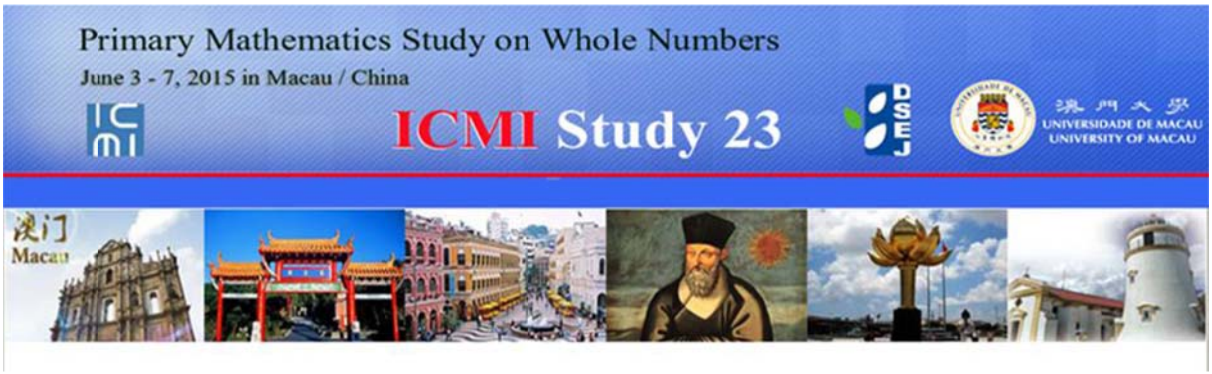
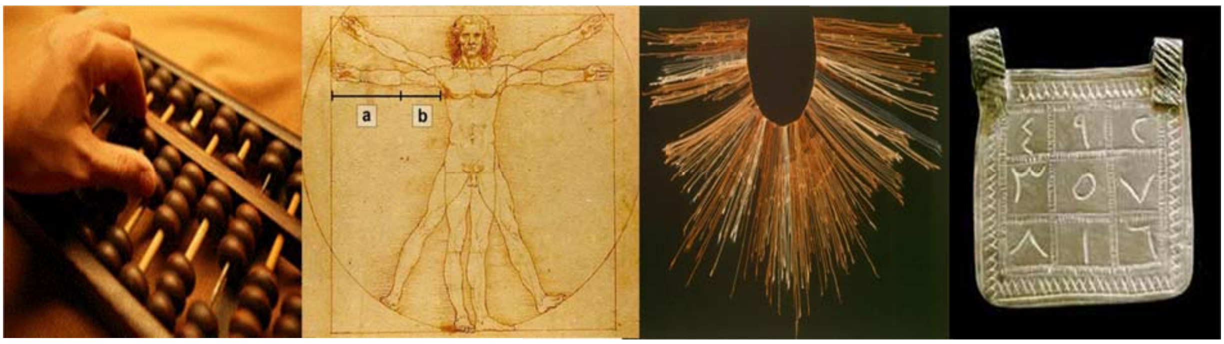


CONFERENCE PROCEEDINGS OF ICMI STUDY 23 : PRIMARY MATHEMATICS STUDY ON WHOLE NUMBERS



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International Commission on Mathematical Instruction

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The Twenty-third ICMI Study:
Primary Mathematics Study on Whole Numbers

Macao, China
University of Macau

June 3 - 7, 2015

Proceedings

Edited by Xuhua Sun, Berinderjeet Kaur and Jarmila Novotná

Macao 2015

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We want to acknowledge the significant work done by all the IPC members, who voluntarily invested time and effort to be part of the IPC on top of their other obligations. They were friendly, and willing to work hard when some speed-up of the process was needed to meet the deadlines (We specially acknowledge Hamsa Venkat for the help in language editing of some parts). We want also to thank the IMU Secretariat and the University of Macau, the Education and Youth Affairs Bureau, Macau SAR (DSEJ; 澳門特別行政區政府教育暨青年局) and Macao mathematics education study association for the generous financial support for both the IPC meeting and the Conference. The colleagues and technical staff of university of Macau contributed effectively to preparation of the conference management system, of the conference and of the proceedings (We specially acknowledge Xitao Fan for effective administration support and leadership of ICMI STUDY 23 conference). Last but not least, Lena Koch (IMU Secretariat at WIAS) helped to solve many problems during the whole process.

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TABLE OF CONTENTS

PREFACE

Xuhua Sun, University of Macau (Macao SAR of China)

With a long history of mathematics theory and practice, the Chinese community has a unique appeal to mathematics educators at the primary education level. Ancient Chinese peoples invented computation tools (the Chinese Suanpan, considered the fifth most important invention in Chinese history, came into widespread use around 1,000 years ago during the Song Dynasty, and was added to UNESCO's list of intangible heritage in 2013). Computation procedures associated with the counting rod tool in SunZi Suanjing (孫子算經) A.D. 5 century (Lam and Ang, 2004), are still used in current textbooks and classrooms. In recent years, the outstanding performance of Chinese students in Shanghai in the OECD PISA mathematics assessment, and Macau SAR's rise from 15th position in 2009 to 6th position in 2013, have attracted much interest from educators around the world.

Macau is unique in its role in Chinese mathematics education. For example, the first Macau`s Jesuits Matteo Ricci translated Euclid`s Elements with Guangqi Xu and the first arithmetic book on European pen calculation, not bead calculation before, Tong Wen SuanZhi, with Zhihao Li, changed Chinese mathematics education and gave Chinese people their first access to real images of western mathematics. We believe that this heritage of mixed traditions under the influence of the Confucian educational heritage can provide a resource for new thinking in global mathematics education development.

With a fascinating history of 400 years of cultural exchanges between the East and the West, Macau is also unique in its cultures and society. It boasts many cultural treasures of all types, including picturesque dwellings in traditional styles, ancient temples built during the Ming and Qing dynasties, buildings with Southern European architectural features, baroque style churches and impressive contemporary structures. In July 2005, the historic district collectively known as the "Historic Centre of Macau" was inscribed on the UNESCO World Heritage List. Today Macau is a Special Administrative Region (SAR) of the People's Republic of China, benefiting from the "one country, two systems" policy. Macau SAR is growing in the number and diversity of its attractions. The greatest of these continues to be Macau's unique society, with communities from the East and the West complementing each other. It offers a perfect environment for an international conference.

We are especially interested in the "Dialogue among Civilizations" relating to whole number arithmetic as a foundational component in mathematics education around the world. It is our great pleasure to have mathematics educators from all over the world come and enjoy Macau, and to make the ICMI STUDY 23 a rich and professionally rewarding conference for all. The conference attendees will have the opportunity to experience the unique characteristics of Chinese mathematics education practice, which is closely connected to the Eastern traditions of didactics of mathematics that has seen important recent developments relating to whole number. We are excited to host ICMI STUDY 23, and we warmly welcome all of you to come to Macau for ICMI STUDY 23, and much more!

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THE ICMI STUDY 23
PRIMARY MATHEMATICS STUDY ON WHOLE NUMBERS

***Maria G. Bartolini Bussi* [1] *and Xuhua Sun* [2]**

[1] ***University of Modena and Reggio Emilia (Italy),***

[2] ***University of Macau (China)***

Introduction

This volume contains the proceedings of the twenty-third study led by the International Commission on Mathematical Instruction (ICMI). The study addresses, for the first time, mathematics teaching and learning in primary school (and pre-school as well) for all, taking into account international perspectives including socio-cultural diversity and institutional constraints. Whole number is the core content area, which is regarded as foundational for later mathematics learning; its teaching and learning are thus very important due to larger impact for later mathematics knowing.

