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Inventory control policy for perishable products under a buyback contract and Brownian demands

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ABSTRACT

In this study, we analyse the inventory management strategy for perishables, where customer demands are assumed to arrive continuously and modelled by a drifted Brownian motion. We apply the well-known (s, S) continuous review inventory policy; in this policy, an order is placed to increase the inventory level to S as soon as it falls below s. Benefitted from highly developed transportation and logistics industries, it is reasonable to employ the zero-leadtime assumption in our model. In addition, a buyback contract is granted for the retailer to recover the loss from perishing. We illustrate the sales cycle through Markov renewal approach and derive closed-form formulas for long-run profit rates. To evaluate the optimal ordering policy which maximises the profit rate of the retailer, we first analyse the condition for accepting backorder and derive the optimal backorder level s as a function of S, and then solve for the global optimal (s, S) policy through numerical studies. Moreover, we develop a heuristic approach to approximate the optimal policy. Sensitivity analyses are conducted to reveal the effect of different parameters on system behaviour. The conclusions provide managerial insights for formulating inventory policies for perishables under Brownian demand.

1. Introduction

Inventory management under uncertain market environments plays a vital role in the performance of any firm. Nowadays, modern digital systems enable retailers to monitor their inventory in real-time. For a more balanced supply and demand, continuous review (s, S) policy is widely employed in real-world applications. In an (s, S) policy, when the inventory level reaches a lower bound s, the retailer immediately places an order to increase the inventory to a maximum level S. Generally, shops that are well stocked hold higher attraction and competitiveness, thus it is feasible to choose a positive s. However, in this study, we focus on the inventory control problem for perishable products, which have limited lifetimes. Representative examples of perishable products include food, drinks, and medical products, they are necessities for people and play an important role in the market. Expired products lose their usability and are no longer qualified for sale. A few ‘special products’, such as blood, also suffer from limited lifetimes. According to a nationwide survey on blood collection and utilisation conducted in the USA, 5.8% of all blood components processed for transfusion were outdated in 2004 (Whitaker and Sullivan, 2005). Due to such limited product lifetimes, an ineffective inventory management can result in high system costs. Usually, the expiration date must be labelled on the body of a product. As a result, in the retailing industry and specifically in supermarkets, it is frequently observed that, when choosing dairy or bakery products, customers are more inclined to select the latest item on the shelf because these products are considered to be fresher. Ideally, it is unwise to choose a positive lower bound s for perishable products, because when two batches of products are on sale simultaneously, the earlier ones are unlikely to be selected, which leads to an increase in perishing risk. When s equals zero, the retailer will immediately replenish when all the products are sold or perished. Moreover, a negative s indicates that backordering is allowed. Typically, there exists a constant cost for each order regardless of the costs associated with order quantity, such as the transportation and labour costs. When the inventory capacity is limited, retailers tend to backorder some demands to lengthen the ordering interval.

The inventory of fluid products is a highly realistic problem with a wide range of applications, such as reservoirs, chemical agents and gasoline in gas stations. Traditional discrete arrival process usually assumes that customers arrive one after another with random inter-arrival

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time, and the quantity of customer demands upon each arrival is a positive integer. However, for the aforementioned products, such discrete processes are inappropriate to describe the arrival of customer demands. Thus, continuous processes so that the total demand within any period of time is randomly distributed will be more relevant. Bombała and Michna (2012) studied an inventory system where the outflow is due to a Lévy process. Yao (2017) employed Brownian motion (also called Wiener process) to model the customer demands, which is a representative example of Lévy process. Under such assumption, the total demand during any time interval follows a normal distribution, whose parameters are determined by the length of the time interval and is independent of any previous events. In general, the Brownian motion is considerably suitable for modelling continuous random demands, and its interesting statistical properties bring convenience to the mathematical analysis. When demand arrival is described as a Brownian motion, the Markov renewal process can be embedded to evaluate the performance measures of the system. The instant at which the inventory level reaches its lower bound is the regenerative point of the underlying process. Based on regenerative processes and the Markov renewal theory, a framework is proposed to compute the long-run profit rates and determine the optimal ordering policy.

The supply contract is an important measure for retailers to maximise their revenue (Cachon, 2003). To achieve an optimal supply chain performance, efficient mechanisms should be enforced between members of the supply chain. Such mechanisms include revenue sharing (Cachon and Lariviere, 2005), quantity flexible contract (Tsay, 1999), and sales rebate contracts (Taylor, 2002). In practice, it has been customary for retailers to return the goods that remain unsold due to expiration to the upstream companies. Hence, retailers tend to pay less attention to their ordering policy, because outdated items could simply be returned to the supplier; however, this places the supplier in a dilemma. Accordingly, the buyback mechanism (Pasternack, 1985) is widely applied, whereby the supplier offers a wholesale price which is higher than usual, but the retailer is allowed to return all or part of the unsold items to the supplier at the end of the selling season, gaining a predetermined full or partial refund per unit. Such contracts have been exploited extensively in various retail sectors where the supplier's salvage value is greater than the retailer's salvage value, for example, fashion apparels, computers and cosmetics. In this study, we consider a supply chain where the supplier provides perishable products to a retailer facing a market in which customers arrive continuously. The retailer places orders according to the (s, S) continuous review policy, and the supplier and retailer are bound by a buyback contract. We develop models to determine the optimal ordering strategy with regard to the long-run profit rates. From our conclusions, we glean some novel insights which are summarised as follows.

Firstly, the optimal maximum inventory level S is mainly determined by customer arrival rate and product lifetime. To avoid the risk of suffering a heavy perishing loss, the optimal S is usually conservative, so that all the items in stock are expected to be sold out right before expiration. Secondly, the backorder policy depends on profit rates during the in-stock and out-of-stock status. When backorder is accepted, the optimal backorder level s can be uniquely represented by a function of S. Thirdly, when the buyback price is increasing, the retailer can retrive more from perished products, thus will order more from the supplier, but the profit rate is not increasing. That is because the major source of profit is always those products sold to customers, which is not affected by the buyback price. However, the supplier's profit rate is significantly decreasing. Finally, most of the system parameters can exert effect on the retailer's ordering policy. In particular, the maximum backorder level s is sensitive to a wider range of parameters, as compared with S. An itemised discussion is also presented based on sensitivity analyses.

The remainder of this paper is organised as follows. Section 2 reviews the existing researches and highlights the novelty of this work. Section 3 presents the model description and the sequence of events. In Section 4, a Markov renewal sales model is constructed, the cost and revenue functions are derived in stages. Based on the revenue rates, in Section 5, we first derive the optimal backorder level s analytically as a function of S, then find out the optimal (s, S) policy through numerical studies. We also provide a heuristic algorithm to approximate the optimal policy. After that, we discuss how the supplier's buyback policy affect the retailer's strategy in Section 6. Moreover, Section 7 reveals the impact of different parameters on the optimal ordering policy through sensitivity analyses. Finally, Section 8 concludes this paper. Proofs are relegated to the Appendix.

2. Literature review

Literature relevant to this study includes the design of (s, S) inventory policies for perishable items, the Brownian motion and continuous demand model, and the contracts between supplier and retailer. In this section, we discuss the novelty of our work in relation to existing literature.

A tremendous amount of works have demonstrated the efficiency of the continuous review policy for inventory management. Traditional inventory models assume that customers arrive according to a Poisson process and each customer demands one unit of item. Liu (1990), in a study representative of the earliest literature on (s, S) policy, presented the stationary probability distribution of the inventory level and established a closed-form of the long-run expected cost function. Numerous researches have demonstrated the optimality of (s, S) policies under linear ordering cost. Scarf (1960) was the first to prove that in a periodic ordering problem, when the ordering cost is linear and holding and shortage costs are convex, the optimal policy is always the (s, S) type without any additional conditions. Iglehart (1963) further proved its optimality in the infinite horizon, and Sethi and Cheng (1997) repeated the results when the distribution of periodic demands is dependent on a Markov chain. Whats more, in recent years, Beyer et al. (2010) and Yao et al. (2015) revealed the optimality of (s, S)-type ordering policies under concave ordering costs and Markovian demand process. Muthuraman et al. (2015) draw the same conclusion for inventory system with stochastic lead times, and Perera et al. (2018) gained the same findings for a inventory model with renewal demand under some mild cost assumptions. Though the optimality of (s, S) policies is beyond the scope of our research, incorporating the same assumptions, we also regard it as a candidate of the optimal ordering strategy in our model. Subsequent investigations attempted different configurations and generated either closed-form solutions or estimations. Liu and Lian (1999) analysed (s, S) inventory models for perishables with a constant lifetime and general renewal demands by embedding a Markov process. Their model is extended to incorporate batch demands (Lian and Liu, 2001). Additionally, there are many researches that comprised random lifetimes. Liu and Shi (1999) studied the (s, S) model for items with exponential lifetime. Gürler and Özkaya (2008) explored the (s, S) policy for perishables with general distributed lifetimes. Such models can be further complemented by considering a positive lead time (Liu and Lian, 1999) or the lost sales (Baron et al., 2017). In the recent years, Kouki et al. (2016) discussed the application of (s, S) policy to multi-item inventory system. Barron (2019) studied continuous-review perishable inventory models allowing for random batch demand under backorders or lost sales. Barron and Baron (2020) investigated a continuous-review perishable inventory system under the (s, S) policy with random lead times and shelf lives under state-dependent Poisson demands. Nonetheless, most of the existing literatures focus on different constructions of customer behaviour and cost structures under discrete demand, while there is still a lack of research based on perishables with continuous demand processes. With regard to this, we consider the (s, S) policy for perishables under a drifted Brownian demand process.

