other hand, as discussed in Section 6 two-spike equilibrium solutions are always linearly stable with respect to the small eigenvalues. In the remainder of this section we consider explicitly the asymptotic construction and linear stability of one- and two-spike solutions.

4.3. Example: symmetric N-spike solutions

By appropriately choosing the spike locations we can explicitly calculate an N-spike solution that is symmetric in the sense that the local profile of each spike is identical. Specifically, letting

$$p_i = -1 + N^{-1}(2i-1), \quad \xi_i = \xi_c \equiv \left(\omega_{s_1} \sum_{k=0}^{N-1} G_D(2N^{-1}k, 0)\right)^{-1} \quad (4.10)$$

for all $i = 1, \ldots, N$, it is then straightforward to show that (4.2) is satisfied and the spike locations are stationary solutions of the slow-dynamics (4.3). Since the resulting matrix $\mathcal{E}^\lambda$ defined by (4.7b) is circulant and its eigenpairs are explicitly given by

$$\begin{aligned}
\boldsymbol{p}_k^\lambda &\equiv \left(1, e^{i\frac{2\pi k}{N}}, \cdots, e^{i\frac{2\pi(N-1)k}{N}}\right)^T, \\
\chi_k^\lambda &= \sum_{j=0}^{N-1} H_j^\lambda e^{i\frac{2\pi jk}{N}} \left(\sum_{j=0}^{N-1} H_j^0\right)^{-1}, \\
H_k^\lambda &\equiv G_D^\lambda(2N^{-1}k, 0)
\end{aligned} \tag{4.11}$$

for each $k = 0, \ldots, N-1$.

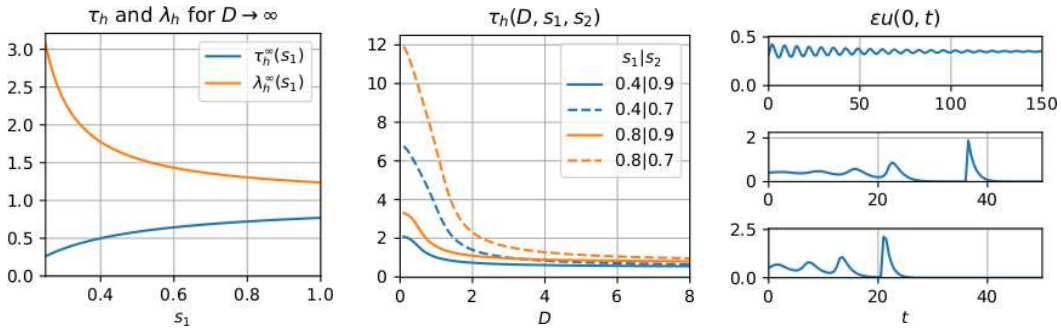
For the remainder of this example we focus exclusively on the calculation of the Hopf bifurcation threshold for a one-spike solution. In particular, by using the winding number argument in [27], we seek conditions under which (4.9) with $k = 0$ admits an unstable solution (i.e with $\Re(\lambda) > 0$). Letting $C_R = \{i\lambda_I \mid -R \leq \lambda_I \leq R\} \cup \{Re^{i\theta} \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ be traversed counterclockwise and noting that $|\chi_0^\lambda| > 0$ for all λ with $\Re(\lambda) \geq 0$ whereas $\mathcal{F}_{s_1}(\lambda)$ has a simple pole on the positive half-plane corresponding to the principal eigenvalue of L_0 we find that the number Z of unstable solutions to (4.9) can be determined by

$$\frac{1}{2\pi i} \lim_{R\to\infty} \oint_{C_R} \frac{d\mathcal{A}_0/d\lambda}{\mathcal{A}_0} d\lambda = Z - 1.$$

Noting that $\chi_0^\lambda \sim \mathcal{O}(\lambda^{1/2s_2-1})$ and therefore $\mathcal{A}_0(\lambda) \sim \mathcal{O}(\lambda^{1-1/2s_2})$ for $|\lambda| \gg 1$ we deduce that the change in argument of $\mathcal{A}_0$ over the semi-circle part of the contour is $\left(1 - \frac{1}{2s_2}\right)\pi$ from which it follows that

$$Z = \frac{3}{2} - \frac{1}{4s_2} - \frac{1}{\pi} \arg \mathcal{A}(i\lambda_I)\big|_{\lambda_I=0}^{\infty}.$$

We note that $\arg \mathcal{A}(i\lambda_I) \to \frac{1}{2}(1 - \frac{1}{2s_2})$ as $\lambda_I \to \infty$ whereas $\mathcal{A}_0(0) = -1$ since $L_0^{-1} w_{s_1}^2 = -w_{s_1}$. Furthermore, numerical evidence suggests that $\Re\mathcal{A}_0(i\lambda_I)$ is monotone increasing in λ_I and so there exists a unique value $0 < \lambda_I^\star < \infty$ such that $\Re\mathcal{A}_0(i\lambda_I^\star) = 0$. It then follows that either $Z = 2$ or $Z = 0$ depending on whether $\Im\mathcal{A}_0(i\lambda_I^\star) > 0$ or $\Im\mathcal{A}_0(i\lambda_I^\star) < 0$ respectively. The Hopf bifurcation threshold can thus be calculated by numerically solving $\mathcal{A}_0(i\lambda_I) = 0$ for $\tau = \tau_h(D, s_1, s_2)$ and $\lambda_I = \lambda_h(D, s_1, s_2)$. By first considering the limit $D \to \infty$ for which $\chi_0^\lambda \to (1+\tau\lambda)^{-1}$ we calculate the Hopf bifurcation threshold $\tau_h^\infty(s_1)$ and accompanying eigenvalue $\lambda_h^\infty(s_1)$, both of which are independent of s_2 and are plotted in Fig. 1(a). In particular we observe that τ_h^∞ is monotone increasing with s_1 and therefore the introduction of Lévy flights for the activator destabilizes the single spike solution as previously observed in [20]. This behaviour persists for finite values of $D > 0$ but we observe that the Hopf bifurcation threshold is monotone decreasing with s_2 and therefore introducing Lévy flights for the inhibitor stabilizes the single spike solution. This behaviour is illustrated in Fig. 1(b) for which we plot the Hopf bifurcation threshold as a function of D for select values of s_1 and s_2. We remark in addition that the Hopf bifurcation's dependence on the inhibitor diffusivity D remains qualitative unchanged with the introduction of Lévy flights: $\tau_h(D, s_1, s_2)$ decreases monotonically with D.



(a) (b) (c)

Figure 1: Hopf bifurcation threshold for a one-spike solution in (a) the shadow limit $D \to \infty$, and (b) for finite $D > 0$ at select values of $s_1 = 0.4, 0.8$ and $s_2 = 0.7, 0.9$. (c) Single spike height obtained from numerical simulations with parameters $\varepsilon = 0.02$, $D = 2$, and $\tau = 1.5$ and exponent sets $(s_1, s_2) = (0.8, 0.7)$, $(0.8, 0.9)$, and $(0.4, 0.7)$ for the top, middle, and bottom plots respectively.

To illustrate the above observations, mainly the destabilization (resp. stabilization) of the single-spike solution with decreasing s_1 (resp. s_2), we numerically solve (1.1) starting with a single spike solution centred at $p_1 = 0$ with $\varepsilon = 0.02, D = 2$, and $\tau = 1.5$ for three distinct pairs of exponents $(s_1, s_2) = (0.8, 0.7), (0.8, 0.9), (0.4, 0.7)$. See Appendix B for details on the numerical calculation. From the numerically calculated threshold we find $\tau_h(2, 0.8, 0.7) \approx 2.306, \tau_h(2, 0.8, 0.9) \approx 1.096$, and $\tau_h(2, 0.4, 0.7) \approx 1.399$ and therefore with $\tau_h = 1.5$ we anticipate the single spike solution to be stable for the first exponent set and unstable for the latter two. The plots of $u(0, t)$ in Fig. 1(c) support these predictions.

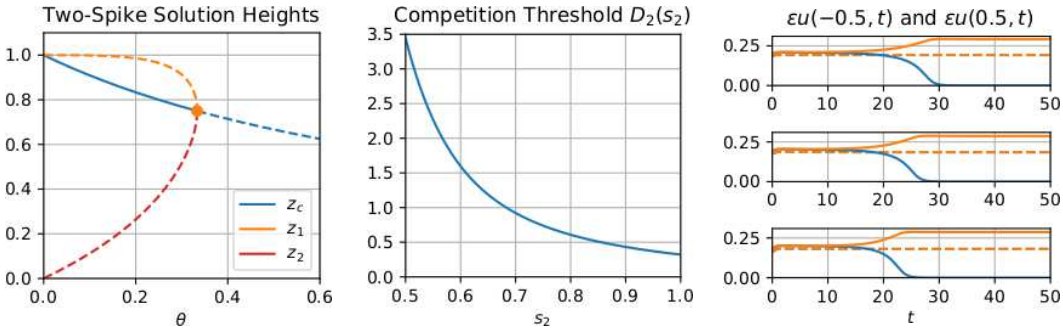
4.4. Example: symmetric and asymmetric two-spike solutions

When $s_1 = s_2 = 1$ it has been shown that the one-dimensional Gierer-Meinhardt model may exhibit asymmetric solutions consisting of spikes with different heights [26, 31]. The gluing method for constructing such asymmetric N-spike solutions relies crucially on the locality of the classical Laplace operator. However, since the fractional Laplace operator $(-\Delta)^s$ is nonlocal for $s < 1$, we cannot use this gluing method to construct asymmetric multi-spike solutions and we are therefore restricted to solving the nonlinear algebraic system (4.2) directly. In this example we restrict our attention to the case of $N = 2$ for which a complete characterization of all two-spike solutions can be obtained directly from the algebraic system (4.2).

Assuming without loss of generality that $-1 < p_1 < p_2 < 1$, we first calculate from (4.3) that

$$\frac{d(p_2 - p_1)}{dt} = -\varepsilon^2 \kappa_s \frac{\xi_1^3 + \xi_2^3}{\xi_1 \xi_2} G_D'\big(|p_2 - p_1|, 0\big),$$

where $G_D'(z, 0) = \frac{dG_D(z,0)}{dz}$. By numerically evaluating $G_D(z, 0)$ (see Appendix C) we observe that it is monotone decreasing for $0 < z < 1$, attains its global minimum at $z = 1$, and is monotone increasing for $1 < z < 2$. Any stationary solution of (4.3)



(a) (b) (c)

Figure 2: (a) Bifurcation diagram showing the rescaled spike heights $z_i = \omega_{s_1} G_D(0,0)\xi_i$ versus θ. Solid (resp. dashed) lines indicate the resulting two-spike solution is linearly stable (resp. unstable) with respect to competition instabilities. (b) The competition instability threshold for a symmetric two-spike solution. (c) Spike heights at $x = -0.5$ (solid blue) and $x = 0.5$ (solid orange) obtained from numerical simulations with initial condition consisting of a symmetric two-spike solution and with parameter values of $s_1 = 0.8$, $\varepsilon = 0.02$, $\tau = 0.05$, and $D = 1.2D_2(s_2)$ where $s_2 = 0.9$ (top), 0.8 (middle), and 0.7 (bottom). The dashed orange line indicates the (common) spike height obtained with the same parameters but with $D = 0.8D_2(s_2)$.

must therefore satisfy $p_2 - p_1 = 1$ and furthermore any such solution is linearly stable with respect to the slow-dynamics with the exception of having a neutral eigenvalue corresponding to translational invariance. Defining

$$z_1 \equiv \omega_{s_1} G_D(0,0)\xi_1, \quad z_2 \equiv \omega_{s_1} G_D(1,0)\xi_2, \quad \theta \equiv \frac{G_D(1,0)}{G_D(0,0)},$$

the algebraic system (4.2) can be rewritten as

$$z_1 - z_1^2 - \theta z_2^2 = 0, \quad z_2 - \theta z_1^2 - z_2^2 = 0. \tag{4.12}$$

This system always admits the symmetric solution for which $z_1 = z_2 = z_c$ where $z_c = (1+\theta)^{-1}$ recovering the result from the previous example for $N = 2$. One the other hand, assuming $z_1 \neq z_2$ we may subtract the first equation from the second to obtain $z_2 = (1-\theta)^{-1} - z_1$. Substituting this expression for z_2 back into the first equation in (4.12) yields a quadratic in z_1 which is readily solved to obtain

$$z_1 = \frac{1/2}{1-\theta}\left(1 + \sqrt{\frac{1-3\theta}{1+\theta}}\right), \quad z_2 = \frac{1/2}{1-\theta}\left(1 - \sqrt{\frac{1-3\theta}{1+\theta}}\right). \tag{4.13}$$

We immediately deduce that an asymmetric two-spike solution exists if an only if $\theta < \frac{1}{3}$ and we obtain the bifurcation diagram shown in Fig. 2(a). Interestingly, the structure of two-spike solutions depends only on the ratio θ depending only on D and the inhibitor exponent s_2.

We conclude this section by considering the linear stability of two-spike solutions with respect to competition instabilities, neglecting the possibility of Hopf bifurcations by assuming that τ is sufficiently small. In view of (4.8) and Theorem 3.1 it suffices to

consider the eigenvalues of

$$\mathcal{E}_0 = \begin{pmatrix} z_1 & \theta z_2 \\ \theta z_1 & z_2 \end{pmatrix}.$$

When $z_1 = z_2 = z_c$ it is easy to see that $\mathcal{E}_0$ has eigenvectors $\boldsymbol{p}_0 = (1,1)$ and $\boldsymbol{p}_1 = (1,-1)$ with corresponding eigenvalues $\chi_0^0 = 1$ and $\chi_1^0 = \frac{(1-\theta)}{(1+\theta)}$. Since $\chi_0^0 > \frac{1}{2}$ by Theorem 3.1 the $k = 0$ mode is always linearly stable. On the other hand the $k = 1$ mode is stable if and only if $\chi_1^0 > \frac{1}{2}$ and in particular the symmetric solution is linearly stable when $\theta < \frac{1}{3}$ and unstable otherwise. Finally, when z_1 and z_2 are given by (4.13) it can be shown that the eigenvalues of $\mathcal{E}_0$ are given by

$$\chi_0^0 = \frac{1/2}{1-\theta}\left(1 + \sqrt{\frac{4\theta^2 - 3\theta + 1}{1+\theta}}\right), \quad \chi_1^0 = \frac{1/2}{1-\theta}\left(1 - \sqrt{\frac{4\theta^2 - 3\theta + 1}{1+\theta}}\right),$$

from which we deduce that $\chi_0^0 > 1$ and $\chi_1^0 < \frac{3-\sqrt{3}}{4} < \frac{1}{2}$ for all $0 < \theta < \frac{1}{3}$. Therefore by Theorem 3.1 the $k = 1$ mode is linearly unstable. In Fig. 2(a) we indicate the values of θ where the two-spike solution is linearly stable (resp. unstable) with respect to competition instabilities by solid (resp. dashed) curves. By numerically solving $\theta = \frac{1}{3}$ for D as a function of s_2 we can calculate the competition instability threshold $D = D_2(s_2)$ for the symmetric two-spike solution and this is shown in Fig. 2(b). Interestingly, the competition instability threshold is independent of the exponent s_1. On the other hand, similarly to the case of a single spike solution there will be an s_1 dependence for the Hopf bifurcation thresholds but this will be qualitatively similar to that found for the single spike solution and for this reason we do not explored this further here. In Fig. 2(c) we illustrate the onset of competition instabilities when $s_1 = 0.8, \varepsilon = 0.02, \tau = 0.05$, and for values of $s_2 = 0.9, 0.8, 0.7$ and $D = 1.2 \times D_2(s_2)$ by performing full numerical simulations of (1.1) (see Appendix B for details). We remark that the accuracy of the leading order approximation to the competition instability calculated above grows increasingly inaccurate as $s_2 \to 0.5$ for a fixed value of $\varepsilon > 0$. Indeed, as described in more detail in the derivation of the slow dynamics found in Appendix D, the first order correction to the quasi-equilibrium solution is $\mathcal{O}(\varepsilon^{2s_2-1})$ and this tends to $\mathcal{O}(1)$ as $s_2 \to \frac{1}{2}$. When $s_2 = \frac{1}{2}$ the Green's function is known to have a logarithmic singularity (see [33, Lemma 2.2.]) and we anticipate that the method of matched asymptotic expansions will lead to an asymptotic expansion in powers of $\nu = -\frac{1}{\log \varepsilon}$ as is often the case for singularly perturbed reaction-diffusion systems in two-dimensions [2, 16].

5. Rigorous proof of the existence results

In this section we shall prove the existence theorem, i.e., Theorem 2.1. We divide the discussion into three sections. In first subsection, we give an approximate solution. Then we apply the classical Lyapunov-Schmidt reduction method to reduce the infinite dimensional problem to a finite dimensional problem in second subsection. In last subsection we solve the finite dimensional problem and thereby prove the Theorem 2.1.

5.1. Study of the approximate solutions

Let $-1 < p_1^0 < p_2^0 < 1$ be two points satisfying the assumptions (H1)-(H3). Let $\hat{\xi}^0 = (\hat{\xi}_1^0, \hat{\xi}_2^0)$ be the solution of (2.6) and let $\mathbf{p}^0 = (p_1^0, p_2^0)$. We shall construct an approximate solution to (2.1) which concentrates near these 2 points.