The study was lauched by ICMI at the end of 2012, with the appointment of two co-chairs (Maria G. (Mariolina) Bartolini Bussi and Xuhua Sun) and of the International Program Committee (IPC), which on behalf of ICMI is responsible for conducting the Study: Berinderjeet Kaur, Hamsa Venkat, Jarmila Novotna, Joanne Mulligan, Lieven Verschaffel, Maitree Inprasitha, Sybilla Beckmann, Sarah González; Abraham Arcavi (ICMI Secretary General), Ferdinando Arzarello (ICMI President), Roger E. Howe (ICMI liason).

During 2013 an intense mail exchange was realised within the IPC, in order to share the rationale, the goals and the steps of the forthcoming ICMI Study.

The process

The IPC meeting in Berlin (January 2014)

In January 2014 (19-24) the IPC meeting took place in Berlin, at the IMU Secretariat, which generously supported the costs. The IPC members were welcomed by Prof. Dr. Jurgen Sprekels, director of the Weierstrass Institute for Applied Analysis and Stochastic (WIAS, Berlin), and by the ICMI President Prof Ferdinando Arzarello, who participated in the whole meeting. The meeting aimed at:

a Sharing the ICMI study guidelines.

b Producing the Discussion Document, in which a number of key issues and sub-themes related to the theme of the Study were identified and described in a preliminary manner.

c Defining, in a preliminary manner, the criteria for identifying participants for the international conference, constituting a working forum that would investigate the theme of the study. Particular emphasis was given to bringing together both experts in the field and newcomers with promising work in progress, as well as to gathering representatives with a variety of backgrounds from different regions, traditions and cultures.

d Defining the way of disseminating the discussion document in order to reach, in the most effective way, the communities of the expected participants in different regions and cultures.

e Defining the dates, the venue and the structure of the study Conference, that mirrored the key issues of the study, to appoint the chairs of the parallel sessions and to select a limited number of keynote speakers to be invited.

The meeting in Berlin took place in a productive and collaborative climate: a draft version of the Discussion Document was agreed and the Conference dates and venue were chosen (June 3 – 7, 2015 at the University of Macau – China). Five themes (each corresponding to a Working Group in the Conference) were identified and assigned to pairs of members of the IPC

1 *The why and what of whole number arithmetic* (Xuhua Sun, Sybilla Beckmann)
2 *Whole number thinking, learning, and development* (Joanne Mulligan, Lieven Verschaffel)
3 *Aspects that affect whole number learning* (Maria G. Bartolini Bussi, Maitree Inprasitha)
4 *How to teach and assess whole number arithmetic* (Berinderjeet Kaur, Jarmila Novotná)
5 *Whole numbers and connections with other parts of mathematics* (Sarah González, Hamsa Venkat).

Three plenary speakers were invited:

Liping Ma: *The theoretical core of whole number arithmetic*
Brian Butterworth: *Low numeracy: from brain to education*
Hyman Bass: *Quantities, numbers, number names, and the real number line*

Three plenary panels were identified:

Traditions in whole number arithmetic (chaired by Ferdinando Arzarello);
Special needs in research and instruction in whole number arithmetic (chaired by Lieven Verschaffel);
Whole numbers arithmetic and teacher education (chaired by Jarmila Novotná).

The Discussion Document

The text of the Discussion Document was finalised in the following weeks and disseminated on April 1st 2014 on the ICMI website[1], on the Conference website[2] by means of the ICMI news[3], through announcements in the most popular international mailing lists and through abridged versions published in some international journals (e. g. Educational Studies in Mathematics, the Newsletter of the European Mathematical Society, International Journal of STEM Education).

In the Discussion Document a special emphasis was given to the importance of cultural diversity and to the effects of this diversity on the early introduction of whole numbers. In order to foster the understanding of the different contexts where authors had developed their studies, each applicant for the Conference was required to include background information about this context.

The call for papers

A conference management system was created by the University of Macau[4]. The initial deadline for submission was extended in order to solicit papers from authors coming from as many contexts as possible.

The Conference

Invited participants

The review and selection processes took place in December 2014 - January 2015. At the end 67 papers were accepted (in many cases with additional supported revisions) and distributed over the five themes. The results were communicated to the submitting authors by the end of February 2015, in line with the announced deadlines. For each accepted paper, a maximum of two (co)authors were invited to participate in the Study Conference. The resulting participation is summarised in Tab. 1.

It is not surprising that Asia (and especially the China area) is well represented, because of the proximity of the venue. The other regions are unequally represented. Actually, in spite of the efforts of the IPC, the equity issue in the participation in this ICMI study is far from being reached, although the themes

[1] http://www.mathunion.org/icmi/conferences/icmi-studies/introduction/
[2] http://www.umac.mo/fed/ICMI23/
[3] http://www.mathunion.org/mailman/listinfo/icmi-news
[4] https://cmt.research.microsoft.com/ICMI2015/Default.aspx

had the potential to involve mathematics educators from non-affluent countries and policy makers. Several obstacles may be identified:

ineffective dissemination: international mailing lists and journals continue to reach a limited portion of the mathematics education community across the world;

language issues: the choice of English as the study language, although unescapable, might have inhibited some authors to apply;

costs: airfares are not strictly related to distances from countries, and we had several comments about prohibitive costs of flights, with these costs affecting less well-represented parts of the globe disproportionately.

Regions	Papers
Europe	28
Asia	17
North America	10
Australia and New Zealand	6
Africa	4
Central and South America	2
Total	67

Tab. 1: Data

Observers

Thanks to generous support from the University of Macau, for the first time, this ICMI study was able to invite observers from non-affluent countries. The choice was to privilege CANP (*Capacity & Networking Project, The Mathematical Sciences and Education in the Developing World*) that is the major development focus of the international bodies of mathematicians and mathematics educators[5].

[5] http://www.mathunion.org/icmi/activities/outreach-to-developing-countries/canp-project/
One representative for each of the following project was invited with a generous financial support:
CANP1, Edi Math (Mali, 2011, with participants from across Sub-Saharan Africa);
CANP2, Central America and the Caribbean (Costa Rica, 2012, with participants from Latin America and the Caribbean);
CANP3, South East Asia (Cambodia, 2013, with participants from ASEAN);
CANP4, East Africa (Tanzania, 2014, with participants from Tanzania, Kenya, Uganda and Rwanda);
CANP5, Andrean Region (Peru, to be held in 2016, with participants from Peru, Ecuador, Paraguay, Bolivia and Amazonian Brazil).

Some policy makers were also invited to join the Conference.

The future steps

The ICMI Study Conference will serve as the basis for the production of the Study Volume. The character of the volume is rather unique to ICMI studies and is different from proceedings, edited books and handbooks. The ICMI Study Volume appears as a book in the New ICMI Studies Series (NISS) whose general editors are the President and the Secretary-General of ICMI. The volume should include the texts of plenary speeches, chapters describing the outcomes from the panels and chapters collectively and consensually produced by each of the groups (under the guidance of their co-leaders, who are members of the IPC) integrating the outcomes from the parallel workshops of the ICMI Study Conference. Although the volume exploits the contributions appearing in the proceedings, the collective production will be started during the Conference, drawing on the discussions and cooperative works of participants. It is planned to present the volume on the occasion of ICME13 in Hamburg (2016)[6].