Continuous models have attracted increasing attention in recent years. In the field of operations management, a variety of continuous models have been designed to illustrate the change of variables with respect to time. Juneja and Shimkin (2013) used a fluid model to address the strategic choice of arrival time in a stochastic queueing model. Liu et al. (2016) applied the Markovian renewal approach to model the age of the oldest stock in a perishable blood bank system with Poisson input and demand streams. Poormoaied (2020) analysed the ageing process of perishables through an embedded Markov renewal process. Regarding the demand process, Yan (2006) conducted a series of studies on continuous models for a production–inventory system, where both production and demand occurred continuously at varying rates. Dai and Yao (2013) analysed an inventory system that fluctuated as a Brownian motion in the absence of control. Wu and Chao (2014) studied a production–inventory system, where both processes were modelled using Brownian motions. Moreover, continuous models are also able to describe the ordering process in inventory systems. Browne and Zipkin (1991) constructed an (r, Q) inventory policy for continuous and stochastic demands. Boxma et al. (2015) studied a fluid EOQ model with two alternate demand rates. Yao et al. (2015) presented the (s, S) inventory model subjected to Brownian demand and a concave ordering cost; their model was extended by considering price-dependent drift (Yao, 2017). Using the fluctuation theory of spectrally one-sided Lévy processes, Yamazaki (2017) proved the optimality of (s, S) policy under a Lévy demand process. He et al. (2017) examined a continuous-review inventory system in which the setup cost of each order is a function of order quantity and the demand process is modelled as a drifted Brownian motion. Cao and Yao (2019) studied a continuous-review inventory system with dual sourcing modes and Brownian demand, and showed that the optimal ordering rate must be strictly less than the expected demand rate. To the best of the authors' knowledge, a continuous inventory model for perishables under Brownian demands has not been introduced thus far. This constitutes a significant difference between the current research and previous works.

Buyback contracts have also been extensively explored in previous studies. Pasternack (1985) considered the channel coordination issue for a seasonal product under a stochastic demand using the newsvendor framework. Padmanabhan and Png (1995) introduced the return policy and discussed conditions for manufacturers to accept a return contract. Lariviere (1999) discussed the possibility of achieving channel coordination under stochastic demands. Hahn et al. (2004) investigated a novel contract in which the retailer obtains a lower wholesale price by renouncing the return of unsold goods to the supplier. Moreover, Duan et al. (2010) analysed the benefit of coordinating a two-level supply chain through a quantity discount strategy considering a fixed lifetime product. In general, most of the existing studies on inventory policies focus on the long-run cost rate while evaluating optimal strategies. When considering the contract between the supplier and retailer, long-run profit rates are more appropriate criteria for decision making. Based on these studies, a framework is proposed to evaluate the retailer's strategy when implementing a buyback contract with the supplier.

3. Model setting and assumptions

We consider a supply chain model with one supplier and one retailer. In the proposed model, we assume that the supplier has unlimited capacity and a stable cooperation with an outside manufacturer, thus can fulfill the retailer's order at any time. A wholesale price w is offered to the retailer. Furthermore, the supplier also grants a buyback contract to the retailer. According to this contract, if any of the items perished before sold, the retailer can return to the supplier and receive a refund of m for each unit returned. Such contract encourages the retailer to place a larger order to serve as many customers as possible, while the supplier also expects to benefit from the bulk order, thus is considered as a win-win cooperation.

The retailer employs an (s, S) continuous review control policy. In this policy, whenever the inventory level reaches the predetermined threshold s, the retailer will place an order to increase the inventory level to its maximum $S(s < S < +\infty)$. Usually, there exists a waiting time from making an order till receiving the ordered products, commonly known as the leadtime. However, in modern society, transportation and logistics are highly developed, the leadtime is relatively short when compared with a sales cycle. Moreover, replenishment can be completed overnight without occupying the opening hours. Thus, it is feasible to assume a zero replenishment leadtime. Under such assumption, there are three reasons that will induce the retailer to choose a non-positive s. Firstly, replenish before all products are sold out will increase the perishing risk of the unsold products. Secondly, too frequent replenishment leads to unnecessary transportation and labour costs. Lastly, by allowing backorders, retailers may wait until they have enough demands to maximise the economics of scale. Besides, we assume that the customers are loyal to their favourite brand, and will not switch to other competitors easily, therefore lost sales can be prevented.

The sequence of events in each sales cycle may develop in two possible directions, as shown in Fig. 1. Assume that a fresh batch of products have an identical lifetime T. Initially, the inventory level at time 0 is S. When the product is in stock, the retailer is available to meet any customer demand upon arrival, until the products are sold out at time T_1, $T_1 \leq T$. After that, the product become out-of-stock, the retailer will backorder any incoming demand until the inventory level reaches the lower bound s at time T_2, and an order is placed to increase the inventory level to S. When replenishment is completed, the supplier will deliver all the backlogged demands, and then come into a new sales cycle. Thereafter, the second batch of products are failed to be sold out before expiration, thus some of them perished at time $T_3 = T_2 + T$, and all the perished items are returned to the supplier. Then the inventory level is cleared to zero, all forthcoming demands are backordered until the inventory level reaches s again at T_4, and the cycle then repeats.

The current research focuses on the retailer's optimal ordering policy. Customer arrival process is assumed independent with the inventory status. Notably, we adopt a complete competition market, that is, the retailer cannot determine a monopoly price arbitrarily. In such context, the retail price of the underlying product usually depends on the manufacturer suggestions and market quotations, thus is also regarded exogenous in our model. To conclude, the retailer's decision variables contain only s and S.

The associated notations and explanations are listed in Table 1. From the perspective of the retailer, there are four main types of associated costs: wholesale price, holding cost, loss, and penalty. Except for the ordering cost paid to the supplier, there exists a fixed set-up cost for each order, usually consists of transportation and labour costs. The holding cost depends on the real-time inventory level. The loss for each perished item is $w - m$. Penalty can be further divided into two parts: the penalty for each backordered unit, which typically embodies a charge of home delivery or discount on the retail price; and the goodwill loss due to delayed delivery. Correspondingly, the profit of the retailer can be computed by subtracting all the costs from the sales revenue. In general, we assume that the retailer is always rational, so that the ordering policy is undoubtedly chosen to maximise their own profit.