Let $-1 < p_1 < p_2 < 1$ be such that $\mathbf{p} = (p_1, p_2) \in B_{\varepsilon^{2s-1}}(\mathbf{p}^0)$. Set

$$r_0 = \frac{1}{10} \min \left\{ p_1^0 + 1,\ 1 - p_2^0,\ \frac{1}{2}|p_1^0 - p_2^0| \right\}$$

and define a cut-off function $\chi(x)$ such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. Letting

$$w_i(y) = w\left(y - \frac{p_i}{\varepsilon}\right) \chi\left(\frac{\varepsilon y - p_i}{r_0}\right), \tag{5.1}$$

where w is the ground state solution of (3.1), it is then straightforward to check that

$$(-\Delta)_y^s w_i(y) + w_i(y) - w_i^2(y) = h.o.t., \tag{5.2}$$

where $h.o.t.$ refers to terms of order ε^{1+2s} in $L^2(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$. Let $\hat{\xi}(\mathbf{p}) = (\hat{\xi}_1, \hat{\xi}_2)$ be defined as in (H1). Fix any function $u \in H^{2s}(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ and let $T[u]$ be the solution of

$$\begin{cases} D(-\Delta)^s T[u] + T[u] - c_\varepsilon u^2 = 0, & x \in (-1, 1), \\ T[u](x) = T[u](x+2), & x \in \mathbb{R}, \end{cases} \tag{5.3}$$

where

$$c_\varepsilon = \left(\varepsilon \int_{\mathbb{R}} w^2(y) dy \right)^{-1}. \tag{5.4}$$

Letting $\mathbf{p} \in B_{\varepsilon^{2s-1}}(\mathbf{p}^0)$ we define

$$w_{\varepsilon,\mathbf{p}} = \sum_{i=1}^{2} \hat{\xi}_i w_i(y)$$

and using (5.3), we compute

$$\begin{aligned} &\tau_i := T[w_{\varepsilon,\mathbf{p}}](p_i) \\ &= \varepsilon c_\varepsilon \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} G_D(p_i, \varepsilon y) w_{\varepsilon,\mathbf{p}}^2(y) dy = \varepsilon c_\varepsilon \sum_{j=1}^{2} \hat{\xi}_j^2 \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} G_D(p_i, \varepsilon y) w_j^2(y) dy \\ &= \varepsilon c_\varepsilon \sum_{j=1}^{2} \hat{\xi}_j^2 \left(G_D(p_i, p_j) \int_{\mathbb{R}} w^2(y) dy \right) + \mathcal{P}_i = \sum_{j=1}^{2} G_D(p_i, p_j) \hat{\xi}_j^2 + \mathcal{P}_i, \end{aligned} \tag{5.5}$$

where $G_D(x, y)$ is defined in (2.3) and $\mathcal{P}_i$ is a number with order ε^{2s-1}. Thus, we have obtained the following system of equations:

$$\tau_i = \sum_{j=1}^{2} G_D(p_i, p_j) \hat{\xi}_j^2 + \mathcal{P}_i. \tag{5.6}$$

According to the assumption (H1)-(H3) and the implicit function theorem, we have the above equation has a unique solution

$$\tau_i = \hat{\xi}_i + \vartheta_i, \quad i = 1, 2, \quad \vartheta_i = \mathcal{O}(\varepsilon^{2s-1}).$$

Hence,

$$T[w_{\varepsilon,\mathbf{p}}](p_i) = \hat{\xi}_i + \mathcal{O}(\varepsilon^{2s-1}).$$

Now for $x = p_i + \varepsilon z$ we calculate

$$\begin{aligned}
&T[w_{\varepsilon,\mathbf{p}}](x) - T[w_{\varepsilon,\mathbf{p}}](p_i) \\
&= c_\varepsilon \int_{-1}^{1} \big[G_D(x,\zeta) - G_D(p_i,\zeta)\big] w_{\varepsilon,\mathbf{p}}^2 \left(\frac{\zeta}{\varepsilon}\right) d\zeta \\
&= c_\varepsilon \hat{\xi}_i^2 \int_{-1}^{1} \big[G_D(x,\zeta) - G_D(p_i,\zeta)\big] w_i^2 \left(\frac{\zeta}{\varepsilon}\right) d\zeta \\
&\quad + c_\varepsilon \sum_{j\neq i} \hat{\xi}_j^2 \int_{-1}^{1} \big[G_D(x,\zeta) - G_D(p_i,\zeta)\big] w_j^2 \left(\frac{\zeta}{\varepsilon}\right) d\zeta \\
&= c_\varepsilon \varepsilon \hat{\xi}_i^2 \int_{\mathbb{R}} \big[G_D(\varepsilon y - \varepsilon z) - G_D(\varepsilon y)\big] w^2(y) dy \\
&\quad + c_\varepsilon \sum_{j\neq i} \hat{\xi}_j^2 \int_{-1}^{1} \big[G_D(x,\zeta) - G_D(p_i,\zeta)\big] w_j^2 \left(\frac{\zeta}{\varepsilon}\right) d\zeta + h.o.t. \\
&= P_i(z) + \varepsilon \sum_{j\neq i} \Big(\hat{\xi}_j^2 z \nabla_{p_i} G_D(p_i,p_j) + \mathcal{O}(\varepsilon z^2)\Big) + h.o.t., \qquad (5.7)
\end{aligned}$$

where

$$P_i(z) = c_\varepsilon \varepsilon \hat{\xi}_i^2 \int_{\mathbb{R}} \big[G_D(\varepsilon y - \varepsilon z) - G_D(\varepsilon y)\big] w^2(y) dy$$

is an even function and of order ε^{2s-1}.

Next we define

$$S[u] := (-\Delta)_y^s u + u - \frac{u^2}{T[u]}, \qquad (5.8)$$

for which we calculate

$$\begin{aligned}
S[w_{\varepsilon,\mathbf{p}}](y) &= (-\Delta)_y^s w_{\varepsilon,\mathbf{p}} + w_{\varepsilon,\mathbf{p}} - \frac{w_{\varepsilon,\mathbf{p}}^2}{T[w_{\varepsilon,\mathbf{p}}]} \qquad (5.9) \\
&= \sum_{j=1}^{2} \hat{\xi}_j \chi\left(\frac{\varepsilon y - p_j}{r_0}\right) (-\Delta)^s w\left(y - \frac{p_j}{\varepsilon}\right) + \sum_{j=1}^{2} \hat{\xi}_j w_j - \frac{w_{\varepsilon,\mathbf{p}}^2}{T[w_{\varepsilon,\mathbf{p}}]} + h.o.t. \\
&= \left[\sum_{j=1}^{2} \hat{\xi}_j w_j^2 - \frac{\left(\sum\limits_{j=1}^{2} \hat{\xi}_j w_j\right)^2}{T[w_{\varepsilon,\mathbf{p}}]}\right] + h.o.t. = E_1 + E_2 + h.o.t. \quad \text{in } L^2\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right),
\end{aligned}$$

where

$$E_1 = \sum_{j=1}^{2} \hat{\xi}_j w_j^2 - \frac{\left(\sum\limits_{j=1}^{2} \hat{\xi}_j w_j\right)^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)}, \quad E_2 = \frac{\left(\sum\limits_{j=1}^{2} \hat{\xi}_j w_j\right)^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)} - \frac{\left(\sum\limits_{j=1}^{2} \hat{\xi}_j w_j\right)^2}{T[w_{\varepsilon,\mathbf{p}}](x)}.$$

Using (5.6) we calculate

$$E_1 = \sum_{j=1}^{2} \hat{\xi}_j w_j^2 - \frac{\left(\sum\limits_{j=1}^{2} \hat{\xi}_j w_j\right)^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)} = \sum_{j=1}^{2} \left(\hat{\xi}_j - \frac{\hat{\xi}_j^2}{\hat{\xi}_i + \vartheta_i}\right) w_j^2 = \mathcal{O}(\varepsilon^{2s-1}) \sum_{j=1}^{2} w_j^2,$$

and therefore

$$\|E_1\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = \mathcal{O}(\varepsilon^{2s-1}). \tag{5.10}$$

In addition since x is close to p_i we see that E_1 can be decomposed into two parts: one part of order ε^{2s-1} and symmetric in $x - p_i$, and the other part of order ε. Next we calculate

$$\begin{aligned}
E_2 &= \sum_{j=1}^{2} \frac{(\hat{\xi}_j w_j)^2}{(T[w_{\varepsilon,\mathbf{p}}](p_i))^2} \big(T[w_{\varepsilon,\mathbf{p}}](x) - T[w_{\varepsilon,\mathbf{p}}](p_i)\big) \\
&\quad \times \left(1 + \sum_{n=1}^{\infty} \left(\frac{T[w_{\varepsilon,\mathbf{p}}](p_i) - T[w_{\varepsilon,\mathbf{p}}](x)}{T[w_{\varepsilon,\mathbf{p}}](p_i)}\right)^n\right) \\
&= \sum_{j=1}^{2} \frac{(\hat{\xi}_j w_j)^2}{(T[w_{\varepsilon,\mathbf{p}}](p_i))^2} P_i(z) \left(1 + \sum_{n=1}^{\infty} \left(-\frac{P_i(z)}{T[w_{\varepsilon,\mathbf{p}}](p_i)}\right)^n\right) \\
&\quad + \varepsilon \sum_{j=1}^{2} \frac{(\hat{\xi}_j w_j)^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)} \sum_{l \neq i} \hat{\xi}_l^2 z \nabla_{p_i} G_D(p_i, p_l) + h.o.t. \\
&= E_{21} + E_{22} + h.o.t.,
\end{aligned} \tag{5.11}$$

where $E_{21} = \mathcal{O}(\varepsilon^{2s-1})$ is symmetric in $x - p_i, i = 1, 2$, and $\|E_{22}\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = \mathcal{O}(\varepsilon)$. We have thus established the following lemma.

Lemma 5.1. *For $x = p_i + \varepsilon z$, $|\varepsilon z| < r_0$, we have the decomposition for $S[w_{\varepsilon,\mathbf{p}}](x)$,*

$$S[w_{\varepsilon,\mathbf{p}}] = S_{1,1} + S_{1,2},$$

where

$$S_{1,1}(z) = \varepsilon w_i^2 \sum_{l \neq i} \hat{\xi}_l^2 z \nabla_{p_i} G(p_i, p_l) + h.o.t.,$$

$$S_{1,2}(z) = \frac{(\hat{\xi}_i w_i)^2}{(T[w_{\varepsilon,\mathbf{p}}](p_i))^2} R_i(z) + h.o.t.,$$

and $R_i(z)$ is even in z and $\|S_{1,2}\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} \leq C\varepsilon^{2s-1}$. Furthermore,

$$S[w_{\varepsilon,\mathbf{p}}] = h.o.t. \quad \textit{for} \quad |x - p_i| \geq r_0, \quad i = 1, 2.$$

5.2. The Lyapunov-Schmidt reduction method

In this subsection, we use the Lyapunov-Schmidt reduction method to solve the problem

$$S[w_{\varepsilon,\mathbf{p}}+\phi]=\sum_{j=1}^{2}c_j\frac{\partial w_j}{\partial y} \tag{5.12}$$

for real constants c_j and a perturbation $\phi\in H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})$ which is small in the corresponding norm. To proceed, we first need to study the linearized operator

$$\tilde{L}_{\varepsilon,\mathbf{p}}\phi:=S'_{\varepsilon}[w_{\varepsilon,\mathbf{p}}]\phi=(-\Delta)^s_y\phi+\phi-2\frac{w_{\varepsilon,\mathbf{p}}}{T[w_{\varepsilon,\mathbf{p}}]}\phi+\frac{w^2_{\varepsilon,\mathbf{p}}}{(T[w_{\varepsilon,\mathbf{p}}])^2}\big(T'[w_{\varepsilon,\mathbf{p}}]\phi\big).$$

For a given function $\phi\in L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})$ we introduce $T'[w_{\varepsilon,\mathbf{p}}]\phi$ as the unique solution of

$$\begin{cases} D(-\Delta)^s\big(T'[w_{\varepsilon,\mathbf{p}}]\phi\big)+T'[w_{\varepsilon,\mathbf{p}}]\phi-2c_\varepsilon w_{\varepsilon,\mathbf{p}}\phi=0, & x\in(-1,1),\\ \big(T'[w_{\varepsilon,\mathbf{p}}]\phi\big)(x)=\big(T'[w_{\varepsilon,\mathbf{p}}]\phi\big)(x+2), & x\in\mathbb{R}. \end{cases} \tag{5.13}$$

The approximate kernel and co-kernel are respectively defined by

$$\mathcal{K}_{\varepsilon,\mathbf{p}}:=\mathrm{Span}\left\{\frac{\partial w_j}{\partial y}\Big|\ j=1,2\right\}\subset H^{2s}\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right),$$
$$\mathcal{C}_{\varepsilon,\mathbf{p}}:=\mathrm{Span}\left\{\frac{\partial w_j}{\partial y}\Big|\ j=1,2\right\}\subset L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right).$$

From the definition of the linear operator L in (3.6) we recall that by Lemma 3.2 we know that

$$L:(X_0\oplus X_0)^\perp\cap\big(H^{2s}(\mathbb{R})\big)^2\to(X_0\oplus X_0)^\perp\cap\big(L^2(\mathbb{R})\big)^2$$

is invertible with a bounded inverse. We shall see that the linear operator L is a limit of the operator $\tilde{L}_{\varepsilon,\mathbf{p}}$ as $\varepsilon\to0$. First we introduce the projection $\pi^\perp_{\varepsilon,\mathbf{p}}:L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})\to\mathcal{C}^\perp_{\varepsilon,\mathbf{p}}$ and study the operator $L_{\varepsilon,\mathbf{p}}:=\pi^\perp_{\varepsilon,\mathbf{p}}\circ\tilde{L}_{\varepsilon,\mathbf{p}}$. Letting $\varepsilon\to0$ we shall show that $L_{\varepsilon,\mathbf{p}}:\mathcal{K}^\perp_{\varepsilon,\mathbf{p}}\to\mathcal{C}^\perp_{\varepsilon,\mathbf{p}}$ is invertible with a bounded inverse provided ε is small enough. This result is contained in the following proposition.

Proposition 5.1. *There exists positive constants $\overline{\varepsilon},\overline{\delta},C$ such that for all $\varepsilon\in(0,\overline{\varepsilon}),(p_1,p_2)\in(-1,1)^2$ with $\min(|1+p_1|,|1-p_2|,|p_1-p_2|)>\overline{\delta}$,*

$$\|L_{\varepsilon,\mathbf{p}}\phi\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})}\geq C\|\phi\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})}.$$

Furthermore, the map

$$L_{\varepsilon,\mathbf{p}}:\mathcal{K}^\perp_{\varepsilon,\mathbf{p}}\to\mathcal{C}^\perp_{\varepsilon,\mathbf{p}} \tag{5.14}$$

is surjective.

Proof. The proof follows the standard method of Lyapunov-Schmidt reduction which was also used in [9, 10, 29–31]. Suppose the proposition is not true. Then there exist sequences $\{\varepsilon_k\}, \{\mathbf{p}^k\}, \phi^k$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$, $\mathbf{p}^k \in (-1,1)^2$, $\min(|1+p_1^k|,\ |1-p_2^k|,\ |p_1^k - p_2^k|) > \overline{\delta}$, and $\phi^k = \phi_{\varepsilon_k} \in \mathcal{K}^{\perp}_{\varepsilon_k,\mathbf{p}^k}$ for all $k \geq 1$ such that

$$\|\phi^k\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = 1, \quad \|L_{\varepsilon_k,\mathbf{p}^k}\phi^k\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.15}$$

We define $\phi_i^k, i = 1, 2$ and ϕ_3^k as follows:

$$\begin{aligned} \phi_i^k(y) &= \phi^k(y)\chi\left(\frac{\varepsilon y - p_i}{r_0}\right), \quad i = 1, 2, \\ \phi_3^k(y) &= \phi^k(y) - \sum_{i=1}^{2}\phi_i^k(y), \quad y \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right). \end{aligned} \tag{5.16}$$

Although each ϕ_i^k is defined only in $(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$, by a standard result they can be extended to $\mathbb{R}$ such that their norms in $H^{2s}(\mathbb{R})$ are still bounded by a constant independent of ε and $\mathbf{p}$ for ε small enough. In the following we shall study the corresponding problem in $\mathbb{R}$. To simplify our notation, we keep the same notation for the extension. Since $\{\phi_i^k\}$ is bounded in $H^{2s}_{\text{loc}}(\mathbb{R})$, it has a weak limit in $H^{2s}_{\text{loc}}(\mathbb{R})$ and therefore also a strong limit in $L^2_{\text{loc}}(\mathbb{R})$ and $L^{\infty}_{\text{loc}}(\mathbb{R})$. We denote the limit by ϕ_i. Then $\Phi = (\phi_1, \phi_2)^T$ solves the system

$$L\Phi = 0.$$

By Lemma 3.1, $\Phi \in \text{Ker}(L) = X_0 \oplus X_0$. Since $\phi^k \perp \mathcal{K}^{\perp}_{\varepsilon_k,\mathbf{p}^k}$, by taking $k \to \infty$ we get $\Phi \in (X_0 \oplus X_0)^{\perp}$ and therefore $\Phi = 0$.