The volume of proceedings

The proceedings are ready, carefully edited by Xuhua Sun, Jarmila Novotna, and Berinderjeet Kaur. The participants are warmly encouraged to exploit the presentation of plenary events and to read in advance at least the parts related to their theme, as a significant part of the working group time will be devoted to discussions between participants, in order to foster the collective production of the volume.

The distribution of papers in different themes (and, accordingly, of authors in different working groups) was not easy at all. At first, we tried to meet the authors‘ first or second choice, but later, in many cases, we had to cluster the papers in different ways, following the Chinese approach of "grasping ways beyond categories" and "categorise in order to unite categories".

Concluding remarks

As participants in other ICMI studies, we believe that this study has some peculiar features that we wish to emphasise:

- the preparation of a context form, to be filled by each participant, to give the background information of the study and/or its theoretical statements,

[6] http://icme13.org

- the invitation to submit video-clips with papers, to exploit the effectiveness of visual data in the age of web communication,
- the participation of IPC members as authors and not only as organisers and co-leaders of working groups,
- the scientific support offered to authors in the revision of their papers,
- the economic support offered to authors from the University of Macau,
- the supported participation of CANP observers,
- the involvement of both the IMU President (Prof. Shigefumi Mori) and the ICMI President (Prof. Ferdinando Arzarello) in the preparation of the Conference.

This collective international effort will lead us in a few weeks to the Macau Conference, as a product of the fruitful cooperation between mathematicians and mathematics educators, when, for the first time in the history of ICMI, the issue of whole numbers arithmetic in primary school is to be addressed.

Reggio Emilia – Macau, March 31 2015

THE ICMI STUDY 23 PLENARY SPEAKERS

Maria G. (Mariolina) Bartolini Bussi

University of Modena and Reggio Emilia, Italy

Since the meeting of the International Program Committee in Berlin (January 2014) the issue of plenary speakers was addressed. As in the case of panels, the IPC agreed on themes to be addressed:

the epistemological issue: as ICMI is a commission of the International Mathematical Union, mathematics must be in the foreground;

the neurocognitive issue: as studies on the development of “number sense” are carried out by neuroscientists, it is timely to develop a serious interdisciplinary work between neuroscientists and mathematics educators;

the cultural issue: because of the importance of the cultural contexts, a thoughtful discussion on different traditions in the teaching and learning of whole number arithmetic is needed.

This decision carried in a natural way the IPC to choose some outstanding researchers which might have represented the above perspectives.

Hyman Bass. His mathematical research covers broad areas of algebra with connections to geometry, topology, and number theory. He is a member of the National Academy of Sciences and the American Academy of Arts and Sciences. He was president of the American Mathematical Society and of the International Commission on Mathematical Instruction (ICMI).

Brian Butterworth. He is Emeritus Professor of Cognitive Neuropsychology at in the Institute of Cognitive Neuroscience at University College London. He is collaborating with colleagues around the world on the neuropsychology and the genetics of mathematical abilities, with a multicultural perspective. A long-term project is to persuade educators and governments to recognize dyscalculia as a serious handicap that needs specialized help.

Ma Liping. She was senior scholar at the Carnegie Foundation for Advancement of Teaching. Her book (Knowing and teaching elementary mathematics) is quoted on all sides of discussions about how to teach mathematics in elementary schools in the United States. She has a Ph.D. from Stanford University and earned a masters degree in education from East China Normal University. She is currently working independently.

QUANTITIES, NUMBERS, NUMBER NAMES, AND THE REAL NUMBER LINE

Hyman Bass*[1]*, University of Michigan, USA

Abstract

This paper describes an approach to developing concepts of number using general notions of quantity and their measurement. This approach, most prominently articulated by Davydov and his colleagues, offers some affordances that are discussed. Some arguments favouring this approach are offered. First is a way of providing coherent connections in the development of whole numbers and fractions. Second is that it makes the geometric number line continuum present from the start of the school curriculum as a useful mathematical object and concept into which real numbers can be eventually explicitly developed. Third, in the Davydov approach, are some opportunities for some early algebraic thinking. I also present an instructional context and approach for the development of place value as a numeration system modelled on the invention of a place value system of number representation.

Key words: measure, number, place value, quantity, real number line

> *... we assumed that the students' creation of a detailed and thorough conception of a real number, underlying which is the concept of quantity, is the purpose of this entire subject, from grade 1 to 10 . . . the teacher, relying on the knowledge previously acquired by the children, introduces number as a . . . representation of a general relationship of quantities, where one of the quantities is taken as a measure and is computing the other.* Vasily Davydov, 1990

Two conceptions of quantity: Counting and measure

Number and operations have two aspects: conceptual (what numbers are) and nominal (how we name and denote numbers). Conceptually, numbers arise from a sense of *quantity* of some experiential species of objects – count (of a set or collection); distance; area; volume; time; rate; etc. And in fact before children enter school, they have already acquired a sense of quantity, of rough comparison of size, as well as of counting. Number is not intrinsically attached to a quantity; rather it arises from *measuring* one quantity by another, taken to be the "unit:" How "much" (or many) of the unit is needed to constitute the given quantity? This is the *measurement framework* in which fractions are often introduced, via part-whole relations, the whole playing the role of the unit, which is a choice to be made, and has to be specified. The discrete (counting) context in which whole numbers are often developed is distinguished by the use of the single-object set as the unit, so that the very concept of the unit, and its

[1] I am greatly indebted to Deborah Ball for critical feedback and for helpful framing of the ideas and perspectives presented here, not all of which we share.

possible variability, is not necessarily subject to conscious consideration. This choice is so natural, and often taken for granted, that the concept of a *chosen unit of measurement* need not enter explicit discussion. If number is first developed exclusively in this discrete context, then fractions, introduced later, might appear to be, conceptually, a new and more complex species of number quite separate from whole numbers. This might make it difficult to see how the two kinds of numbers eventually coherently inhabit the same real number line. Indeed, this integration entails seeing the placement of whole numbers on the number line from the point of view (not of discrete counting, but) of continuous linear measure.

This distinction is further reinforced by the fact that fractions have their own notational representation, distinct from base-10 place value of whole numbers. The operations on numbers likewise have conceptual models, but notational representations of number are needed in order to construct *computational algorithms*. A numerical computation, of say a sum of two numbers, is not about understanding what the sum *means*. Instead, given two numbers A and B in notation system S, a calculation is a construction of a representation of A + B in same notation system S. That is why "2 + 11," though a logically correct answer to, "What is 5 + 8?" is <u>not</u> the correct answer, 13, to: "*Calculate* 5 + 8." Important as the notation is, its emphasis without links to the conceptual foundations can make it seem that quantities are the same as their number names, which could be misleading.

Two possible pathways exist for the development of whole numbers:

Counting:

- Using the discrete context of finite sets, introduce whole numbers as cardinals, and addition as the cardinal of a disjoint union, and the experience of enumerating and comparing sets. (This relies on a discrete model of quantity.)

Measure:

- Using the general context of quantity of various species of experiential objects, and addition as disjoint union or concatenation. This allows discussion of comparison of quantities (which one is more), and, implicitly that the larger quantity equals the smaller plus some other quantity. This can be done before any numerical values have been attached to the quantities, with the relations expressed symbolically.
- Then number is introduced by choice of a unit, and the number attached to a quantity is how much of the unit is needed to constitute the given quantity. Whole numbers then are represented in the form of quantities that are measured exactly by a set of copies of the unit.