4. The Markov renewal sales model

In the current research, the arrival of customer demands are assumed to be continuous over time. As an extension of the normal demand assumption in periodic review inventory model (Rao, 2003), the demand process in this paper is modelled using a drifted Brownian motion with the drift parameter $\mu > 0$ and variance parameter σ. Let $D(t)$ denote the total demand within any time interval of length t, it can be modelled as

$$D(t) = \mu t + \sigma B(t), \quad t \geq 0$$

where $B(t)$ is a standard Brownian motion with zero drift and unit variance. Subsequently, $D(t)$ follows a normal distribution with mean μt and variance $\sigma^2 t$, which indicates that the expected total demand

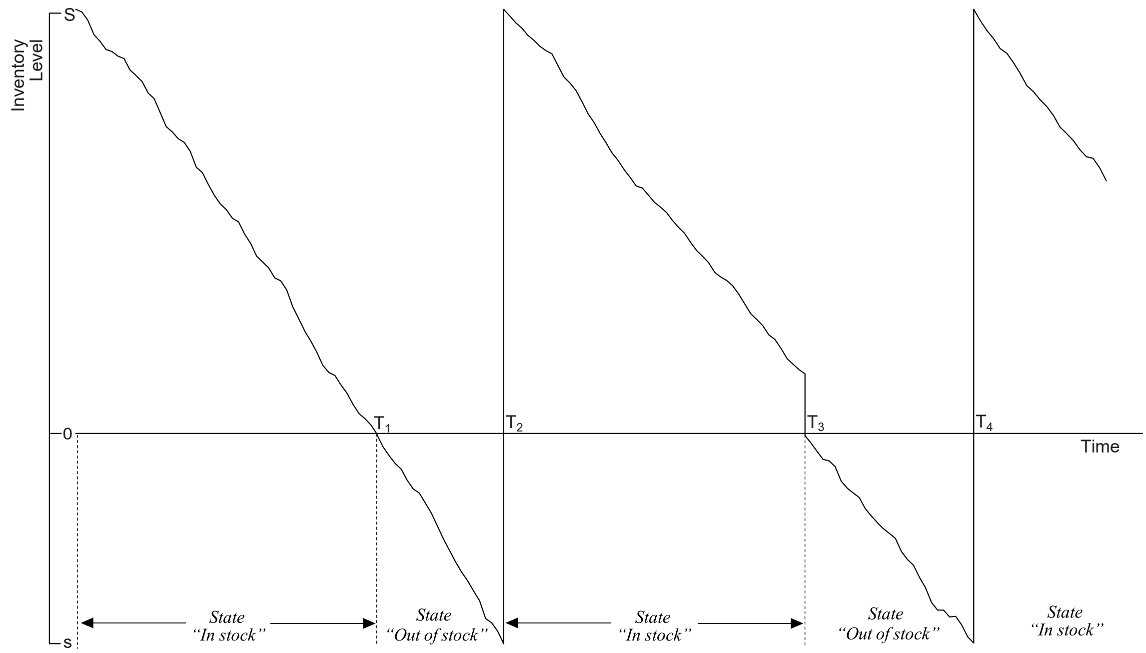

Fig. 1. Example of inventory level trends over time.

Table 1
Notations and explanations.

Notations	Explanations
W	Wholesale price per unit
P	Retail price per unit
M	Buyback price per unit perished
C_o	Order set-up cost per order
C_h	Inventory holding cost per unit per unit time
C_s	Goodwill cost per unit per unit time
C_u	Backorder penalty per unit backordered

increases linearly, but the volatility also grows with respect to time. Notable, the Brownian parameters in this study are chosen such that the occurrence of negative increments is negligible. According to the normal properties, when $\mu t > 3\sigma\sqrt{t}$, or $t > \frac{9\sigma^2}{\mu^2}$, the probability of negative demand is negligibly small. In the remainder of this paper, we always follow the assumption that the product lifetime $T > \frac{9\sigma^2}{\mu^2}$, and the order quantity per cycle $S - s > \frac{9\sigma^2}{\mu}$.

The proposed sales model incorporates a background process and a inventory level process. Let $Z(t)$ denote the inventory level at time t. Assuming that $Z(t)$ is right continuous with left limits, the inventory level $\{Z(t), t \geq 0\}$ is a continuous random process. When the inventory level reaches a threshold, the background process will transit to another state. In the proposed model, the background process comprises the stocking status, and it includes two states: in stock or out of stock. The inventory level $\{Z(t), t \geq 0\}$ modulated by the background process is a regenerative process if we define a cycle as follows. Suppose the cycle starts from S, $Z(t)$ drops with drift $-\mu$ and variance σ^2 until either it reaches 0, or t reaches the lifetime T. Subsequently, the background process transits, and $Z(t)$ continues to reduce until it reaches s, indicating the end of a cycle.

It is evident that the background process is a Markov renewal process. By definition, the occurrence of events after a regenerative point shall be independent of any historical states. Owing to the independence of Brownian motion increments, the epoch when a fresh batch of products arrive is a regenerative point. Besides, The epoch when all the products in stock are sold out or perished is another regenerative point.

The state space of the background process is as follows:

State I: The process is defined to be in *State I* if the products are in stock. It transits into the other state when all the products are either sold out or perished.

State O: The process is defined to be in *State O* if the products are out of stock. It transits into the other state when a fresh batch of products are replenished.

Specially, when $s = 0$, the model degenerates to a process with one regenerative point.

The optimal replenishment policy should maximise the long run profit rate of the retailer, while the total profit in each sales cycle can be regarded as a kind of reward. According to the well-known renewal reward theorem, the profit rate converges with probability one to the ratio between the expected profit in a cycle and the expected time length of a cycle. Moreover, since each cycle consists of two states, the profit and time length in a cycle can be further split into the sum of profits and time lengths in both states. Next, we present the mathematical formulations of expected sojourn time and profit in each state.

In *state I*, assume that all items are used to satisfy the demand until perishing. Let T_S denote the first time when the total demand reaches S, its survival function can be computed as follows:

$$\begin{aligned} P\{T_S > t\} &= P\{\forall 0 < \tau \leq t, D(\tau) < S\} \\ &= P\{\forall 0 < \tau \leq t, \mu\tau + \sigma B(\tau) < S\} \\ &= P\left\{\forall 0 < \tau \leq t, B(\tau) < -\frac{\mu}{\sigma}\tau + \frac{S}{\sigma}\right\}. \end{aligned} \tag{1}$$

Siegmund (1986) revealed that, for any $t > 0$ and constant $a \in R$, $b > 0$,

$$P\{\exists \tau \leq t, \text{ such that } B(\tau) \geq a\tau + b | B(t) = \xi\} = \exp\left\{-\frac{2b(at + b - \xi)}{t}\right\}. \tag{2}$$

Based on the Markovian property of Brownian motion, we have

$$P\{\forall 0<\tau\le t, B(\tau)<a\tau+b\} = E[P\{\forall 0<\tau\le t, B(\tau)<a\tau+b\}|\mathcal{F}_t]$$
$$= \int_{-\infty}^{at+b}\left[1-\exp\left\{-\frac{2b(at+b-x)}{t}\right\}\right]\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2t}\right\}\mathrm{d}x$$
$$= \Phi\left(\frac{at+b}{\sqrt{t}}\right)-\exp\{-2ab\}\Phi\left(\frac{at-b}{\sqrt{t}}\right) \tag{3}$$

where $\Phi(x)$ is the normal distribution function and $\mathcal{F}_t$ is the natural filtration. Thus, the survival function of the hitting time T_S is

$$\overline{F}_S(t) = P\{T_S>t\} = \Phi\left(\frac{S-\mu t}{\sigma\sqrt{t}}\right)-\exp\left\{\frac{2\mu S}{\sigma^2}\right\}\Phi\left(-\frac{S+\mu t}{\sigma\sqrt{t}}\right). \tag{4}$$

Assume that the lifetime of a fresh item in storage is a constant T. The background process transits to *state O* when all the stocks are either sold out or perished, thus, the sojourn time in *state I* can be calculated by truncation $T_I = \min\{T_S, T\}$, and the mean sojourn time in *state I* can be represented by a function of S

$$T_I(S) = E[\min\{T_S, T\}] = \int_0^T t\mathrm{d}F_S(t)+T\overline{F}_S(T) \tag{5}$$

where the survival function $\overline{F}_S(\cdot)$ is defined in (4) and $F_S(\cdot)$ is the corresponding cumulative distribution function.

The revenue in *state I* includes the sales income for sold products and reimbursement for perished products, while the costs consist of set-up cost, ordering cost and holding cost. The instantaneous inventory level at time t is $Z(t) = S - D(t) = S - \mu t - \sigma B(t)$, and the holding cost during the cycle depends on the time until all the items are sold out or perished. The expected total inventory holding cost in a reorder cycle $H(S)$ can be computed through a conditional expectation $H(S) = E\left[E\left[\int_0^{T_I} C_h Z(t)\mathrm{d}t|T_0\right]\right]$, where $Z(t)$ is the inventory level at time t, it follows a normal distribution with mean $S - \mu t$ and variance $\sigma^2 t$. We have

$$H(S) = \int_0^T\int_0^t C_h Z(\tau)\mathrm{d}\tau\mathrm{d}F_S(t)+\overline{F}_S(T)\int_0^T C_h Z(\tau)\mathrm{d}\tau. \tag{6}$$

Return occurs only if some of the items are outdated before being sold out, i.e.,$T_S > T$, otherwise, all the items are sold out before outdating and none of the items perish. In general, the total demand until time T, denoted by $D(T)$, follows a normal distribution with mean μT and variance $\sigma^2 T$. However, under the premise that $T_S > T$, the conditional distribution of $D(T)$ should be truncated at S. Notably, it is assumed that the demand is always non-negative, so the distribution of $D(T)$ should also be truncated at zero. Owing to the high computational complexity of two side truncated distribution, here we provide an approximation scheme under the assumption $\mu T > 3\sigma\sqrt{T}$, so that the occurrence of negative $D(T)$ can be ignored. In this case, the number of perished items in each cycle approximately follows a truncated normal distribution with mean $S - \mu T$ and variance $\sigma^2 T$, truncated at zero. Let $R(S)$ denote its expectation, by the aid of existing conclusions on one side truncated normal distribution, we have

$$R(S) = \overline{F}_S(T)\left(S-\mu T+\frac{\sigma\sqrt{T}}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{S-\mu T}{\sigma\sqrt{T}}\right)\right]}\exp\left\{-\frac{(S-\mu T)^2}{2\sigma^2 T}\right\}\right). \tag{7}$$

For each perished item, the retailer receives a reimbursement of m. The expected reimbursement per cycle can be calculated by $mR(S)$.