By elliptic estimates we get $\|\phi_i^k\|_{H^{2s}(\mathbb{R})} \to 0$ as $k \to \infty$ for $i = 1, 2$. Furthermore, $\phi_3^k \to \phi_3$ in $H^{2s}(\mathbb{R})$, where ϕ_3 solves

$$(-\Delta)^s\phi + \phi = 0 \quad \text{in } \mathbb{R}. \tag{5.17}$$

Therefore, we conclude $\phi_3 = 0$ and $\|\phi_3^k\|_{H^{2s}(\mathbb{R})} \to 0$ as $k \to +\infty$. This contradicts $\|\phi^k\|_{H^{2s}(-\frac{1}{\varepsilon_k},\frac{1}{\varepsilon_k})} = 1$. To complete the proof of Proposition 5.1, we just need to show that the operator conjugate to $L_{\varepsilon,\mathbf{p}}$ (denoted by $L^*_{\varepsilon,\mathbf{p}}$) is injective from $\mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}$ to $\mathcal{C}^{\perp}_{\varepsilon,\mathbf{p}}$. Note that $L^*_{\varepsilon,\mathbf{p}} = \pi_{\varepsilon,\mathbf{p}} \circ \tilde{L}^*_{\varepsilon,\mathbf{p}}$ with

$$\tilde{L}^*_{\varepsilon,\mathbf{p}}\psi = (-\Delta)^s_y\psi + \psi - 2\frac{w_{\varepsilon,\mathbf{p}}}{T[w_{\varepsilon,\mathbf{p}}]}\psi + T'[w_{\varepsilon,\mathbf{p}}]\left(\frac{w^2_{\varepsilon,\mathbf{p}}}{(T[w_{\varepsilon,\mathbf{p}}])^2}\psi\right).$$

The proof for $L^*_{\varepsilon,\mathbf{p}}$ follows exactly the same as the one of $L_{\varepsilon,\mathbf{p}}$ and we omit the details. □

Now we are in position to solve the problem

$$\pi^{\perp}_{\varepsilon,\mathbf{p}} \circ S_{\varepsilon}(w_{\varepsilon,\mathbf{p}} + \phi) = 0. \tag{5.18}$$

Since $L_{\varepsilon,\mathbf{p}}\mid_{\mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}}$ is invertible (call the inverse $L^{-1}_{\varepsilon,\mathbf{p}}$) we can rewrite the above problem as

$$\phi = -\big(L^{-1}_{\varepsilon,\mathbf{p}} \circ \pi^{\perp}_{\varepsilon,\mathbf{p}} \circ S_{\varepsilon}(w_{\varepsilon,\mathbf{p}})\big) - \big(L^{-1}_{\varepsilon,\mathbf{p}} \circ \pi^{\perp}_{\varepsilon,\mathbf{p}} \circ N_{\varepsilon,\mathbf{p}}(\phi)\big) \equiv M_{\varepsilon,\mathbf{p}}(\phi), \tag{5.19}$$

where

$$N_{\varepsilon,\mathbf{p}}(\phi) = S_{\varepsilon}(w_{\varepsilon,\mathbf{p}} + \phi) - S_{\varepsilon}(w_{\varepsilon,\mathbf{p}}) - S'_{\varepsilon}(w_{\varepsilon,\mathbf{p}})\phi$$

and the operator $M_{\varepsilon,\mathbf{p}}$ is defined by $\phi \in H^{2s}(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$. We are going to show that the operator $M_{\varepsilon,\mathbf{p}}$ is a contraction map on

$$B_{\varepsilon,\sigma} := \left\{ \phi \in H^{2s}\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \;\middle|\; \|\phi_{\varepsilon}\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} < \sigma \right\} \tag{5.20}$$

if σ and ε are small enough. We have by the discussion in last section and Proposition 5.1 that

$$\begin{aligned} \|M_{\varepsilon,\mathbf{p}}(\phi)\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} &\le C\left(\big\|\pi^{\perp}_{\varepsilon,\mathbf{p}} \circ N_{\varepsilon,\mathbf{p}}(\phi)\big\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} + \big\|\pi^{\perp}_{\varepsilon,\mathbf{p}} \circ S_{\varepsilon}(w_{\varepsilon,\mathbf{p}})\big\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})}\right) \\ &\le C\big(c(\sigma)\sigma + \varepsilon^{2s-1}\big), \end{aligned} \tag{5.21}$$

where $C > 0$ is a constant independent of $\sigma > 0, \varepsilon > 0$ and $c(\sigma) \to 0$ as $\sigma \to 0$. Similarly we show that

$$\|M_{\varepsilon,\mathbf{p}}(\phi_1) - M_{\varepsilon,\mathbf{p}}(\phi_2)\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} \le C\big(c(\sigma)\sigma\big)\|\phi_1 - \phi_2\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})},$$

where $c(\sigma) \to 0$ as $\sigma \to 0$. If we choose $\sigma = \varepsilon^{\alpha}$ for $\alpha \le 2s-1$ and $\varepsilon > 0$ sufficiently small then $M_{\varepsilon,\mathbf{p}}$ is a contraction map on $B_{\varepsilon,\sigma}$. The existence then follows by the standard fixed point theorem and $\phi_{\varepsilon,\mathbf{p}}$ is a solution to (5.19). We thus proved

Lemma 5.2. *There exists $\overline{\varepsilon} > 0, \overline{\delta} > 0$ such that for every pair of $\varepsilon, \mathbf{p}$ with $0 < \varepsilon < \overline{\varepsilon}, \mathbf{p} \in (-1,1)^2$, and*

$$\min\big\{1 + p_1, 1 - p_2, |p_1 - p_2|\big\} > \overline{\delta},$$

there is a unique $\phi_{\varepsilon,\mathbf{p}} \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}$ satisfying $S_{\varepsilon}(w_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) \in \mathcal{C}_{\varepsilon,\mathbf{p}}$. Furthermore, we have the estimate

$$\|\phi_{\varepsilon,\mathbf{p}}\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} \le C\varepsilon^{\alpha}$$

for any $\alpha \le 2s - 1$.

More refined estimates for $\phi_{\varepsilon,\mathbf{p}}$ are needed. We recall from the discussion in last section that $S[w_{\varepsilon,\mathbf{p}}]$ can be decomposed into the two parts $S_{1,1}$ and $S_{1,2}$ if x is close to the center of spike, where $S_{1,1}$ is in leading order an odd function and $S_{1,2}$ is in leading order an evenly symmetric function. We can similarly decompose $\phi_{\varepsilon,\mathbf{p}}$ as in the following lemma.

Lemma 5.3. *Let $\phi_{\varepsilon,\mathbf{p}}$ be defined in Lemma 5.2. Then for $x = p_i + \varepsilon z, |\varepsilon z| < \delta, i = 1, 2$, we have the decomposition*

$$\phi_{\varepsilon,\mathbf{p}} = \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}, \tag{5.22}$$

where $\phi_{\varepsilon,\mathbf{p},2}$ is an even function in z which satisfies

$$\phi_{\varepsilon,\mathbf{p},2} = \mathcal{O}(\varepsilon^{2s-1}) \quad \textit{in } H^{2s}\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right), \tag{5.23}$$

and

$$\phi_{\varepsilon,\mathbf{p},1} = \mathcal{O}(\varepsilon) \quad \textit{in } H^{2s}\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right). \tag{5.24}$$

Proof. We first solve

$$S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p},2}]-S[w_{\varepsilon,\mathbf{p}}]+\sum_{j=1}^{2} S_{1,2}\left(y-\frac{p_j}{\varepsilon}\right)\in\mathcal{C}_{\varepsilon,\mathbf{p}} \tag{5.25}$$

for $\phi_{\varepsilon,\mathbf{p},2}\in\mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}$. Then we solve

$$S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p},2}+\phi_{\varepsilon,\mathbf{p},1}]-S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p},2}]+\sum_{j=1}^{2} S_{1,1}\left(y-\frac{p_j}{\varepsilon}\right)\in\mathcal{C}_{\varepsilon,\mathbf{p}}, \tag{5.26}$$

for $\phi_{\varepsilon,\mathbf{p},1}\in\mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}$. Using the same proof as in Proposition 5.1, both equations (5.25) and (5.26) have unique solution provided $\varepsilon\ll 1$. By uniqueness, $\phi_{\varepsilon,\mathbf{p}}=\phi_{\varepsilon,\mathbf{p},1}+\phi_{\varepsilon,\mathbf{p},2}$, and it is easy to see that $\phi_{\varepsilon,\mathbf{p},1}$ and $\phi_{\varepsilon,\mathbf{p},2}$ have the required properties. □

5.3. The reduced problem

In this subsection, we solve the reduced problem which finishes the proof of Theorem 2.1. By Proposition 5.1 for every $\mathbf{p}\in B_{\varepsilon^{2s-1}}(\mathbf{p}^0)$ there exists a unique solution $\phi_{\varepsilon,\mathbf{p}}\in\mathcal{K}^{\perp}_{\varepsilon,\mathbf{p}}$ such that

$$S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}]\in\mathcal{C}_{\varepsilon,\mathbf{p}}. \tag{5.27}$$

To complete the proof of Theorem 2.1 we need to determine $\mathbf{p}^{\varepsilon}=(p_1^{\varepsilon},p_2^{\varepsilon})$ near $\mathbf{p}^0$ such that $S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}]\perp\mathcal{C}_{\varepsilon,\mathbf{p}}$, which in turn implies that $S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}]=0$. To this end, let

$$W_{\varepsilon}:=\big(W_{\varepsilon,1}(\mathbf{p}),W_{\varepsilon,2}(\mathbf{p})\big):B_{\varepsilon^{2s-1}}(\mathbf{p}^0)\to\mathbb{R}^2,$$

where

$$W_{\varepsilon,i}(\mathbf{p}):=\varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} S[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}]\frac{\partial w_i}{\partial y}dy,\quad i=1,2.$$

Then $W_{\varepsilon}(\mathbf{p})$ is a map which is continuous in $\mathbf{p}$ and our problem is reduced to finding a zero of the vector field $W_{\varepsilon}(\mathbf{p})$. Let us now calculate $W_{\varepsilon}(\mathbf{p})$

$$W_{\varepsilon,i}(\mathbf{p})=\varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} S_{\varepsilon}[w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}]\frac{\partial w_i}{\partial y}dy$$

$$
\begin{aligned}
&= \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^s(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) + (w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) - \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]+\psi_{\varepsilon,\mathbf{p}}}\right]\frac{\partial w_i}{\partial y}dy \\
&= \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^s(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) + (w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) - \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]}\right]\frac{\partial w_i}{\partial y}dy \\
&\quad - \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[\frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]+\psi_{\varepsilon,\mathbf{p}}} - \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]}\right]\frac{\partial w_i}{\partial y}dy \\
&= I_1 + I_2, \qquad (5.28)
\end{aligned}
$$

where I_1, I_2 are defined by the last equality and $\psi_{\varepsilon,\mathbf{p}}$ satisfies

$$
D(-\Delta)^s\psi_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}} - 2c_\varepsilon w_{\varepsilon,\mathbf{p}}\phi_{\varepsilon,\mathbf{p}} - c_\varepsilon\phi^2_{\varepsilon,\mathbf{p}} = 0. \qquad (5.29)
$$

For I_1, we have by Lemma 5.3

$$
\begin{aligned}
I_1 &= \varepsilon^{-1}\left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^s(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) + (w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}}) - \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)}\right]\frac{\partial w_i}{\partial y}dy \right. \qquad (5.30)\\
&\qquad \left. + \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{(T[w_{\varepsilon,\mathbf{p}}](p_i))^2}\big(T[w_{\varepsilon,\mathbf{p}}](p_i+\varepsilon y) - T[w_{\varepsilon,\mathbf{p}}](p_i)\big)\frac{\partial w_i}{\partial y}dy\right) + \mathcal{O}(\varepsilon^{2s-1}) \\
&= \varepsilon^{-1}\left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^s(\hat{\xi}_i w_i+\phi_{\varepsilon,\mathbf{p}}) + (\hat{\xi}_i w_i+\phi_{\varepsilon,\mathbf{p}}) - \frac{(\hat{\xi}_i w_i+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}](p_i)}\right]\frac{\partial w_i}{\partial y}dy\right) \\
&\quad + \varepsilon^{-1}\left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(\hat{\xi}_i w_i+\phi_{\varepsilon,\mathbf{p},2})^2}{(T[w_{\varepsilon,\mathbf{p}}](p_i))^2}\big(T[w_{\varepsilon,\mathbf{p}}](p_i+\varepsilon y) - T[w_{\varepsilon,\mathbf{p}}](p_i)\big)\frac{\partial w_i}{\partial y}dy\right) + \mathcal{O}(\varepsilon^{2s-1}).
\end{aligned}
$$

Note that, by Lemma 5.3, we have

$$
\begin{aligned}
&\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \big[(-\Delta)^s\phi_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}} - 2w_i\phi_{\varepsilon,\mathbf{p}}\big]\frac{\partial w_i}{\partial y}dy \\
&= \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_{\varepsilon,\mathbf{p},1}\frac{\partial}{\partial y}\left((-\Delta)^s w_i + w_i - w_i^2\right)dy + \mathcal{O}(\varepsilon^{1+2s}) = \mathcal{O}(\varepsilon^{1+2s}), \qquad (5.31)
\end{aligned}
$$

and

$$
\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi^2_{\varepsilon,\mathbf{p}}\frac{\partial w_i}{\partial y}dy = 2\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_{\varepsilon,\mathbf{p},1}\phi_{\varepsilon,\mathbf{p},2}\frac{\partial w_i}{\partial y}dy + h.o.t. = \mathcal{O}(\varepsilon^{2s}). \qquad (5.32)
$$

Now by Lemma 5.3 and Eqs. (5.30) and (5.31) we have

$$
\begin{aligned}
I_1 &= \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} w_i^2\big(T[w_{\varepsilon,\mathbf{p}}](p_i+\varepsilon z) - T[w_{\varepsilon,\mathbf{p}}](p_i)\big)\frac{\partial w_i}{\partial y}dy + \mathcal{O}(\varepsilon^{2s-1}) \\
&= \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} w_i^2\left(P_i(z) + \varepsilon\sum_{j\neq i}\hat{\xi}_j^2 z\nabla_{p_i}G_D(p_i,p_j)\right)\frac{\partial w_i}{\partial y}dy + \mathcal{O}(\varepsilon^{2s-1})
\end{aligned}
$$

$$= -\frac{1}{3}\int_{\mathbb{R}} w^3(y)dy \sum_{j\neq i} \hat{\xi}_j^2 \nabla_{p_i} G_D(p_i,p_j) + \mathcal{O}(\varepsilon^{2s-1}). \tag{5.33}$$

Similarly, we calculate

$$\begin{aligned} I_2 &= \varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[\frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]+\psi_{\varepsilon,\mathbf{p}}} - \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{T[w_{\varepsilon,\mathbf{p}}]}\right]\frac{\partial w_i}{\partial y}dy \\ &= -\varepsilon^{-1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(w_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{(T[w_{\varepsilon,\mathbf{p}}])^2}\psi_{\varepsilon,\mathbf{p}}\frac{\partial w_i}{\partial y}dy + \mathcal{O}(\varepsilon^{2s-1}) \\ &= -\varepsilon^{-1}\hat{\xi}_i\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{1}{3}\frac{\partial w^3}{\partial y}\big(\psi_{\varepsilon,\mathbf{p}} - \psi_{\varepsilon,\mathbf{p}}(p_i)\big)dy + \mathcal{O}(\varepsilon^{2s-1}). \end{aligned} \tag{5.34}$$

Since $\psi_{\varepsilon,\mathbf{p}}$ satisfies (5.29), a similar argument to that used in Lemma 5.3 gives

$$\begin{aligned} &\psi_{\varepsilon,\mathbf{p}}(p_i+\varepsilon z) - \psi_{\varepsilon,\mathbf{p}}(p_i) \\ &= c_\varepsilon \int_{-1}^{1} \big(G_D(p_i+\varepsilon z,\zeta) - G_D(p_i,\zeta)\big)\left(2w_{\varepsilon,\mathbf{p}}\left(\frac{\zeta}{\varepsilon}\right)\phi_{\varepsilon,\mathbf{p}}\left(\frac{\zeta}{\varepsilon}\right) + \phi_{\varepsilon,\mathbf{p}}^2\left(\frac{\zeta}{\varepsilon}\right)\right)d\zeta \\ &= o\bigg(\varepsilon\sum_{j\neq i}\hat{\xi}_j^2 z\nabla_{p_i}G_D(p_i,p_j)\bigg) + \hat{P}_i(z) + h.o.t., \end{aligned} \tag{5.35}$$

where $\hat{P}_i(z)$ is an even function in $z = y - \frac{p_i}{\varepsilon}$. Substituting (5.35) into (5.34) we obtain that

$$I_2 = o\bigg(\sum_{j\neq i}\hat{\xi}_j^2\nabla_{p_i}G_D(p_i,p_j)\bigg) + o(\varepsilon^{2s-1}). \tag{5.36}$$