The measure pathway was articulated in detail by Davydov (1975). My first purpose here is to discuss the measure pathway, and cite some possible virtues that merit our attention. In particular, I will note that it makes available from the beginning the continuous number line as a coherent geometric environment in which all numbers of school mathematics eventually reside.

My second purpose is to discuss our base-10 place value notation for whole numbers (and finite decimals) and their operations, emphasising its extraordinary power and its impact on the progress of mathematics and science. I will also describe a particular instructional model[2] for the development of place value. This model can be seen to provide an activity context for not only conceptual understanding of place value, but also one that models the intellectual need (Harel, 2007) to *invent* some version of positional number notation.

Implications for the development of the real number line:

Two narratives

I propose here some affordances of developing number in the measure context. Most importantly, this approach offers a productive context for developing the real number line across the grades. Relying exclusively on the discrete model of counting leads to what I will call the "construction narrative" of the number line, in which the new kinds of numbers, their notations, and their operations, are added incrementally without sufficient interconnection. In this narrative whole numbers, and their verbal names and symbolic base-10 representations, predominate. New kinds of numbers are added – fractions, negative numbers, a few irrational numbers, and eventually infinite decimals. This process of bringing in these new types of number can lead to "immigration stress," and difficulties of assimilation of the new numbers into one coherent context. In particular, the real number line as a coherent connected number universe with uniformly smooth arithmetic operations is not as explicit as it could be.

In the "measure narrative" the number line, at least as a geometric continuum, is featured as the environment of linear measurement. A premise of this trajectory is that the mathematical resources that children bring include not only discrete counting, but also a sense of measurement of continuous quantity. A possible metaphor for geometric number line is an (indefinitely long) string, flexible but inelastic. Then linear quantities would be "measured" by a segment of string. This would permit comparison of size even before such measures acquire numerical names. An example of an activity drawing on this metaphor is to engage students in considering how far two toy cars travel from a starting point by examining where each car stops along a strip of tape on the floor. In order to compare measures of two things that are remote, one adopts a standard *unit* of

[2] This is based on work by Deborah Ball with teacher candidates, representing work done with primary grade children.

measure, against which both quantities can be compared. And then whole number quantities appear as iterated composites of that unit.

To situate numbers on the number line, the "*oriented unit*" is specified by the choice of an ordered pair of points, called 0 and 1, the unit of linear measure then being the segment, [0, 1], between them. The direction from 0 to 1 then also specifies a positive orientation to the number line (which has an intrinsic linear order defined by the fact that, given any three points, one lies in the interval between the other two), whereupon the whole numbers (and eventually all real numbers) can be located on the number line by juxtaposing replicas of [0, 1] in the positive direction.

Of course the counting approach to whole numbers can be interpreted in measure terms, since cardinal is one particular context of measurement. However, counting is only one such (discontinuous) context, and the unit (a set with one object) must be made explicit to extend to the general concept of unit. Other units in the discrete context are made visible when one later encounters (skip) counting in groups. More general continuous measurement environments for whole numbers are robustly represented with materials such as Cuisenaire rods. Eventually, whole numbers (as cardinals) are so well conceptually assimilated that they seem to become (abstract) entities in their own right.

Fractions are often developed from a measure perspective, with fractions, from the start, being conceived as part-whole relationships, and applied to a wide variety of species of quantities: round food; lengths of ribbon; containers of sugar, or of milk; sets of objects; periods of time; etc. In contrast with whole numbers, it is less common to name a fraction without adding the word "of." Moreover, we do not hesitate to compare the size of whole numbers, while, with fractions, we are more prone to first ask, "fractions of what?" – attending to specification of the unit (or whole).

Operations and the real number line

Addition and subtraction appear to be conceptually similar in both the counting and measure regimes, addition corresponding to combination (composition, and decomposition of quantities) and subtraction to taking away or comparison.

Multiplication is more subtle, and more complex. One model is repeated addition of some fixed quantity, as if applying the counting regime to fixed size groups of unit quantities. One difficulty with this model is that it obscures the commutativity of multiplication. This is sometimes repaired by use of rectangular arrays, eventually evolving into area models. The difficulty of the area model, from a measure perspective, is that numbers and their products then have different units of measure (for example, length and area) so that it is problematic to assign meaning to an expression like a•b+c. One resolution of this is to use a continuous version of repeated addition, which is scaling (magnification and shrinking). This has the advantage of maintaining the species of quantities involved. These are complex conceptual issues, which I do not pursue here.

Suffice it to say here that, from the point of view of quantity (measurement), we can combine (simplify) additive expressions only when they are quantities of the same species (we do not add apples and oranges, unless combined into some larger category, like “fruit”), expressed with a common unit, and then the sum or difference is a quantity expressed with that same unit. When dealing with fractions, a quantity like 3/5 is understood to be three one-fifths, where the latter corresponds to a rescaling of the unit. In adding fractions, finding a “common denominator” is then a process of measuring two quantities with a common unit in order to make simplification of the sum possible. Similarly, in multi-digit addition, the alignment of the base-10 representations of the summands assures that the addition in each column is adding digits with the same base-10 units attached.

On the other hand, for multiplication and division, the units of measurement are not restricted, but simply parallel the operation, leading to compound units, like: kilometres/hour; foot•pounds; pounds per square inch; class•hours.

Once numbers are named and denoted (in base-10, or with fraction notation), then we develop algorithms for the operations in that notational system. The power of the base-10 system is that addition, subtraction, and multiplication can be performed on any pair of whole numbers knowing only how to perform single-digit operations (“basic facts”), plus how to keep track of positional notation. This puts extraordinary computational power instructionally within reach of young children, a major historical development.

Once fractions and integers have been developed, one has the rational numbers, which are densely distributed on the number line: between any two points there is a rational number. The example of irrational numbers, like $\sqrt{2}$, shows that many points remained to be named. Informal arguments of approximation can indicate how all points can eventually be specified by possibly infinite decimal representations. Moreover, informal assurance can be given that the operations extend by continuity to all real numbers, preserving the *basic rules of arithmetic.* This synthesis of the real number line sets the stage for higher mathematics, for example calculus.

The Davydov Curriculum

Davydov, a Vygotzkian psychologist and educator, and his colleagues in the Soviet Union developed, in the 1960s and 1970s, a curriculum based on the measure approach (1990).

In order to develop the concept of number, the Davydov curriculum delayed the introduction of number in school instruction until late in the first grade. Early lessons concentrate on “pre-numerical” material: properties of objects such as colour, shape, and size, and then quantities such as length, volume, area, mass, and amount of discrete objects (i.e. collections of things, but without yet using number to enumerate "how many").

According to Davydov, the fundamental problem solved by the invention of number is the task of taking a given quantity (length, volume, mass, area, amount of discrete objects) and reproducing it at a different time or place. Moxhay (2008) describes the following activity that illustrates this:

On one table, is a strip of paper tape. The task is to go to another table (in a different room) and cut off, from the supply of paper tape, a piece that is exactly the same length as the original one. But one is not allowed to carry the original paper strip over to the other table. In Davydov's experiments, children sometimes just walked over to the second table and cut off a piece of paper of a random size, hoping that it would be the same length as the original one. In such cases, conditions of the task seemed to the children to make a correct solution impossible (except by luck).