After the background process transits into *state O*, the demand still increases according to Brownian motion with drift μ and variance σ^2 until it hits $-s$. For convenience of expression, let $x = -s$ for the remainder of this paper, named after the backorder level. As the expected total demand increases linearly with respect to time, it is straightforward that the expected sojourn time in *state O* is $T_O(x) = x/\mu$.

To compute the expected total goodwill loss, here we employ the conclusions drawn by Karlin and Taylor (1981): for a continuous function $h(\cdot)$, if K denotes the hitting time of s of a Brownian motion whose initial state $y > s$, then the function generated by $w(y) = E\left[\int_0^K h(Z(t))\mathrm{d}t|Z(0)=y\right]$ satisfies the following differential equations:

$$-h(y) = -\mu\frac{\mathrm{d}w}{\mathrm{d}y}+\frac{\sigma^2}{2}\frac{\mathrm{d}^2w}{\mathrm{d}y^2}, \quad w(s)=0. \tag{8}$$

Based on the conclusion of Wu and Chao (2014), we provide a general solution for differential equation (8), and attach a detailed proof in the appendix.

Theorem 1. *The solution to* (8) *is*

$$w(y) = \int_s^y\left(\frac{2}{\sigma^2}\int_u^{+\infty}\exp\left\{-\frac{2\mu(\xi-u)}{\sigma^2}\right\}h(\xi)\mathrm{d}\xi\right)\mathrm{d}u. \tag{9}$$

Proof. *See appendix.* □

The initial state of our model is $y = 0$ and $h(\xi) = -C_s\xi$. The mean goodwill penalty is thus

$$g_p(x) = \int_{-x}^0\left(-\frac{2}{\sigma^2}\int_u^{+\infty}C_s\exp\left\{-\frac{2\mu(\xi-u)}{\sigma^2}\right\}\xi\mathrm{d}\xi\right)\mathrm{d}u. \tag{10}$$

Furthermore, for backordered demands, the retailer will provide a preferential price as $p - C_u$. In other words, the retailer suffers a penalty $C_u x$ due to backordering. The total penalty during a cycle is $g_p(x) + C_u x$.

In summary, the expected total profit for the retailer is

$$\begin{aligned}P_R(x,S) &= p(S-R(S))+mR(S)+(p-C_u)x-g_p(x)-H(S)-w(S+x)-C_0\\ &= (p-w)S-(p-m)R(S)-H(S)+(p-w-C_u)x-g_p(x)-C_0,\end{aligned} \tag{11}$$

and the expected profit rate for the retailer is

$$p_R(x,S) = \frac{(p-w)S-(p-m)R(S)-H(S)+(p-w-C_u)x-g_p(x)-C_0}{T_I(S)+T_O(x)}. \tag{12}$$

5. The retailer's ordering policy

As introduced in Section 3, the decision model is a two-stage game, the supplier determines the wholesale and buyback price at first, then the retailer determines the ordering policy. In this section, we will study the retailer's optimal (s, S) policy for given w and m.

5.1. Optimal backorder policy

Let $x = -s$ be the maximum backorder level, the retailer's problem can be described by

$$\max_{x,S} p_R(x,S), \quad \text{s.t.} \quad x\ge 0, S>0. \tag{13}$$

We solve this optimisation problem sequentially in two stages. Firstly, given the maximum inventory level S, we can express the optimal backorder level $x^*(S)$ as a function of S. Then we can find the solution of the single-variable optimisation problem via simulation study.

When the maximum inventory level S is fixed, the retailer's profit rate $p_R(x, S)$ degenerates to a single variable function of the backorder level x. From the function properties, we can conclude that there exists an optimal backorder level $x^*(S)$ for any given S.

Theorem 2. *When* $C_s > 0$, *unmet demand* $x^*(S) =$

$$\sqrt{(\mu T_I(S))^2 + \frac{2\mu\Delta T_I(S)}{C_s}} - \mu T_I(S) \text{ if}$$

$$\mu(p - w - C_u) + \frac{\sigma^2}{2\mu}C_S > \frac{p(S - R(S)) + mR(S) - wS - H(S) - C_0}{T_I(S)}, \tag{14}$$

and $x^*(S) = 0$ *otherwise. Here* Δ *is obtained by substracting the right side of inequality* (14) *from the left side.* Proof for Theorem 2 can be found in the appendix. We can easily find that the right-hand side of (14) is the retailer's actual profit rate during *state I*, while $\mu(p - w - C_u)$ is the retailer's profit rate during *state O*, regardless of the goodwill loss. Thus, the inequality (14) is essentially a competition between the profit rates during in-stock and out-of-stock status. When it holds, permitting a certain amount of backorder is profitable, until the cumulative goodwill lost is unendurable.

Corollary 1. *When* $C_s = 0$*, the optimal backorder level* $x^*(S)$ *approaches infinity under certain conditions.*

Corollary 1 reveals the significance of the goodwill penalty: a maximum waiting time should be identified to avoid unconstrained backorders, which may irreversibly tarnish the retailer's reputation.

Corollary 2. $x^*(S) > 0$ *always holds when* S *is approaching* 0 *or infinity.*

Corollary 2 is straightforward as for a small S, the order setup cost C_o is expensive in comparison with the sales revenue, whereas for a large S, the retailer is more likely to suffer a heavy loss from perished goods. These factors reduce the profit rate in *state I* and provide a reference for accepting the backorder.

To illustrate the relationship between S and x, simulation studies are designed as follows. The assumed parameter values are listed in Table 2. Firstly, let S varies from 2 to 15 with a fixed interval of 0.1. We generate a string of sales data with these parameters and compute the optimal backorder level $x^*(S)$ through Theorem 2. The replication time for each S is 10,000 times, and the optimal $x^*(S)$ is obtained by taking average of the results in each trial. We plot the $x^*(S)$ against S in Fig. 2(a).

Generally, the plot validates Corollary 2, it resembles a bathtub curve in shape, though it is concave at both ends and flat in the middle part. In the current example, we can conclude that backorder should be permitted when the maximum inventory level S is either smaller than 2.7 or larger than 9.7. Under such backorder policy, one can well imagine that the retailer will suffer a substantial backorder penalty when S is small, and a heavy loss due to perishing when S is large. Thus, the optimal S should locate in the flat segment, or when backorder is not permitted, and a majority of products should be sold at the retail price.

Secondly, for several given S, let the backorder level x vary from 0.1 to 15 with a fixed interval 0.1, and the retailer's profit rates are plotted against x in Fig. 2(b). We can observe that the profit rate decreases in x when S is ranging from 6 to 9 (represented by thinner lines), indicating that no backorder is permitted, which is coincident with the conclusions in Fig. 2(a). When S equals 1.5 or 15 (represented by thicker lines), the profit rate is an unimodal function of x and in particular, when S is significantly large, the peak is not observed owing to space limitations. Furthermore, the expected total demand until expiration is $\mu T = 8$, whereas in the current experiment, the policy $S = 7$ has the highest overall profit rate, followed by $S = 6$, 8 and 9. This indicates that it is a good choice to choose an S which approaches μT but is slightly smaller, so that all the stocks are expected to be sold out right before expiration, while the chance of an expiration is relatively low. The policy $S = 15$ is inferior to all the other policies, owing to the heavy loss caused by perished goods. Additionally, slight fluctuation is caused by the instability of the Brownian random variables.