Combining the estimates for I_1 and I_2, we obtain

$$\begin{aligned} W_{\varepsilon,i}(\mathbf{p}) &= -\frac{1}{3}\int_{\mathbb{R}} w^3(y)dy\sum_{j\neq i}\hat{\xi}_j^2\nabla_{p_i}G_D(p_i,p_j)\big(1+o(1)\big) + \mathcal{O}(\varepsilon^{2s-1}) \\ &= -\frac{1}{3}F_i(\mathbf{p})\int_{\mathbb{R}} w^3(y)dy + \mathcal{O}(\varepsilon^{2s-1}), \end{aligned} \tag{5.37}$$

where $F_i(\mathbf{p})$ is defined in (2.9). From (H3) we have $F(\mathbf{p}^0) = 0$ and from numerical calculations of the Green's function we deduce that $p_2^0 - p_1^0 = 1$. By symmetry we conclude that if there exists $\mathbf{p} = (p_1,p_2)$ such that either one of $W_{\varepsilon,1}(\mathbf{p}) = 0$ or $W_{\varepsilon,2}(\mathbf{p}) = 0$ then $W_\varepsilon(\mathbf{p}) = 0$. For $W_{\varepsilon,i}$ we have

$$\begin{aligned} W_{\varepsilon,i}(\mathbf{p}) = -\frac{1}{3}\int_{\mathbb{R}} w^3(y)dy\,\Big(&\big(p_1-p_1^0\big)\hat{\xi}_2^2\nabla_{p_1}\nabla_{p_1}G_D\big(p_1^0,p_2^0\big) \\ &+\big(p_2-p_2^0\big)\hat{\xi}_2^2\nabla_{p_2}\nabla_{p_1}G_D\big(p_1^0,p_2^0\big)\Big) \\ &+\mathcal{O}\big(|\mathbf{p}-\mathbf{p}^0|^2+\varepsilon^{2s-1}\big). \end{aligned}$$

By assumption (H3) we have $\nabla_{p_1}\nabla_{p_1}G_D(p_1^0,p_2^0)\neq 0$. As a consequence, we can apply Brouwer's fixed point theorem to show that for $\varepsilon \ll 1$ there exists a point $\mathbf{p}^\varepsilon$ such that $W_\varepsilon(\mathbf{p}^\varepsilon)=0$ and $\mathbf{p}^\varepsilon \in B_{\varepsilon^{2s-1}}(\mathbf{p}^0)$. Thus we have proved the following proposition

Proposition 5.2. *For ε sufficiently small there exist points $\mathbf{p}^\varepsilon$ with $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ such that $W_\varepsilon(\mathbf{p}^\varepsilon)=0$.*

Proof of Theorem 2.1. By above proposition, there exists $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ such that $W_\varepsilon(\mathbf{p}^\varepsilon)=0$. In other words, $S[w_{\varepsilon,\mathbf{p}^\varepsilon}+\phi_{\varepsilon,\mathbf{p}^\varepsilon}]=0$. Let $u_\varepsilon = c_\varepsilon(w_{\varepsilon,\mathbf{p}^\varepsilon}+\phi_{\varepsilon,\mathbf{p}^\varepsilon})$, $v_\varepsilon = c_\varepsilon T[w_{\varepsilon,\mathbf{p}^\varepsilon}+\phi_{\varepsilon,\mathbf{p}^\varepsilon}]$. By the Maximum principle, $u_\varepsilon > 0$ and $v_\varepsilon > 0$. Moreover $(u_\varepsilon, v_\varepsilon)$ satisfies all the properties of Theorem 2.1. □

6. Rigorous proof of the stability analysis

The linear stability of the two-spike solution constructed above is determined by two classes of eigenvalues: the large and small eigenvalues satisfying $\lambda_\varepsilon = \mathcal{O}(1)$ and $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$ respectively. In the following two subsections we consider each case separately.

6.1. Stability analysis: large eigenvalues

In this subsection, we consider the stability of the steady state $(u_\varepsilon, v_\varepsilon)$ constructed in Theorem 2.1. Linearizing around the equilibrium states,

$$u = u_\varepsilon + \phi_\varepsilon(x)e^{\lambda_\varepsilon t}, \quad v = v_\varepsilon + \psi_\varepsilon e^{\lambda_\varepsilon t} = T[u_\varepsilon] + \psi_\varepsilon e^{\lambda_\varepsilon t}, \tag{6.1}$$

and substituting the result into (GM) we deduce the following eigenvalue problem:

$$\begin{cases} (-\Delta)_y^s\phi_\varepsilon + \phi_\varepsilon - 2\dfrac{u_\varepsilon}{T[u_\varepsilon]}\phi_\varepsilon + \dfrac{u_\varepsilon^2}{(T[u_\varepsilon])^2}\psi_\varepsilon + \lambda_\varepsilon\phi_\varepsilon = 0, & \text{(6.2a)} \\ D(-\Delta)^s\psi_\varepsilon + \psi_\varepsilon - 2c_\varepsilon u_\varepsilon\phi_\varepsilon + \tau\lambda_\varepsilon\psi_\varepsilon = 0, & \text{(6.2b)} \end{cases}$$

where λ_ε is some complex number. In this section, we study the large eigenvalues, i.e. those for which we may assume that there exists $c>0$ such that $|\lambda_\varepsilon| \geq c > 0$ for ε small. If $\Re(\lambda_\varepsilon) < -c$ then we are done (since these eigenvalues are always stable) and we therefore assume that $\Re(\lambda_\varepsilon) \geq -c$. For a subsequence $\varepsilon \to 0$ and $\lambda_\varepsilon \to \lambda_0$ we shall derive a limiting NLEP satisfied by λ_0.

We first present the case $\tau = 0$. In the end, we shall explain how we proceed when $\tau > 0$ is sufficiently small. By Eq. (6.2b), we have $\psi_\varepsilon = T'[u_\varepsilon](\phi_\varepsilon)$. Let us assume that $\|\phi_\varepsilon\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = 1$ and we cut off ϕ_ε as follows:

$$\phi_{\varepsilon,i}(y) = \phi_\varepsilon(y)\chi\left(\frac{\varepsilon y - p_i^\varepsilon}{r_0}\right), \tag{6.3}$$

where $\chi(x)$ is a given in (5.1) of Section 5. Using Lemma 5.3 together with $\Re(\lambda_\varepsilon) \geq -c$, the asymptotic expansion of u_ε given in Theorem 2.1, and the algebraic decay of w given in Proposition 3.1, we get that

$$\phi_\varepsilon = \sum_{i=1}^{2} \phi_{\varepsilon,i} + h.o.t. \quad \text{in } H^{2s}\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right). \tag{6.4}$$

Then by standard procedure we extend $\phi_{\varepsilon,i}$ to a function defined on $\mathbb{R}$ such that

$$\|\phi_{\varepsilon,i}\|_{H^{2s}(\mathbb{R})} \leq C\|\phi_{\varepsilon,i}\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})}, \quad i = 1, 2. \tag{6.5}$$

Without loss of generality, we may assume that $\|\phi_\varepsilon\|_{H^{2s}(\mathbb{R})} = 1$ and by taking a subsequence of ε, we may also assume that $\phi_{\varepsilon,i} \to \phi_i$ strongly as $\varepsilon \to 0$ in $L^2 \cap L^\infty$ for $i = 1, 2$, on compact subsets of $\mathbb{R}$. Therefore we also have

$$w\phi_{\varepsilon,i} \to w\phi_i \quad \text{as} \quad \varepsilon \to 0, \quad \text{strongly in } L^\infty(\mathbb{R}). \tag{6.6}$$

It is known that

$$\psi_\varepsilon(x) = c_\varepsilon \int_{-1}^{1} G_D(x, \zeta) u_\varepsilon\left(\frac{\zeta}{\varepsilon}\right) \phi_\varepsilon\left(\frac{\zeta}{\varepsilon}\right) d\zeta. \tag{6.7}$$

Now we use the expansion of u_ε to calculate the value of ψ_ε at $x = p_i^\varepsilon$ for each $i = 1, 2$

$$\begin{aligned}\psi_\varepsilon(p_i^\varepsilon) &= 2c_\varepsilon \int_{-1}^{1} G_D(p_i^\varepsilon, \zeta) \sum_{j=1}^{2} \hat{\xi}_j w\left(\frac{\zeta - p_j^\varepsilon}{\varepsilon}\right) \chi\left(\frac{\varepsilon\zeta - p_j}{r_0}\right) \phi_\varepsilon\left(\frac{\zeta}{\varepsilon}\right) d\zeta + h.o.t. \\ &= 2\varepsilon c_\varepsilon \sum_{j=1}^{2} \hat{\xi}_j G_D(p_i, p_j) \int_{\mathbb{R}} w\phi_j dy + o_\varepsilon(1). \end{aligned} \tag{6.8}$$

Substituting (6.8) into Eq. (6.2a) and letting $\varepsilon \to 0$, we obtain the nonlocal eigenvalue problem

$$\begin{aligned}&(-\Delta)^s \phi_i + \phi_i - 2w\phi_i + 2\left(\int_{\mathbb{R}} w^2(y)dy\right)^{-1} \left(\int_{\mathbb{R}} \sum_{j=1}^{2} \hat{\xi}_j G_D(p_i, p_j) w\phi_j dy\right) w^2 \\ &\quad + \lambda_0 \phi_i = 0, \quad i = 1, 2. \end{aligned} \tag{6.9}$$

We can rewrite (6.9) in matrix form as

$$(-\Delta)^s \Phi + \Phi - 2w\Phi + 2\left(\int_{\mathbb{R}} w^2(y)dy\right)^{-1} \left(\int_{\mathbb{R}} w\mathcal{B}\Phi dy\right) w^2 + \lambda_0 \Phi = 0, \tag{6.10}$$

where $\mathcal{B}$ is the matrix introduced in (2.7) and $\Phi = (\phi_1, \phi_2)^T \in (H^{2s}(\mathbb{R}))^2$. We then have the following conclusion

Theorem 6.1. *Let λ_ε be an eigenvalue of* (6.2) *such that $\Re(\lambda_\varepsilon) > -c$ for some $c > 0$.*

(1) Suppose that for suitable sequence $\varepsilon_n \to 0$ we have $\lambda_{\varepsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the problem given in (6.9).

(2) Let $\lambda_0 \neq 0$ with $\Re(\lambda_0) > 0$ be an eigenvalue of the problem given in (6.9). *Then for ε sufficiently small, there is an eigenvalue λ_ε of* (6.1) *with $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$.*

Proof. The proof of (1) follows from a similar asymptotic analysis to that used in Section 5.

To prove part (2) of Theorem 6.1 we follow the argument given in [3, Section 2]. We assume that $\lambda_0 \neq 0$ is an eigenvalue of problem (6.9) with $\Re(\lambda_0) > 0$ and we first note that from the equation for ψ_ε, we can express ψ_ε in terms of ϕ_ε as in (6.7). Then we rewrite Eq. (6.2) as

$$\phi_\varepsilon = -R_\varepsilon(\lambda_\varepsilon)\left[2\frac{u_\varepsilon\phi_\varepsilon}{v_\varepsilon} - \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon\right],$$

where $R_\varepsilon(\lambda_\varepsilon)$ is the inverse of $(-\Delta)^s + (1+\lambda_\varepsilon)$ in $H^{2s}(\mathbb{R})$ and $\psi_\varepsilon = T_\varepsilon'[u_\varepsilon](\phi_\varepsilon)$ is given in Eq. (6.2b). The key observation is that $R_\varepsilon(\lambda_\varepsilon)$ is a Fredholm type operator if ε is sufficiently small. The rest of the argument follows as in [3]. □

By diagonalizing $\mathcal{B}$ we see that the eigenvalue problem (6.10) can be reduced to the nonlocal eigenvalue problems

$$(-\Delta)^s\hat{\phi}_i + \hat{\phi}_i - 2w\hat{\phi}_i + 2\sigma_i\frac{\int_{\mathbb{R}} w\hat{\phi}_i dy}{\int_{\mathbb{R}} w^2 dy}w^2 + \lambda_0\hat{\phi}_i = 0, \quad \hat{\phi}_i \in H^{2s}(\mathbb{R}), \quad i = 1,2, \tag{6.11}$$

where σ_1 and σ_2 are the two eigenvalues of $\mathcal{B}$.

We now study the stability of (6.2) for large eigenvalues explicitly. Suppose that

$$2\min_{\sigma\in\lambda(\mathcal{B})}\sigma < 1. \tag{6.12}$$

Then by Theorem 3.1(1) there exists an unstable eigenvalue of (6.10) and therefore by Theorem 6.1 there exists an eigenvalue λ_ε of (6.2) such that $\Re(\lambda_\varepsilon) > c_0$ for some positive number c_0. This implies that $(u_\varepsilon, v_\varepsilon)$ is unstable. On the other hand if $2\min_{\sigma\in\lambda(\mathcal{B})}\sigma > 1$ then by Theorem 3.1(2) any nonzero eigenvalue λ_0 is stable. Therefore by Theorem 6.1 for ε small enough all nonzero eigenvalues λ_ε of (6.2) for which $|\lambda_\varepsilon| \geq c > 0$ holds, satisfy $\Re(\lambda_\varepsilon) \leq -c < 0$ for ε small enough.

Finally we comment that when $\tau \neq 0$ and τ is small. We shall apply the results of Theorem 3.2. In this case, the matrix $\mathcal{B}$ will have to be replaced by the matrix $\mathcal{B}_{\tau\lambda_\varepsilon}$ which depends on $\tau\varepsilon$. In particular the Green's function G_D is replaced by the Green's function G_D^λ satisfying

$$D(-\Delta)^s G_D^\lambda + (1+\tau\lambda_\varepsilon)G_D^\lambda = \delta_z, \quad G_D^\lambda(x+2,z) = G_D^\lambda(x,z). \tag{6.13}$$

It is then easy to check that the eigenvalues of $\mathcal{B}_{\tau\lambda_\varepsilon}$ satisfy the same properties as those of $\mathcal{B}$ provided that τ is sufficiently small.

6.2. Stability analysis: small eigenvalues

We now study the eigenvalue problem (6.2) with respect to small eigenvalues. Namely, we assume that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. Let

$$\bar{u}_\varepsilon = w_{\varepsilon,\mathbf{p}^\varepsilon} + \phi_{\varepsilon,\mathbf{p}^\varepsilon}, \quad \bar{v}_\varepsilon = T[w_{\varepsilon,\mathbf{p}^\varepsilon} + \phi_{\varepsilon,\mathbf{p}^\varepsilon}], \tag{6.14}$$

where $\mathbf{p}^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon)$. After rescaling, the eigenvalue problem (6.2) becomes

$$\begin{cases} (-\Delta)_y^s \phi_\varepsilon + \phi_\varepsilon - \dfrac{2\bar{u}_\varepsilon}{\bar{v}_\varepsilon}\phi_\varepsilon + \dfrac{\bar{u}_\varepsilon^2}{\bar{v}_\varepsilon^2}\psi_\varepsilon + \lambda_\varepsilon \phi_\varepsilon = 0, \\ D(-\Delta)^s \psi_\varepsilon + \psi_\varepsilon - 2c_\varepsilon \bar{u}_\varepsilon \phi_\varepsilon + \tau\lambda_\varepsilon \psi_\varepsilon = 0, \end{cases} \tag{6.15}$$

where c_ε is given by (5.4). We take $\tau = 0$ for simplicity. As $\tau\lambda_\varepsilon \ll 1$ the results in this section are also valid for τ finite, this is due to the fact that the small eigenvalue are of the order $\mathcal{O}(\varepsilon^2)$, we shall prove it in this subsection.

We cut off $\bar{u}_\varepsilon$ as follows:

$$\tilde{u}_{\varepsilon,i}(y) = \chi\left(\frac{\varepsilon y - p_i^\varepsilon}{r_0}\right)\bar{u}_\varepsilon(y), \quad i = 1, 2, \tag{6.16}$$

where $\chi(x)$ and r_0 are given in Section 5. Similarly to the Section 5 we define

$$\mathcal{K}_{\varepsilon,\mathbf{p},new} := \mathrm{Span}\left\{\tilde{u}'_{\varepsilon,i} \mid i = 1, 2\right\} \subset H^{2s}\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right),$$
$$\mathcal{C}_{\varepsilon,\mathbf{p},new} := \mathrm{Span}\left\{\tilde{u}'_{\varepsilon,i} \mid i = 1, 2\right\} \subset L^2\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right).$$

Then it is easy to see that

$$\bar{u}_\varepsilon(y) = \sum_{i=1}^{2} \tilde{u}_{\varepsilon,i}(y) + h.o.t.. \tag{6.17}$$

Note that

$$\tilde{u}_{\varepsilon,i}(y) \sim \hat{\xi}_i w\left(y - \frac{p_i^\varepsilon}{\varepsilon}\right) \quad \text{in } H^{2s}(-1, 1)$$

and $\tilde{u}_{\varepsilon,i}$ satisfies

$$(-\Delta)^s \tilde{u}_{\varepsilon,i} + \tilde{u}_{\varepsilon,i} - \frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon} + h.o.t. = 0. \tag{6.18}$$

Thus $\tilde{u}'_{\varepsilon,i} := \frac{d\tilde{u}_{\varepsilon,i}}{dy}$ satisfies

$$(-\Delta)_y^s \tilde{u}'_{\varepsilon,i} + \tilde{u}'_{\varepsilon,i} - 2\frac{\tilde{u}_{\varepsilon,i}}{\bar{v}_\varepsilon}\tilde{u}'_{\varepsilon,i} + \varepsilon\frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2}\bar{v}'_\varepsilon + h.o.t. = 0, \tag{6.19}$$

and we have

$$\tilde{u}'_{\varepsilon,i} = \hat{\xi}_i \frac{\partial w}{\partial y}\left(y - \frac{p_i^\varepsilon}{\varepsilon}\right)\big(1 + o(1)\big).$$

Let us now decompose

$$\phi_\varepsilon = \sum_{i=1}^{2} a_i^\varepsilon \tilde{u}'_{\varepsilon,i} + \phi_\varepsilon^\perp, \tag{6.20}$$

where a_i^ε are complex numbers and $\phi_\varepsilon^\perp \perp \mathcal{K}_\varepsilon$. Similarly, we can decompose

$$\psi_\varepsilon = \sum_{i=1}^{2} a_i^\varepsilon \psi_{\varepsilon,i} + \psi_\varepsilon^\perp, \tag{6.21}$$

where $\psi_{\varepsilon,i}$ satisfies

$$D(-\Delta)^s \psi_{\varepsilon,i} + \psi_{\varepsilon,i} - 2c_\varepsilon \bar{u}_\varepsilon \tilde{u}'_{\varepsilon,i} = 0, \quad i = 1, 2, \tag{6.22}$$

and $\psi_\varepsilon^\perp$ satisfies

$$D(-\Delta)^s \psi_\varepsilon^\perp + \psi_\varepsilon^\perp - 2c_\varepsilon \bar{u}_\varepsilon \phi_\varepsilon^\perp = 0. \tag{6.23}$$

We impose periodic boundary conditions on both of these equations.