Davydov and his colleagues explained that a solution might involve taking a third object, such as a string, and cutting it to be just precisely the length of the paper strip, and then carrying this intermediary object (the string) to the other table, where it can be used to lay off a new paper strip of the required length. In this case, the intermediary is equal in length to the object to be reproduced. The curricular approach showed children how to take a given third object, say a piece of wood, and, if it is longer than the paper strip, mark it to show the length of the paper strip. This solution was equivalent to the first one, with the children performing just a different set of operations. But if the only available intermediary object was *smaller* than the paper strip – for example, a wooden block, this was the an interesting case, for then the children could learn that they could use the block as a *unit* – as an intermediary than could be placed repeatedly (each time marking the paper with a pencil), and then *counting up* how many times the unit has been laid down. The unit could then be carried to the other table (together with the number) where it would be laid down on the paper tape the number of times that is necessary to reproduce, by cutting, a paper strip of the required length. Note that, only with this last method – selection of a unit and counting how many of it are needed – that number names make an appearance.

Although this is a particular task, solved by a particular discovery on the part of the children, it is said to lead "genetically" to the solution of all analogous tasks. If the children, working as a collective, grasp the meaning of the construction they have made, then they should (again, collectively, at least at first) be able to attack all analogous problems. Davydov argues that children thus recreate, in brief, the invention of number as a human tool that enables *any* quantity to be reproduced at a different place or time. It is worth noting that this task would lose its force in the discrete context of counting, in which the portability of measure is much simpler to achieve, but therefore also invisible and tacit.

Davydov argued that it was important for children to reflect on, become conscious of, the ideas developed through this activity. He develops this as a collective process, with the teacher guiding the children to ask one another questions like, “How did you do this? Why did you do this? Does your method work? Is that the best method for solving the task?”

Algebra in the context of Davydov (see Schmittau, 2005)

In the Davydov curriculum, children were to study scalar quantities such as the length, area, volume, and weight of real objects, which they can experience visually and tactilely, thus gaining a first access to the real number continuum. Early in the first grade, for example, children were shown that they could make two unequal volumes equal by adding to the smaller or subtracting from the larger the difference between the original quantities.

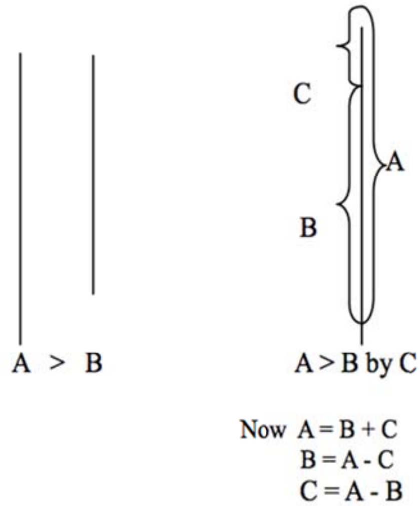

Fig. 1: Schematic representing the change necessary to equalise two volumes, A and B. (Schmittau, 2005, p.19)

They determined that if volume A is greater than volume B, then A = B + C, where C is the difference between A and B (see Fig. 1).

The children would be led to schematise their result with a “length” model, and symbolise it with equations and inequalities.

The following problem, occurring approximately half way through the first grade curriculum, provides another example of the role of the schematic in problem solving: N apples were in a bowl on the table. R people entered the room and each took an apple. How many apples remained? Children first analyse the structure of the problem, identifying it as a part-whole structure, with N as the whole and R as a part. They schematise the quantitative relations expressed in the problem as follows (see Fig. 2).

N

/ \

R ?

Fig. 2: Part-whole structure (Schmittau, 2005, p.19)

Beyond the visibly algebraic form of these equations and relations, introduced quite early, there are further noteworthy features, having to do with the very nature of the "=" sign. When equations are introduced numerically, the first exercises often have the format, 8 + 4 = _, with the result that students gain the habit of reading "=" as "calculate what is on the left, and put the answer on the right." Thus, they will validate the equation 8 + 4 = 12, but question 12 = 8 + 4. Moreover, they may fill the blank in 8 + 4 = _ + 7 with 12. I expect that these confusions would be mitigated with the balancing of quantities approach of the Davydov curriculum. Of course other curricula have ways of accomplishing this as well.

Place Value

> *The greatest calamity in the history of science was the failure of Archimedes to invent positional notation.* - Carl Friedrich Gauss

Davydov emphasised the notion of quantity as being primary, the concept of number being later derived as a measure of one quantity by another (the unit). There then arises the task of providing names and notations for numbers. Although the notion of quantity is in some sense cognitively primordial, the naming of numbers, in contrast, is a cultural construct, and it has been accomplished historically in many different ways (see for example ICMI Study 13, (Leung, Graf and Lopez-Real, 2006)). But the naming of numbers is much more than a cultural convention. It is itself a piece of conceptual technology with huge bearing on the progress of science. Our current Hindu-Arabic system of (base-10) place value notation, now universally used in science, was solidified relatively late in history. It puts within reach of even young children a quantitative power not reached even by the mathematical genius of ancient Greece. (See the above quote from Gauss.)

Howe (2011) offers a critique of elementary curriculum in the U.S., "Place value ... is treated as a vocabulary issue: ones place, tens place, hundreds place. It is described procedurally rather than conceptually." How can one produce in young children, and their teachers, a robust conceptual understanding of place value? I describe here a method developed by Deborah Ball, one that is now an integral part of the teacher education program at my university. Teacher candidates experience this sequence for several purposes, among them to appreciate the structure and meaning of a numeration system, in this case, the base-ten system. This approach fits here since its design echoes the instructional approach of Davydov, Brousseau, and others, who like to introduce a concept using a mathematical problem context whose solution necessitates discovery of that concept.

In this case, *the problem is to collectively count a large collection*. The size of the count is sufficient to require some structural organisation for record keeping, and to make this common across counters so as to be able to coherently combine the different records. It is this need that precipitates the idea of grouping, which leads to a hierarchal structure akin to place value.

The setting here is a methods class for some 25 elementary teacher interns. (The activity is a compressed approximation of what would be done with primary grade children over much longer period of time.) About half of the interns sit in a circle on the floor with the teacher, the others observing and taking notes. On the floor, the teacher pours out a container of over 2,000 sticks. She first invites the interns to guess/estimate how many sticks there are. After a wide range of guesses, she asks, "How could we find out?" and suggests that they count them. So the counting begins, each intern gathering individual sticks from the pile, and lining them up. However, their individual collections quickly become so numerous that they feel a need to somehow consolidate. After some discussion the idea of *grouping* the sticks emerges. Note that this arises, not as a mathematical suggestion, but as a practical necessity, given the large size of the counting task. And with rubber bands that are available, they begin to form what they call "bundles" of sticks. But then the question arises, "How many sticks should be in a bundle?" Several choices are considered (e.g., 2, 5, 10, 25, 60). The small values are judged not to achieve enough efficiency to be worthwhile, and the larger to be possibly unwieldy. It is nonetheless clear that *this is a choice to be made*; it is not mathematically forced. (This opens the space to later contemplate place value in bases other than 10.) More importantly, *this choice should be the same for each person*. Otherwise there would be no coherent way to count the amalgamated collections at the end. The teacher eventually proposes making bundles of ten sticks each.