Table 2
Parameters in the simulation example.

p	w	m	C_0	C_h	C_s	C_u	μ	σ	T
15	8	2	5	0.05	0.1	2	2	0.5	4

5.2. Retailer's optimal replenishment policy

The analysis in Section 5.1 indicates that the optimal backorder level x can be described as a piecewise function of maximum inventory level S. Thus, the profit rate function $p_R(x, S)$ can be simplified by a single variable function $p_R(S)$. In other words, if the retailer makes a decision on his own, the profit rate is determined uniquely by the selection of S. Unfortunately, owing to the intractability of stochastic integral, especially with the involvement of the truncated distributions, it is difficult to obtain a closed-form solution for the optimal value of S. Thus, we need to perform a simulation. Let S vary from 0.1 to 15 with a fixed interval of 0.1, we use the optimal backorder level $x^*(S)$ obtained through Theorem 2. The pair of S and $x^*(S)$ that maximises the retailer's profit rate is considered as the optimal inventory policy. In real applications, the retailer can determine the maximum inventory level according to his/her capacity, and then divide it by a fixed interval. A smaller interval indicates a larger replication time and higher accuracy. In general, the complexity of the simulation is linear in the replication time, and the results can be obtained in a finite period of time.

Assume that the manufacturing cost for each unit of product is a constant c. Similar to equation (12), we can compute the expected profit rate for the supplier through

$$p_S(x, S) = \frac{(w - c)(S + x) - mR(S)}{T_I(S) + T_O(x)}, \tag{15}$$

and the expected profit rate for the total channel is

$$p_T(x, S) = \frac{(p - c)S - pR(S) - H(S) + (p - c - C_u)x - g_p(x) - C_0}{T_I(S) + T_O(x)}. \tag{16}$$

Assume that the manufacturing cost per unit $c = 4$. The profit rates of the retailer, supplier, and whole channel under varying values of S are plotted in Fig. 3.

We can conclude from Fig. 3 that the total profit rate of the channel reaches its maximum at $S = 7.6$. The profit rate of the retailer reaches its maximum earlier, at $S = 6.8$, after which it decreases rapidly due to perishability and is finally surpassed by the profit rate of the supplier at $S = 11.3$. The profit rate of the supplier is less sensitive to S in comparison with that of the retailer. In particular, under the parameter values given in Table 2, the profit rate functions of the retailer and whole channel are convex when backorder is permitted, and concave when backorder is forbidden. However, although the concavity and convexity properties change with the backorder policy, the profit rate functions of the retailer and whole channel still exhibit a quasi-concave property, that is, both of them have an unique ordering policy to optimise the function value individually.

Although we cannot provide a closed-form solution for S^*, an analysis on the marginal revenue and cost can provide some insights on the location of S^*. The retailer's revenue rate and cost rate during each cycle can be represented by

$$Rev(x, S) = \frac{p(S - R(S) + x) + mR(S)}{T_I(S) + T_O(x)}, \quad C(x, S) = \frac{w(S + x) + H(S) + C_u x + g_p(x) + C_0}{T_I(S) + T_O(x)},$$

the profit rate $p_R(x, S) = Rev(x, S) - C(x, S)$. We plot them with respect to S in Fig. 4, the retailer's marginal revenue and cost rates are the slopes of the curves.

We can conclude from Fig. 4 that the revenue rate is overall stable when S is varying. When S is large, since the perished products can only provide some salvage values, the retailer will backlog some unmet demands to maintain the revenue rate. Specially, when S has just exceeded μT, a slight increase on the revenue rate is detected. This is because the customer arrival follows a random process, the actual demand has a

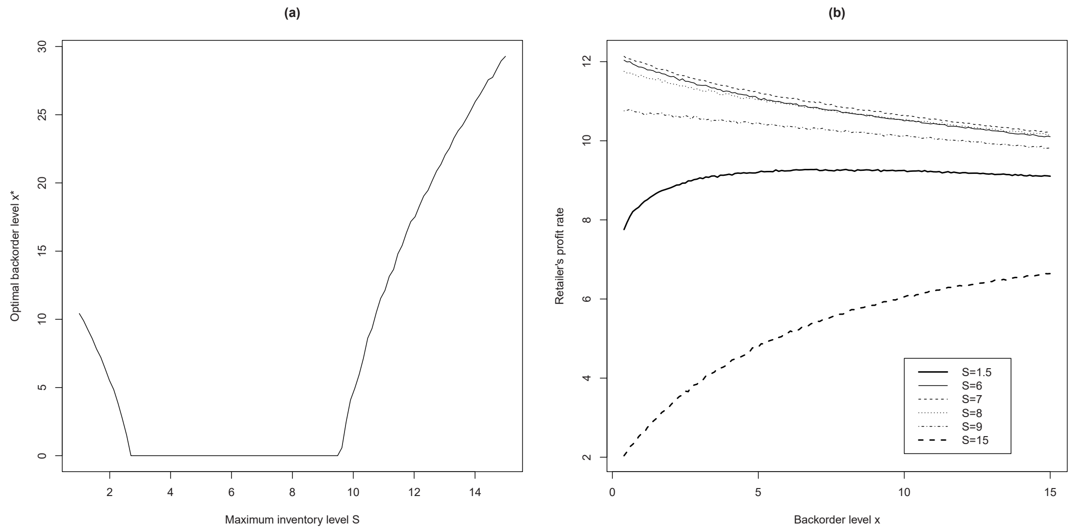

Fig. 2. Relationship between backorder level x and maximum inventory level S.

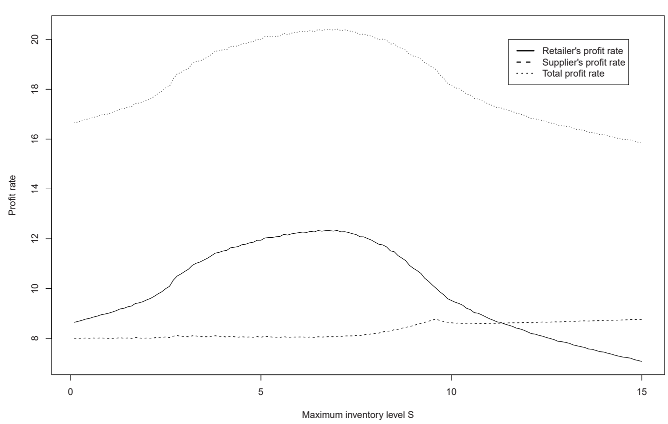

Fig. 3. Profit rates under varying maximum inventory level S.

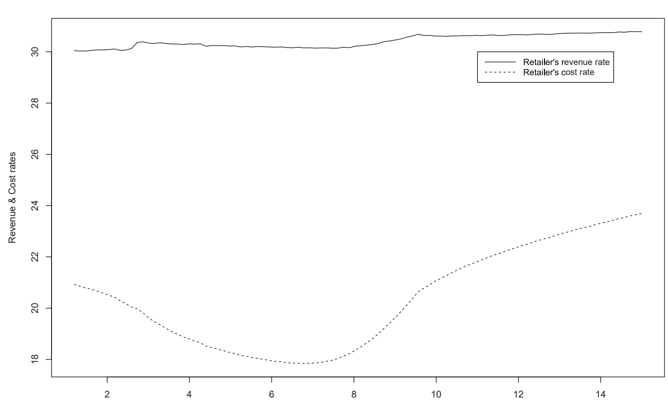

Fig. 4. Sensitivity of retailer's revenue and cost rates with respect to S.

same probability to be higher or lower than expected. In the former case, the revenue rate will be much higher than expected; while in the latter case, the retailer can still earn some surplus value from perished products. On the other side, when S exceeds μT, the cost increases rapidly, this is because the expected sojourn time in *State I* has an upper bound T, but the holding cost and wholesale cost are increasing rapidly with S. As a result, the marginal cost rate is much higher than the marginal revenue rate when $S > \mu T$, which provides a strong evidence for choosing $S < \mu T$ under the current parameter settings.

We glean some observations from the numerical studies: the retailer's profit rate function is unimodal and the optimal value of S is likely to be close to μT. Based on these findings, we can adopt a simple line search algorithm to efficiently approximate the optimal value of S. We firstly choose $S_0 = \mu T$ as the initial value. Owing to a lack of information on the searching direction, we attempt an initial stepsize of 1, that is $S_1 = S_0 + stepsize$. If the profit rate value becomes greater, we keep the same search direction with the same stepsize: $S_2 = S_1 + stepsize$, otherwise if the profit rate becomes smaller, we change the searching direction and shrink the stepsize to be the one-third of the previous stepsize. We keep doing so utill the increment of the profit rate after a move is smaller than a predetermined threshold.

6. The effect of the buyback price

When the retailer orders from the supplier, the retailer needs to consider not only the holding cost, but the shortage cost and the items expired. In many cases, when S is large, the overall marginal cost is greater than the marginal revenue. As a result, the retailer is reluctant to order too much from the supplier. Consequently, the supplier is usually willing to share some risk with the retailer by buying back the expired items in order to attract the retailer to order more, such that these two parties can reach a win-win. In this section, we will do some numerical analysis to investigate how the buyback price m affects the retailer's order policy.