Suppose that $\|\phi_\varepsilon\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = 1$. Then $|a_i^\varepsilon| \le C$ since

$$a_i^\varepsilon = \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_\varepsilon \frac{\partial \tilde{u}_{\varepsilon,i}}{\partial y} dy \left(\hat{\xi}_i^2 \int_{\mathbb{R}} w^2 dy\right)^{-1} + o(1).$$

Substituting the decompositions of ϕ_ε and ψ_ε into (6.15) we have

$$\begin{aligned}(-\Delta)_y^s \phi_\varepsilon^\perp + \phi_\varepsilon^\perp - \frac{2\bar{u}_\varepsilon}{\bar{v}_\varepsilon}\phi_\varepsilon^\perp + \frac{\bar{u}_\varepsilon^2}{\bar{v}_\varepsilon^2}\psi_\varepsilon^\perp + \lambda_\varepsilon \phi_\varepsilon^\perp - \varepsilon \sum_{i=1}^{2} a_i^\varepsilon \left(\frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2}\bar{v}'_\varepsilon - \frac{1}{\varepsilon}\frac{\bar{u}_\varepsilon^2}{\bar{v}_\varepsilon^2}\psi_{\varepsilon,i}\right) \\ + h.o.t. = -\lambda_\varepsilon \sum_{i=1}^{2} a_i^\varepsilon \tilde{u}'_{\varepsilon,i}.\end{aligned} \tag{6.24}$$

Let us first compute

$$\begin{aligned} J_0 &:= \varepsilon \sum_{i=1}^{2} a_i^\varepsilon \left(\frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2}\bar{v}'_\varepsilon - \frac{1}{\varepsilon}\frac{\bar{u}_\varepsilon^2}{\bar{v}_\varepsilon^2}\psi_{\varepsilon,i}\right) \\ &= \varepsilon \sum_{i=1}^{2} a_i^\varepsilon \left(\frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2}\left(\bar{v}'_\varepsilon - \frac{1}{\varepsilon}\psi_{\varepsilon,i}\right)\right) - \sum_{i=1}^{2} a_i^\varepsilon \sum_{j\neq i} \frac{\tilde{u}_{\varepsilon,j}^2}{\bar{v}_\varepsilon^2}\psi_{\varepsilon,i} + h.o.t. \\ &= \varepsilon \sum_{i=1}^{2} a_i^\varepsilon \frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2}\left(-\frac{1}{\varepsilon}\psi_{\varepsilon,i} + \bar{v}'_\varepsilon\right) - \sum_{i=1}^{2}\sum_{j\neq i} a_j^\varepsilon \psi_{\varepsilon,j}\frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_\varepsilon^2} + h.o.t.\,. \end{aligned}$$

We rewrite J_0 as follows

$$J_0 = -\varepsilon \sum_{i=1}^{2}\sum_{j=1}^{2} a_j^{\varepsilon} \frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_{\varepsilon}^2}\left(\frac{1}{\varepsilon}\psi_{\varepsilon,j} - \bar{v}_{\varepsilon}'\delta_{ij}\right) + h.o.t.. \tag{6.25}$$

Let us also set

$$\tilde{L}_{\varepsilon}\phi_{\varepsilon}^{\perp} := -(-\Delta)_y^s \phi_{\varepsilon}^{\perp} - \phi_{\varepsilon}^{\perp} + 2\frac{\bar{u}_{\varepsilon}}{\bar{v}_{\varepsilon}}\phi_{\varepsilon}^{\perp} - \frac{\bar{u}_{\varepsilon}^2}{\bar{v}_{\varepsilon}^2}\psi_{\varepsilon}^{\perp} \tag{6.26}$$

and

$$\mathbf{a}_{\varepsilon} := \big(a_1^{\varepsilon}, a_2^{\varepsilon}\big)^T. \tag{6.27}$$

Multiplying both sides of (6.24) by $\tilde{u}'_{\varepsilon,l}$ and integrating over $\big(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\big)$, we obtain

$$r.h.s. = -\lambda_{\varepsilon}\sum_{i=1}^{2} a_i^{\varepsilon}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \tilde{u}'_{\varepsilon,i}\tilde{u}'_{\varepsilon,l}dy = -\lambda_{\varepsilon}a_l^{\varepsilon}\hat{\xi}_l^2\int_{\mathbb{R}}\big(w'(y)\big)^2 dy\big(1+\mathcal{O}(\varepsilon^{2s+1})\big), \tag{6.28}$$

and

$$\begin{aligned} l.h.s. = &\left(\sum_{i=1}^{2}\sum_{j=1}^{2} a_j^{\varepsilon}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,i}^2}{\bar{v}_{\varepsilon}^2}\big(\psi_{\varepsilon,j} - \varepsilon\bar{v}_{\varepsilon}'\delta_{ij}\big)\tilde{u}'_{\varepsilon,l}dy \right.\\ &\left. + \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_{\varepsilon}^2}\psi_{\varepsilon}^{\perp}\tilde{u}'_{\varepsilon,l}dy - \varepsilon\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_{\varepsilon}^2}\bar{v}_{\varepsilon}'\phi_{\varepsilon}^{\perp}dx\right)\big(1+o(1)\big)\\ = &\big(J_{1,l} + J_{2,l} + J_{3,l})(1+o(1)\big), \end{aligned} \tag{6.29}$$

where $J_{i,l}, i = 1,2,3$, are defined by the last inequality.

We define the vectors

$$\mathbf{J}_i = (J_{i,1}, J_{i,2})^T, \quad i = 1,2,3. \tag{6.30}$$

To give estimates on each $\mathbf{J}_i, i = 1,2,3$ we need the following three lemmas.

Lemma 6.1. *We have*

$$\big(\psi_{\varepsilon,j} - \varepsilon\bar{v}_{\varepsilon}'\delta_{ji}\big)\big(p_i^{\varepsilon}\big) = -\varepsilon\nabla\mathcal{G}_D^T\mathcal{H}^2 + o(\varepsilon). \tag{6.31}$$

Proof. Note that for $i \neq j$, we have

$$\begin{aligned} \big(\psi_{\varepsilon,j} - \varepsilon\bar{v}_{\varepsilon}'\delta_{ji}\big)\big(p_i^{\varepsilon}\big) &= \psi_{\varepsilon,j}\big(p_i^{\varepsilon}\big) = 2c_{\varepsilon}\int_{-1}^{1} G_D\big(p_i^{\varepsilon},\zeta\big)\bar{u}_{\varepsilon}\tilde{u}'_{\varepsilon,j}d\zeta\\ &= -\varepsilon\hat{\xi}_j^2\nabla_{p_j^{\varepsilon}}G_D\big(p_i^{\varepsilon}, p_j^{\varepsilon}\big) + \mathcal{O}(\varepsilon^{1+2s}). \end{aligned} \tag{6.32}$$

Next we compute $\psi_{\varepsilon,i} - \bar{v}_{\varepsilon}'$ near p_i^{ε}

$$\begin{aligned} \bar{v}_{\varepsilon}(x) &= c_{\varepsilon}\int_{-1}^{1} G_D(x,\zeta)\bar{u}^2\left(\frac{\zeta}{\varepsilon}\right)d\zeta\\ &= \varepsilon c_{\varepsilon}\int_{-\infty}^{\infty} G_D(x,\varepsilon z)\tilde{u}_{\varepsilon,i}^2(z)dz + c_{\varepsilon}\sum_{j\neq i}\int_{-1}^{1} G_D(x,\zeta)\tilde{u}_{\varepsilon,j}^2\left(\frac{\zeta}{\varepsilon}\right)d\zeta + \mathcal{O}(\varepsilon^{1+2s}). \end{aligned}$$

So

$$\bar{v}_\varepsilon'(x) = \varepsilon c_\varepsilon \int_{-\infty}^{\infty} \nabla_x G_D(x, \varepsilon z) \tilde{u}_{\varepsilon,i}^2(z) dz + c_\varepsilon \sum_{j \neq i} \int_{-1}^{1} \nabla_x G_D(x, \zeta) \tilde{u}_{\varepsilon,j}^2 \left(\frac{\zeta}{\varepsilon} \right) d\zeta + \mathcal{O}(\varepsilon^{1+2s}).$$

Thus

$$\begin{aligned} \psi_{\varepsilon,i}(x) - \varepsilon \bar{v}_\varepsilon'(x) = 2\varepsilon c_\varepsilon \int_{-\infty}^{\infty} G_D(x, \varepsilon z) \tilde{u}_{\varepsilon,i} \tilde{u}_{\varepsilon,i}' dz - \varepsilon^2 c_\varepsilon \int_{-\infty}^{\infty} \nabla_x G_D(x, \varepsilon z) \tilde{u}_{\varepsilon,i}^2(z) dz \\ - \varepsilon c_\varepsilon \sum_{j \neq i} \int_{-1}^{1} \nabla_x G_D(x, \zeta) \tilde{u}_{\varepsilon,j}^2 \left(\frac{\zeta}{\varepsilon} \right) d\zeta + \mathcal{O}(\varepsilon^{1+2s}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \psi_{\varepsilon,i}\big(p_i^\varepsilon\big) - \varepsilon \bar{v}_\varepsilon'\big(p_i^\varepsilon\big) &= 2\varepsilon c_\varepsilon \int_{-\infty}^{\infty} G_D\big(p_i^\varepsilon, \varepsilon z\big) \tilde{u}_{\varepsilon,i} \tilde{u}_{\varepsilon,i}' dz - \varepsilon^2 c_\varepsilon \int_{-\infty}^{\infty} \nabla_{p_i^\varepsilon} G_D\big(p_i^\varepsilon, \varepsilon z\big) \tilde{u}_{\varepsilon,i}^2(z) dz \\ &\quad - \varepsilon^2 c_\varepsilon \sum_{j \neq i} \int_{-\infty}^{\infty} \nabla_{p_i^\varepsilon} G_D\big(p_i^\varepsilon, \varepsilon z\big) \tilde{u}_{\varepsilon,j}^2(z) dz + \mathcal{O}(\varepsilon^{1+2s}) \\ &= -2\varepsilon \hat{\xi}_i^2 \nabla_{p_i^\varepsilon} G_D\big(p_i^\varepsilon, p_i^\varepsilon\big) - \varepsilon \sum_{j \neq i} \hat{\xi}_j^2 \nabla_{p_i^\varepsilon} G_D\big(p_i^\varepsilon, p_j^\varepsilon\big) + \mathcal{O}(\varepsilon^{2s}) \\ &= -\varepsilon \hat{\xi}_i^2 \nabla_{p_i^\varepsilon} G_D\big(p_i^\varepsilon, p_i^\varepsilon\big) + \mathcal{O}(\varepsilon^{2s}). \end{aligned} \tag{6.33}$$

Eq. (6.31) then follows from solving (6.32) and (6.33). □

Lemma 6.2. *Let q_{ji} be defined as in* (2.16)*. Then we have*

$$\begin{aligned} &\big(\psi_{\varepsilon,i} - \varepsilon \bar{v}_\varepsilon' \delta_{ji}\big)\big(p_j^\varepsilon + \varepsilon z\big) - \big(\psi_{\varepsilon,i} - \varepsilon \bar{v}_\varepsilon' \delta_{ji}\big)\big(p_j^\varepsilon\big) \\ &= -\varepsilon^2 z \left(\nabla_{p_j^\varepsilon} \nabla_{p_i^\varepsilon} G_D\big(p_j^\varepsilon, p_i^\varepsilon\big) + q_{ji} \delta_{ji} \right) \hat{\xi}_i^2 + o(\varepsilon^2). \end{aligned} \tag{6.34}$$

We next study the asymptotic expansion of $\phi_\varepsilon^\perp$. Let us first define

$$\phi_{\varepsilon,i}^1 = \sum_{j=1}^{2} \nabla_{p_i^\varepsilon} \hat{\xi}_j w \left(y - \frac{p_j^\varepsilon}{\varepsilon} \right), \quad \phi_\varepsilon^1 := -\varepsilon \sum_{i=1}^{2} a_i^\varepsilon \phi_{\varepsilon,i}^1. \tag{6.35}$$

Then we have the following lemma.

Lemma 6.3. *Let ε be sufficiently small. Then*

$$\big\| \phi_\varepsilon^\perp - \phi_\varepsilon^1 \big\|_{H^{2s}(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})} = o(\varepsilon). \tag{6.36}$$

Proof. We first derive a relation between $\psi_\varepsilon^\perp$ and $\phi_\varepsilon^\perp$. Note that similar to the proof of Proposition 5.1, $\tilde{L}_\varepsilon$ is invertible from $\mathcal{K}^\perp_{\varepsilon,\mathbf{p},new}$ to $\mathcal{C}^\perp_{\varepsilon,\mathbf{p},new}$. By Lemma 6.1 and the fact that $\tilde{L}_\varepsilon$ is invertible, we deduce that

$$\left\|\phi_\varepsilon^\perp\right\|_{H^{2s}(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = \mathcal{O}(\varepsilon). \tag{6.37}$$

Let us decompose

$$\tilde{\phi}_{\varepsilon,i} = \frac{\phi_\varepsilon^\perp}{\varepsilon}\chi\left(\frac{\varepsilon y - p_i^\varepsilon}{r_0}\right). \tag{6.38}$$

Then

$$\phi_\varepsilon^\perp = \varepsilon\sum_{i=1}^{2}\tilde{\phi}_{\varepsilon,i} + h.o.t.. \tag{6.39}$$

Suppose that

$$\tilde{\phi}_{\varepsilon,i} \to \phi_i \quad \text{in } H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right). \tag{6.40}$$

By the equation for $\psi_\varepsilon^\perp$ (similar to the proof of Lemma 6.1)

$$\begin{aligned}\psi_\varepsilon^\perp\big(p_i^\varepsilon\big) &= 2\varepsilon c_\varepsilon\sum_{j=1}^{2}\int_{-1}^{1}G_D\big(p_i^\varepsilon,z\big)\bar{u}_\varepsilon\tilde{\phi}_{\varepsilon,j}(z)dz + o(\varepsilon)\\ &= 2\varepsilon\sum_{j=1}^{2}G_D\big(p_i^\varepsilon,p_j^\varepsilon\big)\hat{\xi}_j\frac{\int_{\mathbb{R}}w\phi_j dx}{\int_{\mathbb{R}}w^2dx} + o(\varepsilon),\end{aligned} \tag{6.41}$$

and therefore

$$\big(\psi_\varepsilon^\perp(p_1^\varepsilon),\psi_\varepsilon^\perp(p_2^\varepsilon)\big)^T = 2\varepsilon\mathcal{G}_D\mathcal{H}\frac{\int_{\mathbb{R}}w\Phi_0 dx}{\int_{\mathbb{R}}w^2dx} + o(\varepsilon), \tag{6.42}$$

where $\Phi_0 = (\phi_1,\phi_2)^T$. Substituting (6.42) into (6.24) and using Lemma 6.1, we have that Φ_0 satisfies

$$(-\Delta)^s\Phi_0 + \Phi_0 - 2w\Phi_0 + 2\mathcal{G}_D\mathcal{H}\frac{\int_{\mathbb{R}}w\Phi_0 dx}{\int_{\mathbb{R}}w^2dx}w^2 - (\nabla\mathcal{G}_D)^T\mathcal{H}^2\mathbf{a}^0w^2 = 0, \tag{6.43}$$

where

$$\mathbf{a}^0 = \lim_{\varepsilon\to0}\mathbf{a}^\varepsilon = \lim_{\varepsilon\to0}\big(a_1^\varepsilon,a_2^\varepsilon\big)^T.$$

Thus

$$\Phi_0 = -\left(I - 2\mathcal{G}_D\mathcal{H}\right)^{-1}(\nabla\mathcal{G}_D)^T\mathcal{H}^2\mathbf{a}^0w = -\mathcal{P}(\nabla\mathcal{G}_D)^T\mathcal{H}^2\mathbf{a}^0w, \tag{6.44}$$

where

$$\mathcal{P} = (I - 2\mathcal{G}_D\mathcal{H})^{-1}.$$

Now we compare Φ_0 with ϕ_ε^1. By definition,

$$\phi_\varepsilon^1 = -\varepsilon \sum_{j=1}^{2}\sum_{i=1}^{2} a_i^\varepsilon \nabla_{p_i^\varepsilon}\hat{\xi}_j w\left(y - \frac{p_j^\varepsilon}{\varepsilon}\right). \tag{6.45}$$

On the other hand,

$$\phi_\varepsilon^\perp = \varepsilon \sum_{i=1}^{2} \tilde{\phi}_{\varepsilon,i} + h.o.t. = \varepsilon \sum_{i=1}^{2} \phi_i\left(y - \frac{p_i^\varepsilon}{\varepsilon}\right) + o(\varepsilon). \tag{6.46}$$

The lemma is proved by using (6.44) and comparing (6.45) and (6.46). □

From Lemma 6.3 we have that

$$\left(\psi_\varepsilon^\perp(p_1^\varepsilon), \psi_\varepsilon^\perp(p_2^\varepsilon)\right)^T = -2\varepsilon \mathcal{G}_D \mathcal{H}\mathcal{P}(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mathbf{a}^0 + o(\varepsilon) \tag{6.47}$$

and

$$\psi_\varepsilon^\perp\left(p_i^\varepsilon + \varepsilon z\right) - \psi_\varepsilon^\perp\left(p_i^\varepsilon\right) = 2\varepsilon^2 z \sum_{j=1}^{2} \nabla_{p_i^\varepsilon} G_D\left(p_i^\varepsilon, p_j^\varepsilon\right)\hat{\xi}_j \frac{\int_{\mathbb{R}} w\phi_j dx}{\int_{\mathbb{R}} w^2 dx} + o(\varepsilon^2). \tag{6.48}$$

With the above three lemmas we can now derive the following results concerning the three terms $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ defined in (6.30).