Then the counting continues, and the interns make a bundle as soon as ten loose sticks are available to do so. At any given moment, an intern's collection has the form of a certain number of bundles, together with at most 9 loose sticks. However, the big pile is so numerous that the interns confront the same problem again, this time with their bundles instead of individual sticks. A discussion similar to the earlier one then ensues about grouping the bundles, to form "bundles of bundles," or "super-bundles," as they came to be called. Again the question arose: "How many bundles should there be in a super-bundle?" It was noted that this choice could, in principle, be independent of the first. But it was decided that there would be some mathematical merit in again choosing ten for the number of bundles in a super-bundle. And these could again be bound together with rubber bands. At this point, each intern's collection consists of a modest number of super-bundles, at most 9 bundles, and at most 9 loose sticks.

Finally, when the big pile was exhausted, the collections of the different interns were brought together. Then the many loose sticks were bundled until at most 9 loose sticks remained. In turn then, the bundles were super-bundled until at most 9 bundles remained. Finally, there being over twenty super-bundles, it was decided to make two "mega-bundles," each composed of ten super-bundles. In the end then, the original pile had been organized into 2 mega-bundles, 4 super-bundles, 7 bundles, and 6 loose sticks. Thus, the cardinal of this huge collection of could be specified by a list of just four small numbers, (2, 4, 7, 6), specifying the numbers of mega-bundles, super-bundles, bundles, and loose sticks, respectively. By construction, the number of sticks in a bundle is 10, in a super-bundle $10^2 = 100$, and in a mega-bundle, $10^3 = 1{,}000$. Thus the very concise "coding" (2, 4, 7, 6) tells us that the total number of sticks is 2,000 + 400 + 70 + 6 = 2,476 (in base-10 notation).

Fig. 3: Counting a large collection

This activity, with modest scaffolding, *simulated the invention of the place value system of recording numbers.* Moreover it dramatically and physically presented the compressive power of the system: four small digits suffice to specify this perceptually very large quantity. In the course of the activity, the teacher could pose a number of questions, about representing particular numbers with the sticks and their bundles, and also how to identify numbers represented by various configurations of bundled sticks, modelling the sorts of interactions that would be carried out with children.

Attention was further drawn to the fact that the bundled sticks remained an authentic representation of quantity, since they could be unbundled to reproduce the original collection. This was put in contrast with other physical representations of number, such as Dienes blocks, for which the ten-rod could not be decomposed into to ten little cubes; rather, this would require a trade.

These physical models of base-10 provide concrete models for the arithmetic operations. The correspondence with the symbolic base-10 notation can then be extended to provide concrete meaning to the algorithms for arithmetic computation.

Conclusion

I have argued that the measure-based introduction to number, as developed for example by Davydov, supports a possibly more coherent development of the real number line. Moreover I suggest that it smooths the transition from whole numbers to fractions, and it provides an early introduction to algebraic thinking. Finally, I have described an instructional activity, developed by Ball, that simulates the conceptual development of place value.

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LOW NUMERACY: FROM BRAIN TO EDUCATION

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Abstract

It is widely agreed that humans inherit a numerical competence, though the exact nature of this competence is disputed. I argue that it is the inherited competence with whole numbers (the 'number module') that is foundational for arithmetical development. This is clear from a longitudinal study of learners from kindergarten to Year 5. Recent research has identified a brain network that underlies our capacity for numbers and arithmetic, with whole number processing a core region of this network. A twin study shows a strong heritable component in whole number competence, its link to arithmetical development and to the brain region. These findings have implications for improving numeracy skills especially among low-attaining learners.

Key words: dyscalculia, innate capacities, intervention design, numerosity processing, parietal lobes

Introduction

Leopold Kronecker is quoted famously as making the ontological claim that "God made the integers, all else is the work of man.' This is not a testable hypothesis. Kronecker may or may not have been a believer in the supernatural when he made this statement. He was born a Jew but converted to Christianity a year before his death. He apparently believed that only integers and objects constructed from them actually existed. This included rational numbers but excluded the reals, π, transcendental numbers more generally, and infinities, all of which may be mathematically useful, but didn't really exist.

If God did make the integers, how did we come to know them? This is a problem that has exercised the best philosophical minds since the time of Plato. However, if we take his apothegm more metaphorically, he may be arguing that our *knowledge* of maths depends on our *knowledge* of integers. That is, we re-cast his ontological claim as an epistemological one. We can go further, and re-cast God as evolution. That is to say, is there an evolutionary basis for our knowledge of integers? Here we need to step back from the term 'integer', which includes negative numbers, and restrict ourselves to positive whole numbers, the so-called 'natural numbers'.

It is now widely acknowledged that the typical human brain is endowed by evolution with a mechanism for representing and discriminating numbers. It is important to be clear right at the outset, that when I talk about numbers I do not mean just our familiar symbols – counting words and 'Arabic' numerals, I include any representation of the number of items in a collection, more formally the cardinality of the set, including unnamed mental representations. Evidence comes from a variety of sources.

Human infants notice changes in the number of objects they can see, when other dimensions of the objects are controlled. In the first study of this kind, infants of five to six months noticed when a successive displays of two dots was followed by a display of three dots, and when successive displays of three dots was followed by a display of two dots. However, they did not notice a change from four to six dots or from six to four dots (Starkey and Cooper, 1980). With larger numbers of dots, infants need a ratio of 2:1 to notice a change in the number of dots (Xu and Spelke, 2000). Recently studies have shown that infants notice the matches between the number of sounds and the number of objects on the screen (Izard et al., 2009; Jordan and Brannon, 2006), suggesting that the infant's mental representation of number is relatively abstract – that is, independent of modality of presentation.

There is also evidence for individual differences in various measures of this ability, at least in older children (Geary et al., 2009; Piazza et al., 2010; Reeve et al., 2012). Twin studies suggest that differences appear to be at least partly genetic (Geary et al., 2009; Piazza et al., 2010; Reeve et al., 2012). The genetic factor is reinforced by the finding that certain kinds of genetic anomaly, such as Turner's Syndrome, affects numerical abilities, including very basic abilities such as selecting the larger of two numbers or giving the number of dots in an array, even when general cognitive ability is normal or even superior (Bruandet al., 2004; Butterworth et al., 1999; Temple and Marriott, 1998).

Another line of evidence comes from the studies of other species. Many of those in which numerical abilities have been tested, show performance comparable with or significantly better than human infants. Chimpanzees are able to match the correct digit to a random display of dots up to at least ten (Matsuzawa, 1985; Tomonaga and Matsuzawa, 2002). Monkeys are able to select the larger numerosity of two displays even when the elements in the display are novel. Moreover, they show a very similar 'distance effect' to humans – that is, the more different the numbers, the more likely they are to select the larger correctly (Brannon and Terrace, 1998). Birds have been known to be good at number tasks for 80 years or more. Numerical abilities have been demonstrated in elephants, cats, rats, salamanders and even fish (Agrillo et al., 2012).

Neuropsychological studies of patients with brain damage reveal a complex network in the brain that supports arithmetical processes. Damage to the frontal lobes affects the ability to solve novel problems, while damage to the parietal lobe, usually the left parietal lobes, affects the ability to do routine tasks or to recall previously learned facts (Cipolotti and van Harskamp, 2001) and (Butterworth, 1999, Chapter 4) for reviews. Neuroimaging shows that the parietal lobes are activated by very simple tasks, such as selecting the larger of two numbers or the display with more dots (Dehaene et al., 2003; Pinel et al., 2001). In fact, small regions in the left and right parietal lobes (the intraparietal sulci) are specific for processing the numerosity of displays (Castelli, Glaser and Butterworth, 2006). These regions are part of a brain network involving both the parietal and frontal lobes that are activated almost every time we carry out a numerical calculation, routine or novel (Andres et al., 2011) . These findings link numerosity processing and arithmetical calculation in the brain. See

(Butterworth and Walsh, 2011) for a review of the neural basis of mathematics. I will return to the question of whether individual differences in brain structure and functioning can be linked to individual differences in arithmetical competence.