We conduct a test on the system performance under fixed wholesale price $w = 8$ and varying buyback prices. We acquiesce in the restriction that buyback price can never exceed the wholesale price, otherwise, the retailer can benefit from expiration, which creates a condition for commercial fraud. Let the buyback price m varies from 0 to 8 with an interval 0.5. We anticipate the retailer's ordering policy and obtain the profit rates of the whole channel, supplier and retailer under such policy. Some results are plotted in Fig. 5.

The following observations can be made from Fig. 5:

(i) $m = 0$ indicates that no contract is signed between the retailer and supplier. In comparison with the case $m = 2$ (Fig. 3) and $m = 4$, there is no significant difference in the retailer's optimal ordering policy S^*, while the supplier's profit rate under S^*, say $p_S(x^*, S^*)$, is also nearly constant. This is because the retailer is conservative when buyback price is low, as a result, the chance of expiration is low and buyback rarely happens.

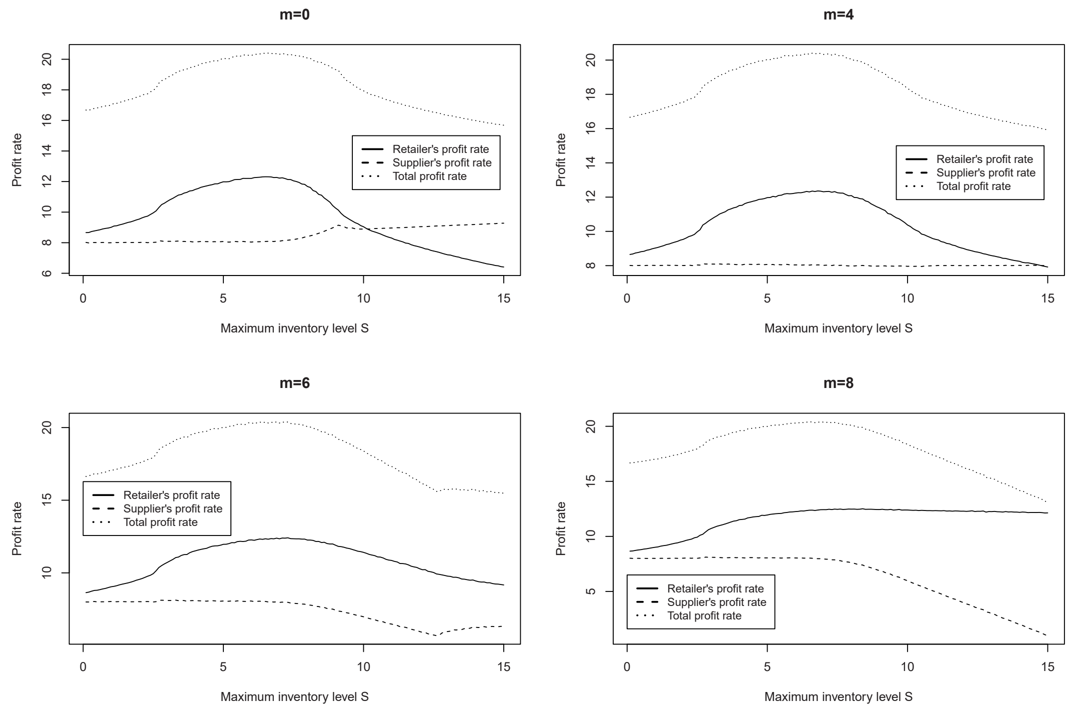

Fig. 5. Profit rates under varying m and retailer's optimal policy (x^*, S^*).

(ii) It is worth mentioning that when $m = 4$, the retailer's optimal policy $S = 7.6$ also maximises the supply chain's total profit rate. In this case, the retailer and supplier share the highest total profit rate, and we say the supply chain is coordinated (Cachon, 2003). When m is greater than 4, the retailer assumes an advantageous position in the channel because the supplier's profit from perished items $w - m$ is lower than its manufacturing cost c.

(iii) When m increases, the retailer can retrive more from perished products, and the loss from perishing decreases accordingly. This can be interpreted as a higher marginal revenue when S is large. As a result, the policy $S^* > \mu T$ may become feasible. Consequently, the supplier's profit rate $p_S(x^*, S^*)$ falls.

(iv) Finally, when $m = 8$, the supplier must refund the retailer for any items perished. In other words, the retailer does not suffer any loss from perished products. Thus, $p_R(x, S)$ is nearly constant when S is large, until the inventory holding cost is high enough make an impact. When S is extremely large, the retailer takes almost all the profits in the channel, while the supplier faces a deficit. Consequently, it is irrational to guarantee a buyback contract that allows unlimited returns and full refund.

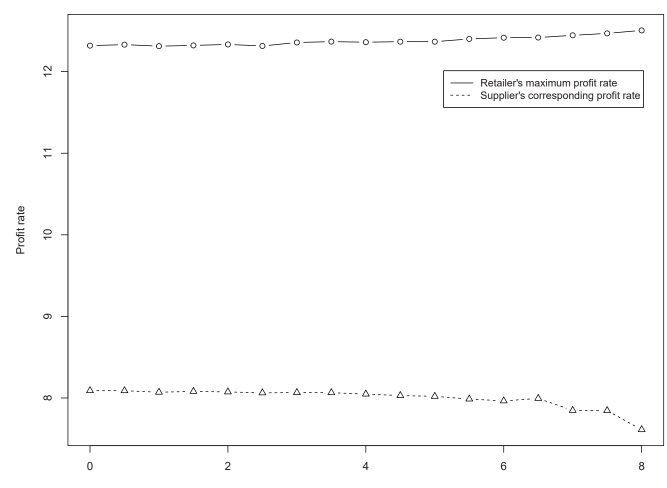

Fig. 6. Retailer's and Supplier's profit rates under varying m and (x^*, S^*).

Moreover, Fig. 6 presents the retailer's maximal profit rate $p_R(x^*, S^*)$ and the supplier's corresponding profit rate $p_S(x^*, S^*)$ under different m. It may seem counter-intuitive that $p_R(x^*, S^*)$ is rarely affected by the buyback price m. Even if $m = w$, the retailer still cannot benefit much from it. The is because when m is small, the optimal policy is conservative and the chance of expiration is low. However, even when m is large, owing to the property of normal distributions, the occurrence probability of a extremely large demand is negligible. An $S > (\mu + 3\sigma)T$ will almost certainly lead to unnecessary waste on the inventory holding cost. Generally, the major source of profit is always those products sold to customers, which is not affected by m. However, the supplier's profit rate $p_S(x^*, S^*)$ significantly decreases when m is large. To conclude, an excessive high m unilaterally hurts the supplier's benefit, and thus damages the social welfare.

In all, when the wholesale price w is fixed, there does not exist a buyback policy m that is superior to others. However, it can be imagined that the supplier will benefit from a higher wholesale price w, and consequently, the retailer expects a lower w. The effect of w to (x^*, S^*) will be discussed in the following section. In real-world problems, the final settlement always depends on the bargaining power of the supplier and retailer.

7. Sensitivity analysis and insights

In this section, we analyse the impact of different parameters in our model on the optimal ordering policy through a series of sensitivity analyses. The optimal backorder level x^* is zero in our previous example, which is inappropriate as a reference. Thus, we select another set of parameters in this section so that x^* is positive, and then both upward and downward changes on the backorder policy can be detected. The updated parameters and corresponding optimal policy are listed in Table 3.

Table 3
Parameters and optimal policy as a frame of reference.

p	w	m	C_0	C_h	C_s	C_u	μ	σ	T	S^*	x^*	$p_R(x^*, S^*)$
10	6	2	5	0.05	0.1	1	2	0.5	3	5.27	2.734	5.829

First of all, we study the effects of retail price p and wholesale price w on the optimal policy. The results are listed in Table 4.

We can conclude the following results from Table 4.