Lemma 6.4. *Let $\mathcal{G}_D, \mathcal{H}, \mathcal{Q}$, and $\mathbf{a}_\varepsilon$ be given by* (2.4)*,* (2.15)*,* (2.16)*,* (6.27) *respectively. Then*

$$\mathbf{J}_1 = c_1\varepsilon^2 \mathcal{H}(\nabla^2 \mathcal{G}_D + \mathcal{Q})\mathcal{H}^2 \mathbf{a}_\varepsilon + o(\varepsilon^2), \tag{6.49a}$$

$$\mathbf{J}_2 = 2c_1\varepsilon^2 \mathcal{H}\nabla \mathcal{G}_D \mathcal{H}\mathcal{P}(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mathbf{a}_\varepsilon + o(\varepsilon^2), \tag{6.49b}$$

$$\mathbf{J}_3 = o(\varepsilon^2),$$

where $c_1 = \frac{1}{3}\int_{\mathbb{R}} w^3 dy$.

Proof. The computation of $\mathbf{J}_1$ follows from Lemma 6.2. In fact since $\bar{v}_\varepsilon' = o(1)$

$$\begin{aligned}
J_{1,l} &= \sum_{j=1}^{2} a_j^\varepsilon \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\left(\psi_{\varepsilon,j} - \varepsilon \bar{v}_\varepsilon' \delta_{jl}\right)\tilde{u}_{\varepsilon,l}' dy \\
&= \sum_{j=1}^{2} a_j^\varepsilon \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\Big(\left[\psi_{\varepsilon,j}(y) - \varepsilon \bar{v}_\varepsilon'(y)\delta_{jl}\right] - \left[\psi_{\varepsilon,j}(p_l^\varepsilon) - \varepsilon \bar{v}_\varepsilon'(p_l^\varepsilon)\delta_{jl}\right]\Big)\tilde{u}_{\varepsilon,l}' dy + o(\varepsilon^2) \\
&= -\varepsilon^2 \hat{\xi}_l \int_{\mathbb{R}} \left(y w^2 w'(y)\right) dy \sum_{j=1}^{2} a_j^\varepsilon \left(\nabla_{p_l^\varepsilon}\nabla_{p_j^\varepsilon} G_D\left(p_l^\varepsilon, p_j^\varepsilon\right) + q_{lj}\delta_{lj}\right)\hat{\xi}_j^2 + o(\varepsilon^2) \\
&= c_1\varepsilon^2 \hat{\xi}_l \sum_{j=1}^{2} a_j^\varepsilon \left(\nabla_{p_l^\varepsilon}\nabla_{p_j^\varepsilon} G_D\left(p_l^\varepsilon, p_j^\varepsilon\right) + q_{lj}\delta_{lj}\right)\hat{\xi}_j^2 + o(\varepsilon^2),
\end{aligned} \tag{6.50}$$

which, by Lemma 6.1, proves estimate (6.49a). The estimate for $\mathbf{J}_2$ follows from

$$\begin{aligned} J_{2,l} &= \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\psi_\varepsilon^\perp \tilde{u}'_{\varepsilon,l}dy \\ &= \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\psi_\varepsilon^\perp\big(p_l^\varepsilon\big)\tilde{u}'_{\varepsilon,l}dy + \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\big(\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(p_l^\varepsilon)\big)\tilde{u}'_{\varepsilon,l}dy \\ &= \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\tilde{u}_{\varepsilon,l}^2}{\bar{v}_\varepsilon^2}\big(\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(p_l^\varepsilon)\big)\tilde{u}'_{\varepsilon,l}dx + o(\varepsilon^2), \end{aligned} \tag{6.51}$$

together with (6.44), (6.47), (6.48). The estimate on $J_{3,l}$ follows from Lemma 6.3, the fact that $\bar{v}_\varepsilon(p_l^\varepsilon) = \hat{\xi}_l + \mathcal{O}(\varepsilon^{2s-1})$ at p_l^ε and the leading order of $\bar{v}'_\varepsilon(p_l^\varepsilon + \varepsilon y) - \bar{v}'_\varepsilon(p_l^\varepsilon)$ is an odd function of order ε. □

We can now provide an estimate on the small eigenvalue. From Lemma 6.4 we have

$$\begin{aligned} \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 &= c_1\varepsilon^2\mathcal{H}\left((\nabla^2\mathcal{G}_D + \mathcal{Q})\mathcal{H}^2 + 2\nabla\mathcal{G}_D\mathcal{H}\mathcal{P}(\nabla\mathcal{G}_D)^T\mathcal{H}^2\right)\mathbf{a}_\varepsilon + o(\varepsilon^2) \\ &= c_1\varepsilon^2\mathcal{H}^2\mathcal{M}(\mathbf{p}^\varepsilon)\mathbf{a}_\varepsilon + o(\varepsilon^2), \end{aligned}$$

and therefore by combining (6.28) and (6.29), we obtain

$$c_1\varepsilon^2\mathcal{H}^2\mathcal{M}(\mathbf{p}^\varepsilon)\mathbf{a}_\varepsilon + o(\varepsilon^2) = -\lambda_\varepsilon\mathcal{H}^2\mathbf{a}_\varepsilon\int_{\mathbb{R}}\big(w'(y)\big)^2dy\big(1+o(1)\big). \tag{6.52}$$

From this equation we see that

$$\lambda_\varepsilon = -\varepsilon^2 c_2\lambda_{\mathcal{M}(\mathbf{p}^0)}\big(1+o(1)\big),$$

where c_2 is a positive constant and $\lambda_{\mathcal{M}(\mathbf{p}^0)}$ is the non-zero left eigenvalue of $\mathcal{M}(\mathbf{p}^0)$ given in (2.19). In Appendix C we derive a quickly converging series expression for the Green's function for which we can interchanging summation and differentiation to calculate its second derivatives. Numerical calculations then indicate that $\partial_x^2 G_D(x,0) > 0$ at $x = 1$ and using (2.19) we therefore deduce that the non-zero left eigenvalue of $\mathcal{M}(\mathbf{p}^0)$ is positive. The small λ_ε is therefore negative (stable) so that the two spike pattern is linearly stable with respect to the small eigenvalues. In particular, linear stability is determined solely by the eigenvalues of $\mathcal{B}$ and the proof of Theorem 2.2 is therefore complete.

7. Conclusion and open problems

In this paper we have proven the existence and rigorously analyzed the stability of both symmetric and asymmetric two spike equilibrium solutions of the fractional one-dimensional Gierer-Meinhardt system (1.1) with periodic boundary conditions. In addition, by using a combination of formal asymptotic and numerical methods we have

calculated asymptotic approximations for N-spike quasi-equilibrium solutions and derived a system of ODEs governing their slow dynamics on an $\mathcal{O}(\varepsilon^{-2})$ timescale as well as a system of NLEPs governing their linear stability on an $\mathcal{O}(1)$ timescale. Our findings indicate that a single spike solution may be destabilized or stabilized with respect to oscillatory instabilities by decreasing the fractional exponents for the activator, s_1, or inhibitor, s_2, respectively. On the other hand we found that decreasing the fractional exponent for the inhibitor, s_2, has a stabilizing effect on the stability of symmetric two-spike solutions with respect to competition instabilities. Finally we determined that asymmetric two-spike solutions are always linearly unstable with respect to competition instabilities. In all one- and two-spike cases we found that the equilibrium spike patterns are linearly stable with respect to the slow dynamics and that this is a consequence of the choice of periodic boundary conditions.

We conclude this section with an outline of open problems and directions for future research. The first open problem is to prove the existence and to provide a complete classification of all N-spike equilibrium solutions to the fractional one-dimensional Gierer-Meinhardt model. In particular a key question is whether, as in the classical Gierer-Meinhardt model [31], asymmetric N-spike solutions are generated by sequences of spikes of two types. Second, in this paper we have chosen to use periodic boundary conditions to reduce the technical difficulties typically encountered when implementing Dirichlet or Neumann boundary conditions (see [17]). However, we believe our results can be extended to these more general cases by appropriately modifying the relevant Green's function satisfying (2.3) with Dirichlet or Neumann boundary conditions as well as by extending our analysis to provide regularity estimates at the boundaries $x = \pm 1$. Another interesting direction for future research is to investigate the behaviour of solutions to the fractional GM model in the $D \ll 1$ regimes for which the classical GM model is known to exhibit distinct behaviour such as spike splitting and clustering. Finally a detailed analysis, either rigorous or formal, of localized solutions for different reaction-kinetics as well as in higher-dimensional domains would be a fruitful direction of future research. Indeed the analysis of localized solutions to the classical Gierer-Meinhardt model is markedly different in one-, two-, and three-dimensions [8, 13, 29] due, at least in part, to the different singular behaviour of the related Green's functions and we suspect that this will also be the case for the higher-dimensional analogues of (1.1).

Appendix A. The nonlocal eigenvalue problem

In this section, we prove Theorem 3.1. We consider the eigenvalue problem

$$(-\Delta)^s \phi + \phi - 2w\phi + \gamma \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} + \alpha\phi = 0, \quad \phi \in H^{2s}(\mathbb{R}). \tag{A.1}$$

Our aim is to show that the above eigenvalue problem has an eigenvalue with real part when $\gamma \in (0, 1)$ and the real part of the eigenvalue is always negative if $\gamma > 1$ and $s > \frac{1}{4}$.

Before we give the proof of Theorem 3.1, we first present the following result.

Proposition A.1 ([4]). *The eigenvalue problem*

$$(-\Delta)^s\phi + \phi - 2w\phi + \mu\phi = 0 \quad \textit{in } \mathbb{R}, \quad \phi \in H^{2s}(\mathbb{R}) \tag{A.2}$$

admits the following set of eigenvalues:

$$\mu_1 > 0, \quad \mu_2 = 0, \quad \mu_3 < 0, \quad \dots. \tag{A.3}$$

Moreover, the eigenfunction corresponding to μ_1 is radial and of constant sign.

Proof of Theorem 3.1(1). The original problem is equivalent to finding a positive zero root of the function $\mathcal{F}(\alpha)$ defined by

$$\mathcal{F}(\alpha) = \int_{\mathbb{R}} w^2 dx + \gamma \int_{\mathbb{R}} w(L_0 + \alpha)^{-1} w^2 dx,$$

where

$$L_0\phi = (-\Delta)^s\phi + \phi - 2w\phi.$$

By the above proposition, L_0 has a unique eigenvalue $\mu_1 > 0$ with an eigenfunction of constant sign. We now consider $\mathcal{F}(\alpha)$ in the interval $(0, \mu_1)$. Since $L_0^{-1}w^2 = -w$, we deduce that

$$\mathcal{F}(0) = (1-\gamma)\int_{\mathbb{R}} w^2 dx > 0, \tag{A.4}$$

provided $\gamma < 1$. Next, as $\alpha \to \mu_1^-$, we have that

$$\int_{\mathbb{R}} w(L_0 + \alpha)^{-1} w^2 dx \to -\infty. \tag{A.5}$$

Hence, we get from (A.5) that

$$(\alpha) \to -\infty \quad \text{as} \quad \alpha \to \mu_1^{-1}, \tag{A.6}$$

when $\gamma \in (0,1)$. By (A.4), (A.6) and the continuity of (α), we can find a $\alpha_0 \in (0, \mu_1)$ such that $f(\alpha_0) = 0$ whenever $\gamma \in (0,1)$. □

Next, we shall study (A.1) when $\gamma > 1$. We shall prove that the real part of the eigenvalue is negative in any case. To this end, we introduce some notation and make some preparations. Set

$$L\phi := L_0\phi + \gamma \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w^2, \quad \phi \in H^{2s}(\mathbb{R}). \tag{A.7}$$

According to the definition of L, we can easily see that L is not self-adjoint. Let

$$X_0 := \ker\{L_0\} = \mathrm{Span}\left\{\frac{\partial w}{\partial x}\right\}.$$

Then

$$L_0 w = -w^2, \quad L_0\left(w + \frac{1}{2s} x \cdot \nabla w\right) = -w. \tag{A.8}$$

Hence,

$$\int_{\mathbb{R}} \left(L_0^{-1} w\right) w dx = \int_{\mathbb{R}} \left(-\frac{1}{2s} x \cdot \nabla w - w\right) w dx = \frac{1-4s}{4s} \int_{\mathbb{R}} w^2 dx, \tag{A.9}$$

and

$$\int_{\mathbb{R}} \left(L_0^{-1} w\right) w^2 dx = -\int_{\mathbb{R}} L_0^{-1} w L_0 w dx = -\int_{\mathbb{R}} w^2 dx. \tag{A.10}$$

Before we give the proof of Theorem 3.1. We present the following important lemma.

Lemma A.1. *Let L_1 be an operator defined by*

$$L_1\phi = L_0\phi + \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w^2 + \frac{\int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2 dx} w - \frac{\int_{\mathbb{R}} w^3 dx \int_{\mathbb{R}} w\phi dx}{\left(\int_{\mathbb{R}} w^2 dx\right)^2} w. \tag{A.11}$$

Then we have

(1) L_1 is self-adjoint and the kernel of L_1 (denoted by X_1) is $\mathrm{Span}\{w, \frac{\partial w}{\partial x}\}$.

(2) There exists a positive constant $a_1 > 0$ such that

$$\begin{aligned} L_1(\phi,\phi) := &\int_{\mathbb{R}} \left(\left|(-\Delta)^{\frac{s}{2}}\phi\right|^2 + \phi^2 - 2w\phi^2\right) dx + 2\frac{\int_{\mathbb{R}} w\phi dx \int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2 dx} \\ &- \frac{\int_{\mathbb{R}} w^3 dx \left(\int_{\mathbb{R}} w\phi dx\right)^2}{\left(\int_{\mathbb{R}} w^2 dx\right)^2} \\ &\geq a_1 d^2_{L^2(\mathbb{R})}(\phi, X_1) \end{aligned} \tag{A.12}$$

for all $\phi \in H^{2s}(\mathbb{R})$, where $d_{L^2(\mathbb{R})}$ means the distance in L^2-norm.

Proof. By (A.12), L_1 is self-adjoint. It is easy to see that $w, \frac{\partial w}{\partial y} \in \mathrm{Ker}\{L_1\}$. On the other hand, if $\phi \in \mathrm{Ker}\{L_1\}$, then by Proposition A.1

$$L_0\phi = -c_1(\phi)w - c_2(\phi)w^2 = c_1(\phi)L_0\left(w + \frac{1}{2s} x \cdot \nabla w\right) + c_2(\phi)L_0(w),$$

where

$$c_1(\phi) = \frac{\int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2 dx} - \frac{\int_{\mathbb{R}} w^3 dx \int_{\mathbb{R}} w\phi dx}{\left(\int_{\mathbb{R}} w^2 dx\right)^2}, \quad c_2(\phi) = \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx}.$$

Hence,

$$\phi - c_1(\phi)\left(w + \frac{1}{2s} x \cdot \nabla w\right) - c_2(\phi)w \in \mathrm{Ker}\{L_0\}. \tag{A.13}$$

Note that

$$\begin{aligned} c_1(\phi) &= c_1(\phi)\frac{\int_{\mathbb{R}} w^2(w+(x/2s)\cdot\nabla w)dx}{\int_{\mathbb{R}} w^2dx} \\ &\quad - c_1(\phi)\frac{\int_{\mathbb{R}} w^3dx\int_{\mathbb{R}} w(w+(x/2s)\cdot\nabla w)dx}{\left(\int_{\mathbb{R}} w^2dx\right)^2} \\ &= c_1(\phi) - c_1(\phi)\left(1-\frac{1}{4s}\right)\frac{\int_{\mathbb{R}} w^3dx}{\int_{\mathbb{R}} w^2dx} \end{aligned}$$

by (A.9) and (A.10). This implies that $c_1(\phi)=0$ for $s>\frac{1}{4}$. By (A.13) and Proposition A.1, we prove the first conclusion.