Various environmental factors can all be associated with lower mathematics attainment, including socioeconomic status and minority ethnic status, as well as gender, which should perhaps be considered a social rather than genetic factor in this context (Royer and Walles, 2007). Although it is difficult to assess the role of poor or inappropriate teaching, the fact that the introduction of detailed new national primary school strategy for numeracy in the UK had only a minor and possibly nonsignificant effect on numeracy for the group studied is indicative (Gross, Hudson and Price, 2009). It should also be noted that there are wide individual differences on even very simple tasks that depend relatively little on the quality of educational experience, such as comparison of the magnitude of two single-digit numbers or enumerating a small array of objects (Reigosa-Crespo et al., 2012; Wilson and Dehaene, 2007).

Taken together, the evidence presented here suggests that factors specific to the domain of numbers and arithmetic make a major independent contribution to low arithmetic attainment. This is supported by findings from studies that have found low attainment in learners matched for IQ and Working Memory. In a longitudinal study by Geary and colleagues, tests on understanding the numerosity of sets and on estimating the position of a number on a number line were two important predictors of low achievement in mathematics, affecting some 50% of the sample, and of mathematics learning disability, affecting approximately 7% of the sample (Geary et al., 2009). In a sample of 1500 pairs of monozygotic and 1375 pairs of dizygotic 7-year-old twins, Kovas and colleagues found that approximately 30% of the genetic variance was specific to mathematics (Kovas et al., 2007). In another genetic study, this time of first-degree relatives of dyslexic probands, it was found that numerical abilities constituted a separate factor (Schulte-Körne et al., 2007). In fact, recent reviews have proposed that developmental dyscalculia follows from a core deficit in this domain-specific capacity (Butterworth, 2005; Rubinsten and Henik, 2009; Wilson and Dehaene, 2007).

One obvious question arises: how do our numerical innate capacities relate to the learner's ability to acquire arithmetic?

Innate capacities

Now it will come as no surprise to teachers of the first three years of school, that children's numerical competence begins with whole numbers. However, recent research on the innate mechanisms available to humans and many other species propose two foundational 'core systems' that do not involve whole numbers. Deficiencies in these core systems it has been argued could contribute to low numeracy.

1. A mechanism for keeping track of the objects of attention. This is sometimes referred to as the 'object tracking system' and has limit of three or four objects. It is thought to underlie the phenomenon of 'subitizing' – making an accurate estimate of one to four objects without serial enumeration (Feigenson, Dehaene and Spelke, 2004). It is proposed that the objects to be enumerated are held in working memory and that constitutes a representation with 'numerical content' (Carey, 2009; Le Corre and Carey, 2007).
2. A mechanism for the analogue representation of the approximate number objects in a display. This is referred to as the 'analogue number system' (ANS). The internal representations of different numerical magnitudes can be thought of as Gaussian distributions of activation on a 'mental number line'. It is typically tested by tasks involving clouds of dots (or other objects) typically too numerous to enumerate exactly in the time available. One common task is to compare two clouds of dots. (Addition and subtraction tasks for which the solution is compared with a third cloud of dots are also used). Individual differences are described in terms of a psychometric function, such as the Weber fraction, the smallest proportional difference between two clouds that can be reliably distinguished by the individual (Feigenson et al., 2004).

There has been considerable interest, indeed excitement, in many studies that show the performance on tasks designed to measure competence in the approximate number system correlates significantly with arithmetical performance in both children and adults (Barth et al., 2006; Gilmore, McCarthy and Spelke, 2010; Halberda et al., 2012; Halberda, Mazzocco and Feigenson, 2008). But as we all know, correlation is not cause, and no plausible mechanism for the relationship has been proposed and accepted.

Now there are various problems with both core systems from the point of view of learning arithmetic. In the case of 1., there is an upper limit of 4. Now one key property of the number system is that a valid operation on its elements always yields another element in the same system. If one such operation is addition, and if 3 is an element, then 3 + 3 should yield an element in the system, but it cannot, since the limit is 4. To get round this, it has been proposed that noticing the number of object being tracked can be linked to the number words a child hears, and that they will be able to generalise – 'bootstrap' – from these experiences to numbers above the limit (Carey, 2009; Le Corre and Carey, 2007). The problem is that the object-tracking system is designed to keep track of particular objects with as much detail as is required by the task, not abstract away from them (Bays and Husain, 2008).

The problem with 2 is that it deals only in approximate quantities, whereas ordinary school arithmetic deals with exact quantities, and the transition from approximations to exact whole number arithmetic is still mysterious. These problems are well-known.

While we do not doubt that these systems exist in the brain of human infants and other species, we have argued that a quite different core system underlies the development of arithmetic. We and others have proposed a mechanism that can represent the ‘numerosity’ of a collection of objects; that is, the number of objects exactly, not approximately, up to a limit imposed by the developing brain. In a pioneering exploration, Gelman and Gallistel called these representations ‘numerons’, and argued that learning to count is a process of learning how to map number words consistently onto numerons (Gelman and Gallistel, 1978). I have argued, following Gelman and Gallistel, that humans inherit a ‘number module’ to deal with sets and their numerosity and that some developmental weaknesses in arithmetical development can be traced to deficiencies in the module (Butterworth, 1999, 2005).

We have shown that a neural network computer simulation of the number module using what we have called a ‘numerosity code’ accurately models the ‘size effect’ in addition. This where accuracy and speed are a function of the addends – that is, the larger the addends, or their sum, the longer it takes to retrieve or calculate the answer (Butterworth et al., 2001; Zorzi, Stoianov and Umilta, 2005).

In the next section, I describe briefly some studies we have carried out that stress the importance of whole number competence in the subsequent development of arithmetic, using a very simple test: how quickly and accurately can the child enumerate a display of dots and say the answer.

Longitudinal study of arithmetical development from Kindergarten to Grade 5

This is a study carried out in Melbourne, Australia, led by Robert Reeve and his lab. The sample comprised one hundred fifty-nine 5.5- to 6.5-year olds (95 boys). The children attended one of seven independent schools in middle-class suburbs of a large Australian city and, at the beginning of the study, were halfway through their first year of formal schooling. The children were interviewed individually on seven occasions over a 6-year period as part of a larger study. On each occasion they completed a series of tests, including those reported here. The mean ages for the test times were (a) 6 years (5.5 – 6.5 years) Kindergarten; (b) 7 years (6.5 – 7.5 years); (c) 8.5 years (8 – 9 years); (d) 9 years (8.5 – 9.5 years); (e) 9.5 years (9 – 10 years); (f) 10 years (9.5 – 10.5 years) and (g) 11 years (10.5 – 11.5 years). For full details, see (Reeve et al., 2012). Here I will focus on two aspects of the study: competence in numerosity processing as measured by the speed and accuracy of dot enumeration and age appropriate arithmetic accuracy.

Using cluster analysis, dot enumeration competence revealed three clusters at each age, which we labelled Fast (31% of the children), Medium (50%) and Slow (19%). These were relatively stable on re-testing over the period of the study. That is, although children in each cluster improved with age, each tended to stay in the same cluster.