(i) Both S^* and x^* are insensitive to p. We can observe from equation (11) that the term which includes p in the total profit $P_R(x, S)$ represents the sales revenue. The coefficient of p in this term is the actual sales. As a result, the coefficient of p in the long run profit rate $p_R(x, S)$ should be the actual sales rate. Obviously, $p\mu$ acts as a constant term in $p_R(x, S)$, thus changes in p will not affect its maximum point (x^*, S^*).
(ii) For each perished product, the retailer suffers an actual loss $w - m$. A higher w will prevent the retailer from holding too many stocks. As a result, S^* decreases with w.
(iii) When S^* is decreasing, the retailer tends to moderately increase the backorder level to lengthen the replenishment interval. But the increase has gradually slowed down as the goodwill loss become another non-negligible problem.
(iv) Generally, since S^* is chosen so that no item is expected to perish, the profit rate increases with p almost linearly. Similarly, it decreases with w almost linearly. In particular, when p is considerably low or w is considerably high, the profit rate can take a negative value and the retailer faces a deficit.

Next, we study the effects of two types of backorder penalties, C_s and C_u, on the optimal policy. The results are listed in Table 5.

We can conclude the following results from Table 5.

(i) x^* decreases significantly with both C_s and C_u. This is straightforward as a higher penalty prevents the retailer from accepting too many backorders.
(ii) When $x^* > 0$, the profit rate $p_R(x^*, S^*)$ decreases with both C_s and C_u, among which C_u deals a greater influence than C_s. That is, the goodwill loss is slighter when compared with the backorder penalty.
(iii) S^* is not sensitive to C_s or C_u. However, when x^* is considerably small or even equal to zero, slight fluctuation can be observed in S^* and the profit rate $p_R(x^*, S^*)$, this is because the sales cycle is excessively short, and the results are therefore unstable.

Next, we study the effects of demand arrival rate μ and product lifetime T on the optimal policy. The results are listed in Table 6.

We can conclude the following results from Table 6.

(i) S^* increases with both μ and T. Generally, we always have $S^* < \mu T$, which reiterates our previous conclusions.
(ii) According to Section 4, the customer arrival rate μ affects most of the costs and revenues, thus x^* may not be monotone with μ, though it is decreasing with μ here. On the other side, from Theorem 2, the profit rate in *state O* is completely independent of T, however, in *state I*, the profit rate increases with T. Consequently, x^* decreases with T.
(iii) The maximum profit rate $p_R(x^*, S^*)$ increases rapidly with the demand arrival rate μ. When $x^* = 0$, $p_R(x^*, S^*)$ increases almost linearly with μ as expected.
(iv) The profit rate $p_R(x^*, S^*)$ increases with T, mainly because of two reasons: Firstly, a longer T can weaken the effect of the order setup cost C_0. Secondly, due to the Brownian properties, the variance of the total demand per cycle is increasing with T. As a result, the retailer's policy becomes more conservative when T is larger. Intuitively, $\mu T - S^*$ is increasing with T, so the expected perishing loss per cycle is decreasing with T.

Finally, we explore the effect of μ and T under the assumption that μT is constant, that is, μ and T vary simultaneously. Additionally, we study the system performance under changes in the variance parameter σ of demand arrival process. The results are listed in Table 7.

We can conclude the following results from Table 7.

(i) When μT is fixed, S^* still increases with μ, and consequently decreases with T. Intuitively, μ does not affect the variance of customer demands, while T is proportional to the variance of customer demands within the shelf life. Thus, to guarantee that all the products are sold out before expiration, the retailer's policy becomes more conservative when T is increasing. As a result, $\mu T - S^*$ is increasing with T.
(ii) The variance of customer demands during a fixed time interval is increasing with σ. Thus, smaller σ can relieve the sensitivity of S^* to T, that is, when μ and T are fixed, $\mu T - S^*$ decreases with σ. Consequently, x^* is lower and $p_R(x^*, S^*)$ is higher under a smaller σ.
(iii) In a periodic review problem, the optimal order quantity per cycle can be represented by $\mu + SF\sigma$, where $SF\sigma$ represents the safety stock coped with unexpectedly large demand, and SF is a stocking factor determined by the demand and cost parameters (Petruzzi and Dada, 1999). However, in the current model, since the products are perishable, setting a safety stock might increase the risk of suffering a heavy perishing loss, whose benefit is rarely

Table 4
Optimal policies and maximum profit rates under varying p or w.

p	w	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$	p	w	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$
12.5	6	(2.673, 5.33)	0.67	10.839	10	9	(3.709, 5.06)	0.94	−0.239
12		(2.646, 5.28)	0.72	9.847		8.5	(3.584, 5.09)	0.91	0.770
11.5		(2.722, 5.33)	0.67	8.839		8	(3.394, 5.15)	0.85	1.783
11		(2.749, 5.27)	0.73	7.836		7.5	(3.292, 5.12)	0.88	2.795
10.5		(2.728, 5.25)	0.75	6.831		7	(3.149, 5.18)	0.82	3.805
10		(2.734, 5.27)	0.73	5.829		6.5	(2.908, 5.19)	0.81	4.817
9.5		(2.729, 5.24)	0.76	4.827		6	(2.734, 5.27)	0.73	5.829
9		(2.730, 5.30)	0.70	3.832		5.5	(2.553, 5.24)	0.76	6.842
8.5		(2.746, 5.21)	0.79	2.825		5	(2.424, 5.25)	0.75	7.862
8		(2.731, 5.24)	0.76	1.826		4.5	(2.082, 5.46)	0.54	8.878
7.5		(2.703, 5.21)	0.79	0.825		4	(1.894, 5.42)	0.58	9.904
7		(2.691, 5.30)	0.70	−0.176		3.5	(1.412, 5.61)	0.39	10.928

Table 5
Optimal policies and maximum profit rates under varying C_s or C_u.

C_s	C_u	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$	C_s	C_u	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$
0.025	1	(8.051, 5.21)	0.79	5.863	0.1	0	(11.254, 5.19)	0.81	6.931
0.05		(4.747, 5.24)	0.76	5.849		0.5	(7.866, 5.18)	0.82	6.298
0.075		(3.544, 5.21)	0.79	5.839		1	(2.734, 5.27)	0.73	5.829
0.1		(2.734, 5.27)	0.73	5.829		1.5	(0, 5.21)	0.79	5.826
0.125		(2.310, 5.27)	0.73	5.824		2	(0, 5.27)	0.73	5.840
0.15		(1.975, 5.21)	0.79	5.827		2.5	(0, 5.21)	0.79	5.833
0.25		(1.326, 5.27)	0.73	5.820		3	(0, 5.30)	0.70	5.836
0.5		(0.717, 5.30)	0.70	5.816		3.5	(0, 5.27)	0.73	5.836
1		(0.387, 5.29)	0.71	5.795		4	(0, 5.27)	0.73	5.840

Table 6
Optimal policies and maximum profit rates under varying μ or T.

μ	T	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$	μ	T	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$
1.2	3	(6.798, 2.96)	0.64	3.051	2	1.5	(9.252, 2.58)	0.44	5.162
1.6		(5.210, 4.15)	0.65	4.394		2	(7.217, 3.44)	0.56	5.372
2		(2.734, 5.27)	0.73	5.829		2.5	(5.004, 4.33)	0.67	5.589
2.4		(0, 6.33)	0.87	7.461		3	(2.734, 5.27)	0.73	5.829
2.8		(0, 7.46)	0.94	9.066		3.5	(0.478, 6.10)	0.90	6.073
3.2		(0, 8.54)	1.06	10.661		4	(0, 7.06)	0.94	6.311
3.6		(0, 9.71)	1.09	12.255		4.5	(0, 7.68)	1.32	6.465
4		(0, 10.60)	1.40	13.850		5	(0, 8.58)	1.42	6.593

Table 7
Optimal policies and maximum profit rates under fixed μT and varying μ, T, σ

$\sigma = 0.5$					$\sigma = 0.25$				
μ	T	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$	μ	T	(x^*, S^*)	$\mu T - S^*$	$p_R(x^*, S^*)$
1	6	(3.490, 4.91)	1.09	2.819	1	6	(1.320, 5.43)	0.57	2.901
1.5	4	(3.131, 5.12)	0.88	4.321	1.5	4	(0.843, 5.54)	0.46	4.438
2	3	(2.734, 5.27)	0.73	5.829	2	3	(0.516, 5.63)	0.37	5.981
2.5	2.4	(2.478, 5.36)	0.64	7.358	2.5	2.4	(0, 5.64)	0.36	7.573
3	2	(2.007, 5.45)	0.55	8.880	3	2	(0, 5.67)	0.33	9.138
3.5	1.714	(1.780, 5.42)	0.58	10.402	3.5	1.714	(0, 5.67)	0.33	10.588
4	1.5	(1.189, 5.48)	0.52	11.961	4	1.5	(0, 5.67)	0.33	12.261
4.5	1.33	(0.419, 5.49)	0.51	13.495	4.5	1.33	(0, 5.68)	0.32	13.615
5	1.2	(0.235, 5.49)	0.51	15.096	5	1.2	(0, 5.73)	0.27	15.398
5.5	1.091	(0, 5.52)	0.48	16.725	5.5	1.091	(0, 5.75)	0.25	17.002
6	1	(0, 5.58)	0.42	18.240	6	1	(0, 5.79)	0.21	18.531

worth the cost. As a result, the optimal S is usually smaller than μT, their difference depends on the demand and cost parameters. Unfortunately, owing to the intractability of stochastic integrals, it is hard to give an explicit formula of $\mu T - S^*$. Nonetheless, we can observe from Table 7 that when the backorder policy is consistent (the first three and last three cases), the value of $\mu T - S^*$ when $\sigma = 0.5$ is nearly twice as much when $\sigma = 0.25$.