It remains to prove (2). Suppose it is not true. Then by the first conclusion there exists (α,ϕ) such that (i) α is real and positive, (ii) $\phi\perp w,\ \phi\perp\frac{\partial w}{\partial x}$, (iii) $L_1(\phi)+\alpha\phi=0$.

We shall show the above conclusion is not possible. From (ii) and (iii) we have

$$(L_0+\alpha)\phi+\frac{\int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2dx}w=0. \tag{A.14}$$

First we claim that $\int_{\mathbb{R}} w^2\phi\neq 0$. In fact if $\int_{\mathbb{R}} w^2\phi=0$, then $-\alpha<0$ is an eigenvalue of L_0. By Proposition A.1, $-\alpha=\mu_1$ and ϕ has constant sign. This contradicts with the fact that $\phi\perp w$. Therefore $-\alpha\neq\mu_1,0$ and hence $L_0+\alpha$ is invertible in $X_0^{\perp}$. So (A.14) implies

$$\phi=-\frac{\int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2}(L_0+\alpha)^{-1}w.$$

Thus

$$\int_{\mathbb{R}} w^2\phi dx=-\frac{\int_{\mathbb{R}} w^2\phi dx}{\int_{\mathbb{R}} w^2dx}\int_{\mathbb{R}}\big((L_0+\alpha)^{-1}w\big)w^2dx,$$

which implies

$$\int_{\mathbb{R}} w^2dx=-\int_{\mathbb{R}}\big((L_0+\alpha)^{-1}w\big)w^2dx=\int_{\mathbb{R}}\big((L_0+\alpha)^{-1}w\big)\big((L_0+\alpha)w-\alpha w\big)dx,$$

hence

$$\int_{\mathbb{R}}\big((L_0+\alpha)^{-1}w\big)wdx=0. \tag{A.15}$$

Let

$$h_1(\alpha)=\int_{\mathbb{R}}\big((L_0+\alpha)^{-1}w\big)wdx,$$

then

$$h_1(0)=\int_{\mathbb{R}}\big(L_0^{-1}w\big)wdx=-\int_{\mathbb{R}}\left(w+\frac{1}{2s}x\cdot\nabla w\right)w=\left(\frac{1}{4s}-1\right)\int_{\mathbb{R}} w^2dx<0,$$

due to $s>\frac{1}{4}$. Moreover,

$$h_1'(\alpha) = -\int_{\mathbb{R}} (L_0+\alpha)^{-2} ww = -\int_{\mathbb{R}} \big((L_0+\alpha)^{-1}w\big)^2 dx < 0.$$

This shows that $h_1(\alpha) < 0$ for all $\alpha \in (0, \mu_1)$. Clearly, $h_1(\alpha) > 0$ for all $\alpha \in (\mu_1, \infty)$ since $\lim_{\alpha\to+\infty} h_1(\alpha) = 0$. This is a contradiction to (A.15) and we finish the proof. □

Proof of Theorem 3.1(2)-(3). We now finish the proof of Theorem 3.1(2) and (3). First, we prove (2). Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \ker\{L_0\}$. Then we can obtain two equations

$$\begin{cases} L_0\phi_R + \gamma \dfrac{\int_{\mathbb{R}} w\phi_R dx}{\int_{\mathbb{R}} w^2 dx} w^2 = -\alpha_R\phi_R + \alpha_I\phi_I, & \text{(A.16a)} \\ L_0\phi_I + \gamma \dfrac{\int_{\mathbb{R}} w\phi_I dx}{\int_{\mathbb{R}} w^2 dx} w^2 = -\alpha_R\phi_I - \alpha_I\phi_R. & \text{(A.16b)} \end{cases}$$

Multiplying Eq. (A.16b) by ϕ_R and Eq. (A.16b) by ϕ_I and adding them together, we obtain

$$\begin{aligned} &-\alpha_R \int_{\mathbb{R}} \big(\phi_R^2 + \phi_I^2\big) dx \\ &= L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I) + \frac{\int_{\mathbb{R}} w^3 dx}{\big(\int_{\mathbb{R}} w^2 dx\big)^2} \left[\left(\int_{\mathbb{R}} w\phi_R dx \right)^2 + \left(\int_{\mathbb{R}} w\phi_I dx \right)^2 \right] \\ &\quad + (\gamma-2) \left(\int_{\mathbb{R}} w\phi_R dx \int_{\mathbb{R}} w^2\phi_R dx + \int_{\mathbb{R}} w\phi_I dx \int_{\mathbb{R}} w^2\phi_I dx \right) \left(\int_{\mathbb{R}} w^2 dx \right)^{-1}. \end{aligned} \quad \text{(A.17)}$$

Multiplying Eqs. (A.16) by w and adding together, we get

$$\int_{\mathbb{R}} w^2\phi_R dx - \gamma \frac{\int_{\mathbb{R}} w\phi_R dx}{\int_{\mathbb{R}} w^2 dx} \int_{\mathbb{R}} w^3 dx = \alpha_R \int_{\mathbb{R}} w\phi_R dx - \alpha_I \int_{\mathbb{R}} w\phi_I dx, \quad \text{(A.18a)}$$

$$\int_{\mathbb{R}} w^2\phi_I dx - \gamma \frac{\int_{\mathbb{R}} w\phi_I dx}{\int_{\mathbb{R}} w^2 dx} \int_{\mathbb{R}} w^3 dx = \alpha_R \int_{\mathbb{R}} w\phi_I dx + \alpha_I \int_{\mathbb{R}} w\phi_R dx. \quad \text{(A.18b)}$$

We multiply Eq. (A.18a) by $\int_{\mathbb{R}} w\phi_R dx$ and Eq. (A.18b) by $\int_{\mathbb{R}} w\phi_I dx$ and add them together, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} w\phi_R dx \int_{\mathbb{R}} w^2\phi_R dx + \int_{\mathbb{R}} w\phi_I dx \int_{\mathbb{R}} w^2\phi_I dx \\ &= \left(\alpha_R + \gamma \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) \left(\left(\int_{\mathbb{R}} w\phi_R dx \right)^2 + \left(\int_{\mathbb{R}} w\phi_I dx \right)^2 \right). \end{aligned} \quad \text{(A.19)}$$

Therefore, we have

$$-\alpha_R \int_{\mathbb{R}} \big(\phi_R^2 + \phi_I^2\big) dx = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+\frac{\int_{\mathbb{R}} w^3dx}{(\int_{\mathbb{R}} w^2dx)^2}\left[\left(\int_{\mathbb{R}} w\phi_R dx\right)^2+\left(\int_{\mathbb{R}} w\phi_I dx\right)^2\right]$$
$$+(\gamma-2)\left(\alpha_R+\gamma\frac{\int_{\mathbb{R}} w^3dx}{\int_{\mathbb{R}} w^2dx}\right)\frac{(\int_{\mathbb{R}} w\phi_R dx)^2+(\int_{\mathbb{R}} w\phi_I dx)^2dx}{\int_{\mathbb{R}} w^2dx}. \quad \text{(A.20)}$$

Set

$$\phi_R = c_R w+\phi_R^{\perp}, \quad \phi_R^{\perp}\perp X_1,$$
$$\phi_I = c_I w+\phi_I^{\perp}, \quad \phi_I^{\perp}\perp X_1.$$

Then

$$\int_{\mathbb{R}} w\phi_R dx = c_R\int_{\mathbb{R}} w^2dx, \quad \int_{\mathbb{R}} w\phi_I dx = c_I\int_{\mathbb{R}} w^2dx,$$

and

$$d^2_{L^2(\mathbb{R})}(\phi_R,X_1)=\left\|\phi_R^{\perp}\right\|^2_{L^2}, \quad d^2_{L^2(\mathbb{R})}(\phi_I,X_1)=\left\|\phi_I^{\perp}\right\|^2_{L^2}.$$

By some simple computations we have

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)+\alpha_R(\gamma-1)\big(c_R^2+c_I^2\big)\int_{\mathbb{R}} w^2dx$$
$$+\big(c_R^2+c_I^2\big)(\gamma-1)^2\int_{\mathbb{R}} w^3dx+\alpha_R\left(\left\|\phi_R^{\perp}\right\|^2_{L^2}+\left\|\phi_I^{\perp}\right\|^2_{L^2}\right)=0.$$

By Lemma A.1,

$$\alpha_R(\gamma-1)\big(c_R^2+c_I^2\big)\int_{\mathbb{R}} w^2dx+(\gamma-1)^2\big(c_R^2+c_I^2\big)\int_{\mathbb{R}} w^3dx$$
$$+\alpha_R\left(\left\|\phi_R^{\perp}\right\|^2_{L^2}+\left\|\phi_I^{\perp}\right\|^2_{L^2}\right)\le 0.$$

Since $\gamma>1$, we have $\alpha_R<0$, which proves Theorem 3.1(2).

It remains to prove the last conclusion. Since ϕ satisfies

$$(-\Delta)^s\phi+\phi-2w\phi+\gamma\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2dx}w^2=0. \quad \text{(A.21)}$$

Then $L_0\phi=-c_3(\phi)w^2$, where

$$c_3(\phi)=\gamma\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2dx}.$$

Hence, $\phi-c_3(\phi)w\in \mathrm{Ker}\{L_0\}$. Thus

$$c_3(\phi)\gamma=\gamma\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2dx}=c_3(\phi). \quad \text{(A.22)}$$

So if $\gamma\neq 1$, we get $c_3(\phi)=0$. Then $\phi\in\mathrm{Ker}\{L_0\}$ and we complete the proof. □

Appendix B. Overview of numerical calculations

In this section we briefly outline the numerical calculation of solutions to the core problem (3.1) and the time-dependent fractional GM system with periodic boundary conditions (1.1). In both cases we use the finite difference-quadrature discretization for the fractional Laplacian with piecewise linear interpolants developed by Huang and Oberman [12]. When discretizing (3.1) we approximate the fractional Laplacian on a truncated domain using the far-field behaviour presented in Proposition 3.1 to capture the nonlocal behaviour outside the truncated domain. On the other hand, when spatially discretizing (1.1) we use the spatial periodicity of the system to simplify the expression for the discrete fractional Laplacian. Time stepping of the spatially discretized system is then performed using a second-order semi-implicit backwards difference scheme [25]. In the remainder of this section we provide additional details for both of these cases.

First we consider the numerical calculation of solutions to the core problem (3.1). Since the domain for (3.1) is $-\infty < y < \infty$ we need to both truncate and then discretize the truncated domain to obtain a numerical calculation. Outside of the truncated domain we use the far-field behaviour from Proposition 3.1 to impose a Dirichlet boundary condition. Specifically, letting $L > 0$ we approximate solutions to (3.1) by solving the truncated problem

$$(-\Delta)^s U + U - U^2 = 0, \quad |y| < L, \quad U(y) = U(L)(L/y)^{1+2s}, \quad |y| \geq L,$$

where we have replaced $\mathfrak{b}_s$ with $U(L)L^{1+2s}$ since we do not yet know the value of $\mathfrak{b}_s$. To account for the nonlocal contributions outside of the truncated domain we discretize a computational domain that extends beyond the truncated domain. Specifically we discretize the computational domain $-2L \leq y \leq 2L$ by letting $y_i = ih$ for $i = -2N, \ldots, 2N$ where $h = \frac{L}{N}$. Seeking symmetric solutions we impose $U_i = U_{|i|}$ for all $i = -2N, \ldots, 2N$ which reduces the unknown values to $U_0, \ldots, U_N$. Note in addition that $U_i = (\frac{L}{y_{|i|}})^{1+2s} U_N$ for all $|i| > N$. The fractional Laplacian can then be approximated by (see [12, Section 5])

$$\begin{aligned} &(-\Delta)^s U(y_i) \approx (-\Delta_h)^s U_i \\ &= \sum_{j=-2N}^{2N} (U_i - U_{|i-j|}) w_j + C^{II} U_i - C_i^{III} U_N, \quad i = 0, \ldots, N, \end{aligned} \tag{B.1}$$

where the first term accounts for integration inside of the truncated domain and

$$\begin{aligned} w_j = &\frac{C_s}{2s(2s-1)h^{2s}} \\ &\times \begin{cases} 2^{1-2s} - 2 + (1-s)^{-1} s, & j = \pm 1, \\ |j+1|^{1-2s} - 2|j|^{1-2s} + |j-1|^{1-2s}, & \text{otherwise}, \end{cases} \quad (j = \pm 1, \pm 2, \ldots), \end{aligned} \tag{B.2}$$

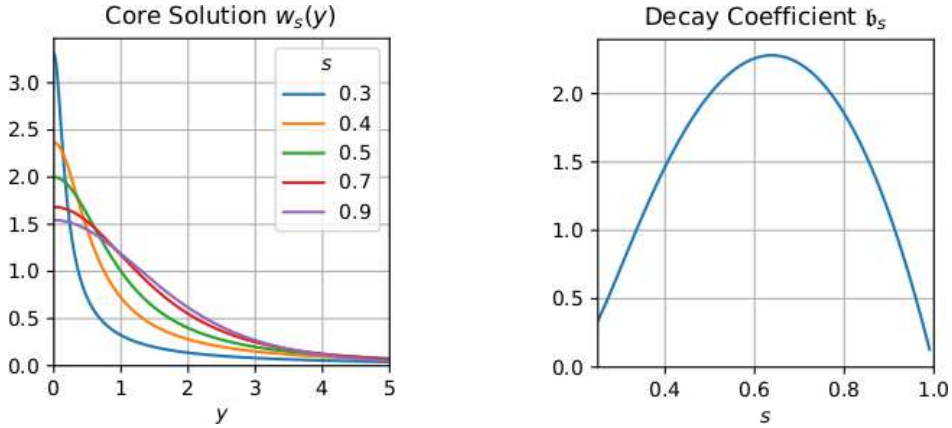
where we note that the value of w_0 is never needed in the discretization. Note that these weights require that $s \neq \frac{1}{2}$ but a simple modification is needed for $s = \frac{1}{2}$ (see [12, Section 3] for details). The remaining two terms C^{II} and C_i^{III} account for contributions outside of the computational domain and are respectively given by

$$C^{II} = \frac{C_s}{s(2L)^{2s}},$$
$$C_i^{III} = \frac{C_s L^{2s+1}}{(4s+1)(2L)^{4s+1}} \left({}_2F_1\left(2s+1, 4s+1; 4s+2, \frac{y_i}{2L}\right) + {}_2F_1\left(2s+1, 4s+1; 4s+2, -\frac{y_i}{2L}\right)\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

With the above discretization it is then possible to approximate solutions to (3.1) by solving the nonlinear algebraic system (B.1) for the $N+1$ unknowns $U_0, \ldots, U_N$. To numerically solve this nonlinear system we use the `fsolve` function in the Python 3.6.8 SciPy library. Our initial guess for the nonlinear solver is obtained by numerical continuation in s starting with $s = \frac{1}{2}$ for which the exact solution $w_s = \frac{2}{1+y^2}$ is known. In this way we may numerically calculate the core solution for an arbitrary value of s and in Fig. 3(a) we plot the resulting core solutions for select values of s where we have used $N = 2000$ for the spatial discretization. From these solutions we may also extract the value of the far-field decay coefficient $\mathfrak{b}_s$ and this is plotted in Fig. 3(b). We conclude by remarking that no nontrivial solution to the core problem (3.1) exists for $s \leq \frac{1}{6}$ (see for example [4]) and our numerical computations failed to yield solutions for $s \approx 0.2$ and below because of this.

Turning now to the numerical solution of (1.1) we discretize the interval $-1 < x < 1$ into N uniformly distributed points given by $x_i = -1 + 2ih$ for $i = 0, \ldots, N-1$



(a) (b)

Figure 3: (a) Sample plots of numerically computed solutions to the core problem (3.1). (b) Far-field decay coefficient $\mathfrak{b}_s$ in the core problem (3.1).

where $h = \frac{1}{N}$. Assuming that $\phi(x)$ is a 2-periodic function on $-1 < x < 1$ and letting $\phi_i \equiv \phi(x_i)$ for each $i = 0, \dots, N-1$ we calculate (see [12, Eq. (FLh)])

$$(-\Delta)^s \phi(x_i) \approx (-\Delta_h)^s \phi_i = \sum_{j=-\infty}^{\infty} (\phi_i - \phi_{i-j}) w_j = \sum_{j=0}^{N-1} W_{i-j}(\phi_i - \phi_j), \tag{B.3}$$

where the final equality follows from the periodicity of ϕ and where

$$W_\sigma \equiv w_\sigma + \sum_{k=1}^{\infty} (w_{\sigma+Nk} + w_{\sigma-Nk})$$

with each weight $w_i, i \in \mathbb{Z}$ being given by (B.2). In our numerical calculations we truncate the sum after 500 terms. From (B.3) it is then straightforward to deduce the entries of the matrix $(-\Delta_h)^s$ which we remark is dense in contrast to the tridiagonal matrix obtained by applying a finite-difference approximation to the one-dimensional Laplacian. With this spatial discretization we can then approximate (1.1) with the $2N$-dimensional system of ODEs

$$\frac{d\Phi}{dt} + \mathcal{A}\Phi + \mathcal{N}(\Phi) = 0, \tag{B.4}$$

where

$$\Phi(t) = \big(u_0(t), \dots, u_{N-1}(t), v_0(t), \dots, v_{N-1}(t)\big)^T,$$
$$\mathcal{A} = \operatorname{diag}\big(\varepsilon^{2s_1}(-\Delta_h)^{s_1}, \tau^{-1} D (-\Delta_h)^{s_2}\big),$$

and $\mathcal{N}(\Phi)$ is the $2N$-dimensional array that accounts for the nonlinearities in (1.1). To integrate (B.4) we employ a second-order semi-implicit backwards difference scheme (2-SBDF) [25] that uses second-order backward difference time-stepping for the fractional Laplace term and explicit (forward) time-stepping for the nonlinear term. Specifically, given a time-step size $\Delta t > 0$ and denoting by $\Phi_n = \Phi(t_n)$ where $t_n = n\Delta t$ the 2-SBDF scheme becomes

$$(3\mathcal{I} - 2\Delta t A)\Phi_{n+1} = 4\Phi_n - \Phi_{n-1} + 4\Delta t \mathcal{N}(\Phi_n) - 2\Delta t \mathcal{N}(\Phi_{n-1}), \tag{B.5}$$

where $\mathcal{I}$ is the $2N \times 2N$ identity matrix. Given an initial condition Φ_0 (based on the asymptotic approximations of Section 4) we also need Φ_1 to initiate time-stepping with 2-SBDF. We calculate Φ_1 by using a first-order semi-implicit backwards difference scheme (1-SBDF) [25] given by

$$(\mathcal{I} - \Delta t \mathcal{A})\Phi_{n+1} = \Phi_n + \Delta t \mathcal{N}(\Phi_n) \tag{B.6}$$

with which we perform five time steps with a step size that is one-fifth that used in our main 2-SBDF scheme. Throughout the numerical simulations of (1.1) in Sections 4.3 and 4.4 we used a mesh consisting of $N = 2000$ points and a time-step size of $\Delta t = 0.001$.