8. Concluding remarks

To draw a conclusion from the paper, we outline the procedures to obtain the optimal (s, S) inventory policy as follows.

(i) Determine the pricing policies with the supplier and estimate the demand arrival rate and other cost rates, including holding and penalty rates.
(ii) For any S candidate, evaluate whether it is necessary to permit backorder according to Theorem 2.
(iii) If backorder is permitted, express the optimal s as a function of S through Theorem 2, and then represent the profit rate of the retailer as a single variable function of S by (12).
(iv) Try $S = \mu T$ initially, and approximate the optimal S that maximises the profit rate of the retailer through a line search algorithm. Then, the optimal policy is $(-x^*(S^*), S^*)$.

In this paper, we introduced an (s, S) inventory policy for perishable items with a continuous demand process and a buyback contract. Drifted Brownian motion was applied to model the arrival of customers. The Markov renewal process was utilised to characterise the sales cycle, and closed-form profit rate functions were derived. The results indicate that the optimal backorder level depends on the maximum inventory level, and profit rates are highly sensitive to changes in the backorder policy. The optimal ordering policy is obtained through numerical analysis, and an algorithm is developed to approximate the optimum. Typically, a moderate S to ensure that all stocks are sold out right before expiration can maximise the profit rate of the retailer. If the retailer persists to choose a larger S to be on the safe side, then backlog a greater amount of demands is a more informed choice. Conversely, when S is fairly small, the retailer should also increase the backorder level to relieve the pressure from the setup cost. Furthermore, the supplier rarely benefits from a low buyback price, because the optimal policy is conservative when buyback price is low, as a result, the chance of expiration is also low and the profit rate is actually not affected by the buyback price. Finally, one can well imagine that both the customer arrival distribution and supplier's pricing policies have effects on the retailer's strategy. Through sensitivity analyses, we demonstrated the effects of different parameters on the optimal replenishment policy.

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Appendix

This section includes the proofs of the results presented in the preceding sections.

proof of Theorem 1. It is evident that $w(s) = 0$. Define $\lambda = \frac{2\mu}{\sigma^2}$ for the remainder of the proof, then

$$\frac{\mathrm{d}w}{\mathrm{d}y} = \frac{2}{\sigma^2}\int_y^{+\infty} \exp\{-\lambda(\xi - y)\}h(\xi)\mathrm{d}\xi.$$

In other words, $-\mu\frac{\mathrm{d}w}{\mathrm{d}y} = -\lambda\int_y^{+\infty}\exp\{-\lambda(\xi - y)\}h(\xi)\mathrm{d}\xi$. Furthermore,

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2w}{\mathrm{d}y^2} = \frac{\mathrm{d}\left(\int_y^{+\infty}\exp\{-\lambda(\xi - y)\}h(\xi)\mathrm{d}\xi\right)}{\mathrm{d}y}.$$

Let $t = \xi - y$, we have

$$\begin{aligned}\frac{\sigma^2}{2}\frac{\mathrm{d}^2w}{\mathrm{d}y^2} &= \frac{\mathrm{d}\left(\int_0^{+\infty}\exp\{-\lambda t\}h(y+t)\mathrm{d}t\right)}{\mathrm{d}y}\\ &= \int_0^{+\infty}\exp\{-\lambda t\}\mathrm{d}h(y+t)\\ &= [\exp\{-\lambda t\}h(y+t)]_0^{+\infty} - \int_0^{+\infty}h(y+t)\mathrm{d}\exp\{-\lambda t\}\\ &= -h(y) + \lambda\int_0^{+\infty}\exp\{-\lambda t\}h(y+t)\mathrm{d}t\\ &= -h(y) + \lambda\int_y^{+\infty}\exp\{-\lambda(\xi - y)\}h(\xi)\mathrm{d}\xi.\end{aligned}$$

Stated thus, the equation $-\mu\frac{\mathrm{d}w}{\mathrm{d}y} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2w}{\mathrm{d}y^2} = -h(y)$ holds, which concludes the proof.

Proof of Theorem 2. Firstly, we can infer from Theorem 1 that

$$\begin{aligned}\mu g_p(x) &= \int_{-x}^0\left(-\lambda\int_u^{+\infty}C_s\exp\{-\lambda(\xi-u)\}\xi\mathrm{d}\xi\right)\mathrm{d}u = \int_{-x}^0\int_u^{+\infty}C_s\xi\mathrm{d}\exp\{-\lambda(\xi-u)\}\mathrm{d}u\\ &= \int_{-x}^0[C_s\xi\exp\{-\lambda(\xi-u)\}]_u^{+\infty}\mathrm{d}u - \int_{-x}^0\int_u^{+\infty}C_s\exp\{-\lambda(\xi-u)\}\mathrm{d}\xi\mathrm{d}u\\ &= -\int_{-x}^0 C_s u\mathrm{d}u + \int_{-x}^0\left[\frac{C_s}{\lambda}\exp\{-\lambda(\xi-u)\}\right]_u^{+\infty}\mathrm{d}u\\ &= \frac{1}{2}C_s x^2 - \frac{\sigma^2}{2\mu}C_s x,\end{aligned}$$

the computation here incorporated the higher divergence rate of an exponential function than that of a linear function. Denote $A = \mu[(p-w)S - (p-m)R(S) - H(S) - C_0]$, $B = \mu T_I(S)$, $a = \frac{1}{2}C_s$ and $b = \mu(p - w - C_u) + \frac{\sigma^2}{2\mu}C_S$, we can conclude from equation (12) that

$$p_R(x) = \frac{-ax^2 + bx + A}{x + B}, p_R^{'}(x) = \frac{-ax^2 - 2aBx + bB - A}{(x+B)^2}.$$

The numerator of $p_R^{'}(x)$ is a concave quadratic function with symmetry axis $x = -B < 0$. That is, when $bB - A < 0$, we have $p_R^{'}(x) < 0$ for any $x > 0$, and no backorder should be permitted. Otherwise, when $bB - A > 0$, the equation $p_R^{'}(x) = 0$ has an unique positive solution $x^*(S) = \sqrt{B^2 + \frac{bB-A}{a}} - B$ which satisfies $p_R^{'}(x) > 0$ for any $0 < x < x^*(S)$ and $p_R^{'}(x) < 0$ for any $x > x^*(S)$. Therefore, $x^*(S)$ maximises the profit function and is the optimal backorder level.

Moreover, $p_R''(x) = -\frac{2(aB^2+bB-A)}{(x+B)^3}$. When a positive optimal backorder level $x^*(S)$ exists, we always have $p_R''(x) < 0$, which concludes the proof.

Proof of Corollary 1. When $C_s = 0$, we have

$$p_R(x) = \frac{A + \mu(p - w - C_u)x}{B + x},\ p_R^{'}(x) = \frac{B\mu(p - w - C_u) - A}{(B + x)^2},$$

if $A < B\mu(p - w - C_u)$, we always have $p_R^{'}(x) > 0$, and $p_R(x)$ becomes an increasing function of x, so all the demands should be backordered.

Proof of Corollary 2. Firstly, it is straightforward that

$$\lim_{S\to 0} \frac{p(S - R(S)) + mR(S) - wS - H(S) - C_0}{T_I(S)} = -\lim_{S\to 0} \frac{C_0}{T_I(S)} = -\infty,$$

while we can infer from (5) and (7) that $\lim_{S\to +\infty} T_I(S) = T$ and $\lim_{S\to +\infty} S - R(S) = \mu T$, that is

$$\lim_{S\to +\infty} \frac{p(S - R(S)) + mR(S) - wS - H(S) - C_0}{T_I(S)} = (p - m)\mu - \lim_{S\to +\infty} \frac{(w - m)S + H(S) + C_0}{T} = -\infty.$$

On the other side, the left side of (14) is a positive constant. As a result, (14) always holds when S approaches 0 or infinity.

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