Appendix C. A rapidly converging series for the fractional Green's function

In this section we provide a quickly converging series expansion of the Green's function $G_D(x,z)$ satisfying (2.3). In particular, by adding and subtracting appropriate multiples of $|x-z|^{2s-1}$ and $|x-z|^{4s-1}$ as outlined below we obtain the series expansion

$$\begin{aligned} G_D(x,z) = \mathfrak{a}_s\left(|x-z|^{2s-1} - \frac{1}{2s}\right) - \mathfrak{c}_s\left(|x-z|^{4s-1} - \frac{1}{4s}\right) \\ - \frac{1}{2}\big((2s-1)\mathfrak{a}_s - (4s-1)\mathfrak{c}_s\big)\left(|x-z|^2 - \frac{1}{3}\right) \\ + \frac{1}{2} + \frac{1}{D^3}\sum_{n=1}^{\infty}\left(1 + \frac{1}{D(n\pi)^{2s}}\right)^{-1}\frac{\cos n\pi|x-z|}{(n\pi)^{6s}} \\ + 2\sum_{n=1}^{\infty}\left(\frac{\mathfrak{c}_s b_n}{(n\pi)^{4s}} - \frac{\mathfrak{a}_s a_n}{(n\pi)^{2s}}\right)\cos n\pi|x-z|, \end{aligned} \tag{C.1}$$

where

$$\mathfrak{a}_s = -\frac{2}{\pi D}s\Gamma(-2s)\sin(\pi s), \quad \mathfrak{c}_s \equiv -\frac{4}{\pi D^2}s\Gamma(-4s)\sin(2\pi s), \tag{C.2}$$

and

$$a_n = (2s-1)(2s-2)\int_{n\pi}^{\infty} x^{2s-3}\cos x dx, \tag{C.3}$$

$$b_n = -(4s-1)(4s-2)(4s-3)\left((-1)^n(n\pi)^{4s-4} + (4s-4)\int_{n\pi}^{\infty} x^{4s-5}\cos x dx\right). \tag{C.4}$$

The key reason for considering this expansion is that the coefficients of $\cos n\pi|x-z|$ converge to zero sufficiently fast to allow the order of summation and second-differentiation to be interchanged. In particular using (C.1) we can numerically calculate that $\partial_x^2 G_D(x,0)$ is strictly positive at $x=1$.

To derive (C.1) we use integration by parts to calculate the coefficients in the Fourier series

$$|x|^{\beta-1} = \frac{1}{\beta} + 2\sum_{n=1}^{\infty}\frac{c_{n,\beta}}{(\pi n)^{\beta}}\cos n\pi x, \quad c_{n,\beta} = \int_0^{n\pi} x^{\beta-1}\cos x dx, \tag{C.5}$$

where $\beta = 2s \in (1,2)$ or $\beta = 4s \in (2,4)$. Specifically we calculate

$$\begin{aligned} c_{n,2s} &= -(2s-1)\int_0^{n\pi} x^{2s-2}\sin x dx \\ &= -(2s-1)\int_0^{\infty} x^{2s-2}\sin x dx + (2s-1)\int_{n\pi}^{\infty} x^{2s-2}\sin x dx \\ &= -(2s-1)\int_0^{\infty} x^{2s-2}\sin x dx + (-1)^n(2s-1)(n\pi)^{2s-2} + a_n \end{aligned}$$

for $\beta = 2s$ and

$$c_{n,4s} = (4s-1)(4s-2)(4s-3)\int_0^\infty x^{4s-4}\sin x dx + (-1)^n(4s-1)(n\pi)^{4s-2} + b_n$$

for $\beta = 4s$ and where a_n and b_n are defined by (C.3). The definite integrals appearing in $c_{n,2s}$ and $c_{n,4s}$ can then be written in terms of $\mathfrak{a}_s$ and $\mathfrak{c}_s$ respectively by using the integral representation of the Gamma function

$$\int_0^\infty x^{z-1}\sin x dx = \Gamma(z)\sin\left(\frac{\pi z}{2}\right)$$

for $-1 < \Re(z) < 1$ together with the reflection formula

$$z\Gamma(z)\Gamma(-z) = -\frac{\pi}{\sin \pi z}$$

for all $z \notin \mathbb{Z}$ (see [22, Eqs. (5.9.7), (5.5.3)] respectively).

Appendix D. Derivation of the slow dynamics

In this appendix we outline the derivation of the system of ODEs (4.3) governing the slow dynamics of the multi-spike quasi-equilibrium solutions considered in Section 4.1. Letting $x = x_i + \varepsilon y$ with $y = \mathcal{O}(1)$ we obtain (4.1b) together with (C.1) (with $s = s_2$)

$$v \sim \varepsilon^{-1}\omega_{s_1}\left(\sum_{j=1}^N \xi_j^2 G_D(x_i,x_j) + \mathfrak{a}_{s_2}\xi_i^2\varepsilon^{2s_2-1}|y|^{2s_2-1} + \varepsilon b_i y + \mathcal{O}(\varepsilon^{\min\{2,4s_2-1\}})\right), \quad \text{(D.1)}$$

where $b_i \equiv \sum_{j\neq i}\xi_j^2\nabla_{x_i}G_D(x_i,x_j)$. It follows that the first order correction term in the inner expansion must be $\mathcal{O}(\varepsilon^{2s_2-1})$ and in particular for $x = x_i + \varepsilon y$ and $y = \mathcal{O}(1)$

$$u \sim \varepsilon^{-1}\big(\xi_i w_{s_1}(y) + \varepsilon^{2s_2-1}U_{i1} + o(\varepsilon^{2s_2-1})\big),$$
$$v \sim \varepsilon^{-1}\big(\xi_i + \varepsilon^{2s_2-1}V_{i1} + o(\varepsilon^{2s_2-1})\big).$$

By repeatedly using the method of matched asymptotic expansions we determine that the fractional power ε^{2s_2-1} initiates a chain of corrections at powers of ε that are multiples of $2s_2-1$. In particular for each $i = 1,\dots,N$ the inner expansion when $x = x_i + \varepsilon y$ with $y = \mathcal{O}(1)$ takes the form

$$u \sim \varepsilon^{-1}\left(\xi_i w_{s_1}(y) + \sum_{k=1}^{k_{\max}-1}\varepsilon^{k(2s_2-1)}U_{ik} + \varepsilon U_{ik_{\max}} + o(\varepsilon)\right), \quad \text{(D.2)}$$

$$v \sim \varepsilon^{-1}\left(\xi_i + \sum_{k=1}^{k_{\max}-1}\varepsilon^{k(2s_2-1)}V_{ik} + \varepsilon V_{ik_{\max}} + o(\varepsilon)\right), \quad \text{(D.3)}$$

where $k_{\max}$ is the smallest integer such that $k_{\max}(2s_2-1)\geq 1$. Importantly, since $V_{ik}\sim C_k|y|^{2s_2-1}$ as $|y|\to\infty$ for $1\leq k<k_{\max}$ each of these corrections are even in y. On the other hand when $k=k_{\max}$ we have the far-field behaviour

$$V_{ik_{\max}}\sim \omega_{s_1}b_i y+\delta_{1,k_{\max}(2s_2-1)}C_{k_{\max}}|y|^{2s_2-1},\quad |y|\to\infty, \tag{D.4}$$

where $\delta_{i,j}$ is the discrete Kronecker delta function. Therefore we can write $V_{ik_{\max}}=\omega_{s_1}b_i y+V^e_{ik_{\max}}$ where $V^e_{ik_{\max}}$ is an even function in y. Assuming that each $x_i=x_i(t)$ and substituting (D.2) into (1.1) with $x=x_i+\varepsilon y$ we obtain

$$\begin{aligned}-\frac{1}{\varepsilon}\xi_i\frac{dw_{s_1}}{dy}\frac{dx_i}{dt}+\sum_{k=1}^{k_{\max}-1}\varepsilon^{k(2s_2-1)}L_0U_{ik}+\varepsilon L_0U_{ik_{\max}}+\mathcal{N}_\varepsilon\\ +\varepsilon w_{s_1}^2\left(\omega_{s_1}b_i y+V^e_{ik_{\max}}\right)+o(\varepsilon)=0,\end{aligned} \tag{D.5}$$

where $\mathcal{N}_\varepsilon$ is an even function of y that consists of the residual nonlinear combinations of U_{ik} and V_{ik} for $1\leq k<k_{\max}$. Recalling that $\frac{dw_{s_1}}{dy}$ spans the kernel of L_0 we impose a solvability condition on (D.5) by multiplying it with $\frac{dw_{s_1}}{dy}$ and integrating to obtain

$$\frac{dx_i}{dt}=\varepsilon^2\frac{\omega_{s_1}b_i\int_{-\infty}^{\infty}w_{s_1}^2(dw_{s_1}/dy)ydy}{\xi_i\int_{-\infty}^{\infty}|dw_{s_1}/dy|^2dy}=-\varepsilon^2\frac{\omega_{s_1}\int_{-\infty}^{\infty}w_{s_1}^3}{3\xi_i\int_{-\infty}^{\infty}|dw_{s_1}/dy|^2dy}b_i,$$

where we have used integration by parts to obtain the second equality. This establishes (4.3).

References

[1] P. BRESSLOFF, *Stochastic Processes in Cell Biology*, in: *Interdisciplinary Applied Mathematics*, Springer International Publishing, 2014.

[2] W. CHEN AND M. J. WARD, *The stability and dynamics of localized spot patterns in the two-dimensional Gray-Scott model*, SIAM J. Appl. Dyn. Syst. 10 (2011), 582–666.

[3] E. N. DANCER, *On stability and Hopf bifurcations for chemotaxis systems*, Methods Appl. Anal. 8(2) (2001), 245–256.

[4] R. L. FRANK AND E. LENZMANN, *Uniqueness of non-linear ground states for fractional Laplacians in* $\mathbb{R}$, Acta Math. 210 (2013), 261–318.

[5] A. GIERER AND H. MEINHARDT, *A theory of biological pattern formation*, Kybernetik 12 (1972), 30–39.

[6] A. A. GOLOVIN, B. J. MATKOWSKY, AND V. A. VOLPERT, *Turing pattern formation in the brusselator model with superdiffusion*, SIAM J. Appl. Math. 69(1) (2008), 251–22.

[7] D. GOMEZ, M. J. WARD, AND J. WEI, *The linear stability of symmetric spike patterns for a bulk-membrane coupled Gierer-Meinhardt model*, SIAM J. Appl. Dyn. Syst. 18 (2019), 729–768.

[8] D. GOMEZ, M. J. WARD, AND J. WEI, *An asymptotic analysis of localized three-dimensional spot patterns for the gierer–meinhardt model: Existence, linear stability, and slow dynamics*, SIAM J. Appl. Math. 81 (2021), 378–406.

[9] C. GUI AND J. WEI, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Differential Equations 158 (1999), 1–27.
[10] C. GUI, J. WEI, AND M. WINTER, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), 47–82.
[11] B. HENRY, T. LANGLANDS, AND S. WEARNE, *Turing pattern formation in fractional activator-inhibitor systems*, Physical Review. E 72(2) (2005), p. 026101.
[12] Y. HUANG AND A. OBERMAN, *Numerical methods for the fractional laplacian: A finite difference-quadrature approach*, SIAM J. Numer. Anal. 52 (2013).
[13] D. IRON, M. J. WARD, AND J. WEI, *The stability of spike solutions to the one-dimensional Gierer-Meinhardt model*, Phys. D 150 (2001), 25–62.
[14] H. K. KHUDHAIR, Y. ZHANG, AND N. FUKAWA, *Pattern selection in the Schnakenberg equations: From normal to anomalous diffusion*, Numer. Methods Partial Differ. Equ. (2021), 1–18.
[15] T. KOLOKOLNIKOV, F. PAQUIN-LEFEBVRE, AND M. J. WARD, *Stable asymmetric spike equilibria for the Gierer–Meinhardt model with a precursor field*, IMA J. Appl. Math. 85 (2020), 605–634.
[16] T. KOLOKOLNIKOV, M. J. WARD, AND J. WEI, *Spot self-replication and dynamics for the Schnakenberg model in a two-dimensional domain*, J. Nonlinear Sci. 19 (2009), 1–56.
[17] A. LISCHKE, G. PANG, M. GULIAN, F. SONG, C. GLUSA, X. ZHENG, Z. MAO, W. CAI, M. M. MEERSCHAERT, M. AINSWORTH, AND G. E. KARNIADAKIS, *What is the fractional Laplacian? A comparative review with new results*, J. Comput. Phys. 404 (2020), p. 109009.
[18] R. METZLER AND J. KLAFTER, *The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A 37 (2004), R161–R208.
[19] J. D. MURRAY, *Mathematical Biology II: Spatial Models and Biomedical Applications*, in: *Interdisciplinary Applied Mathematics*, Springer-Verlag, 2003.
[20] Y. NEC, *Spike-type solutions to one dimensional Gierer–Meinhardt model with Lévy flights*, Stud. Appl. Math. 129 (2012), 272–299.
[21] Y. NEC AND M. WARD, *Dynamics and stability of spike-type solutions to a one dimensional Gierer–Meinhardt model with sub-diffusion*, Physica D 241 (2012), 947–963.
[22] *NIST Digital Library of Mathematical Functions*, `http://dlmf.nist.gov/`, Release 1.0. 28, 2020-09-15.
[23] F. OLIVEIRA, R. FERREIRA, L. LAPAS, AND M. VAINSTEIN, *Anomalous diffusion: A basic mechanism for the evolution of inhomogeneous systems*, Front. Phys. 7 (2019).
[24] J. REVEREY, J.-H. JEON, H. BAO, M. LEIPPE, R. METZLER, AND C. SELHUBER-UNKEL, *Superdiffusion dominates intracellular particle motion in the supercrowded space of pathogenic acanthamoeba castellanii*, Sci. Rep. 5 (2015), p. 11690.
[25] S. J. RUUTH, *Implicit-explicit methods for reaction-diffusion problems in pattern formation*, J. Math. Biol. 34 (1995), 148–176.
[26] M. J. WARD AND J. WEI, *Asymmetric spike patterns for the one-dimensional Gierer-Meinhardt model: equilibria and stability*, European J. Appl. Math. 13 (2002), 283–320.
[27] M. J. WARD AND J. WEI, *Hopf bifurcation and oscillatory instabilities of spike solutions for the one-dimensional Gierer-Meinhardt model*, J. Nonlinear Science 13 (2003), 209–264.
[28] J. WEI, *On the interior spike layer solutions to a singularly perturbed Neumann problem*, Tohoku Math. J. (2), 50 (1998), 159–178.
[29] J. WEI AND M. WINTER, *Spikes for the two-dimensional Gierer-Meinhardt system: the weak*

coupling case, J. Nonlinear Sci. 11 (2001), 415–458.
[30] J. WEI AND M. WINTER, *Spikes for the Gierer-Meinhardt system in two dimensions: the strong coupling case*, J. Differential Equations 178 (2002), 478–518.
[31] J. WEI AND M. WINTER, *Existence, classification and stability analysis of multiple-peaked solutions for the Gierer-Meinhardt system in* $\mathbf{R}^1$, Methods Appl. Anal. 14 (2007), 119–163.
[32] J. WEI AND M. WINTER, *Mathematical Aspects of Pattern Formation in Biological Systems*, in: *Applied Mathematical Sciences Series*, Vol. 189, Springer, 2014.
[33] J. WEI AND W. YANG, *Multi-bump ground states of the fractional Gierer-Meinhardt system on the real line*, J. Dynam. Differential Equations 31 (2019), 385–417.
[34] M. WINTER AND J. WEI, *On the Gierer-Meinhardt system with precursors*, Discrete Contin. Dyn. Syst. 25 (2009), 363–398.