It turns out that the cluster established in Kindergarten predicts age-appropriate arithmetic up to the age of 11 at least. I give below the results for three-digit calculations at ages 10-11 years.

	Dot Enumeration Cluster established in Kindergarten					
	Slow		Medium		Fast	
	M	*SD*	*M*	*SD*	*M*	*SD*
Subtraction	46.67	7.38	81.25	2.90	90.65	2.58
Multiplication	60.56	6.53	85.10	2.15	87.07	3.57
Division	41.67	7.02	75.62	2.88	84.86	2.97

Tab. 1: Three-digit Subtraction, Three-digit Multiplication and Three-digit Division accuracy at age 10-11 years

Our recent analyses show that from Kindergarten to Year 2, the clusters are the main predictors of the strategies use in single–digit addition, with Fast clusters more likely to recall answers from memory and use decomposition for sums over 10 in Kindergarten, whereas the Slow cluster children are only recalling the answers and decomposing in Year 2, and then less than 30% of the time.

The neural and genetic basis of low numeracy

This is a study of 104 monozygotic twins and 56 same-sex dizygotic twins aged 8 to 14 years. (Zygosity was assessed using molecular genetic methods). For more further details, see (Ranpura et al., 2013; Ranpura et al., submitted)). All the twins in the study had brain scans and carried out a battery of 40 cognitive and numerical tests. Using factor analysis, we extracted four factors, with Numerical Processing accounting for 24% of the variance, and had the highest loading. It comprised three timed arithmetic scores (addition, subtraction, multiplication), together with dot enumeration speed and the standardised WOND-Numerical Operations (Wechsler, 1996) score. Thus a second factor (19%o of the variance) included measures of general intelligence and working memory; a third factor (12%) included processing speed and performance IQ; while the fourth factor (9%), included tests of motor praxis and finger gnosis. Thus, the factor analysis reveals that that core number skills and arithmetic correlate well with each other, and segregate from general cognitive and performance measures.

We replicated other research in finding a difference in grey matter in the brains of children with low numeracy or dyscalculia in the brain Region of Interest for numerosity processing (Isaacs et al., 2001). See Fig. 1.

We were also able to establish the heritability of both competence and grey matter density by comparing MZ with DZ twins: if the concordance between pairs of MZ twins is significantly higher than between pairs of DZ twins this indicates a genetic factor.

1. Grey matter density is moderately heritable ($h^2 = 0.28$), but common environmental and unique environmental factors are also significant. Shared environment (c^2) is usually thought of as home background and schooling which applies to both twins; unique environment (e^2) is thought of as factors specific to one of the twins.
2. Arithmetical competence and dot enumeration are both heritable. See Tab. 2.

	h^2 **Genetic factor**	c^2 **Shared environment**	e^2 **Unique Environment**
Timed addition	0.54	0.28	0.17
Timed subtraction	0.44	0.38	0.18
Timed multiplication	0.55	0.31	0.15
Dot enumeration	0.47	0.15	0.38

Tab. 2: Heritability of arithmetic and dot enumeration

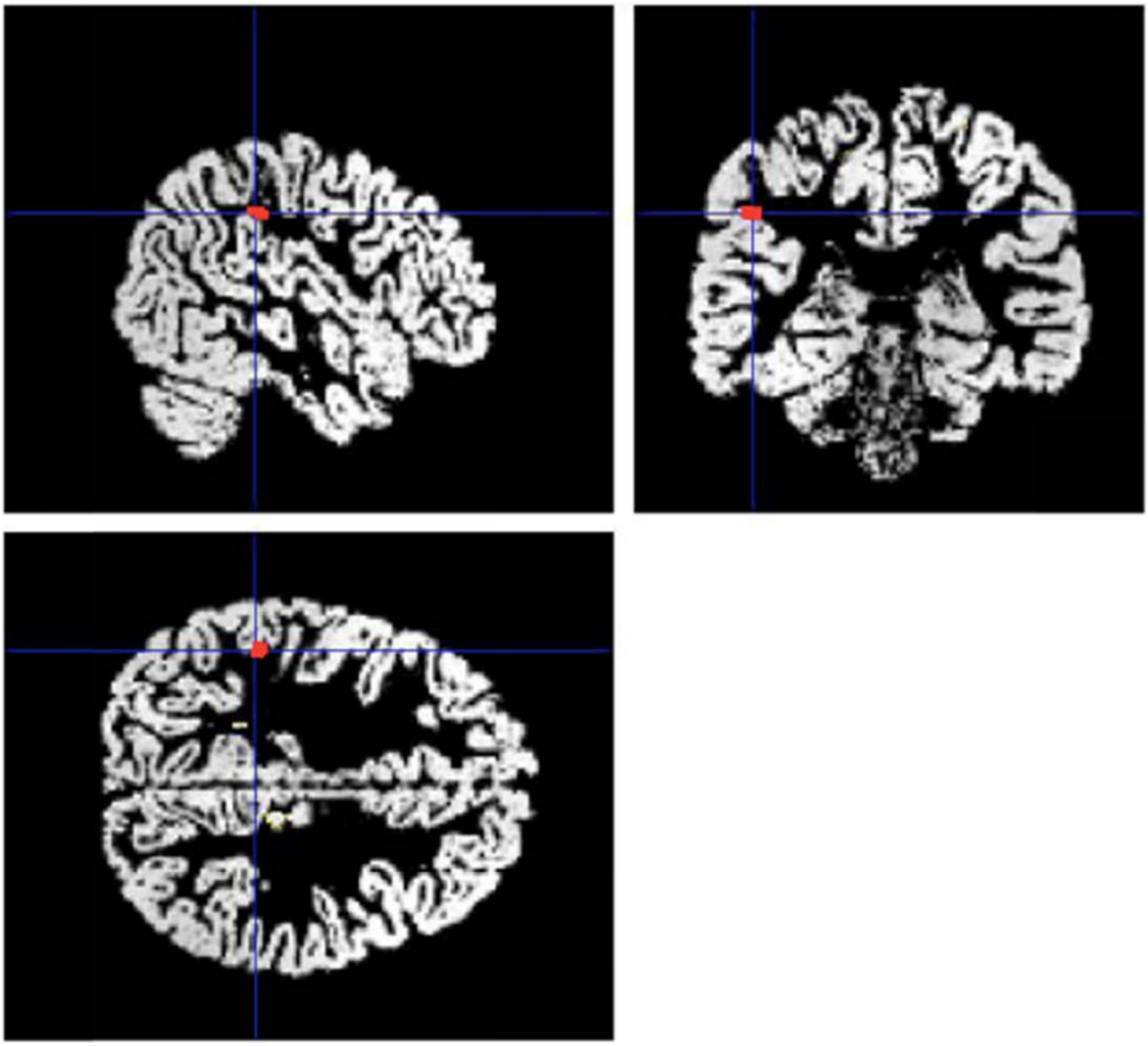

Fig. 1: Voxel-based morphometry (structural brain imaging) identifies a left parietal cluster that correlates with core number skill. (35 voxels with a peak at MNI -48, -36, 34, pFWE-corrected < 0.05)

3. The link between dot enumeration and both arithmetical competence and the Region of Interest are both heritable. Using a different way of analysing the heritability data, called 'cross-twin, cross-trait correlation', we found that the correlation of dot enumeration with timed addition was substantially heritable, with over 50% of that correlation attributable to genetic factors ($h^2h^2r_G = 0.54$, rho = 0.76, $p < 0.05$). Moreover, the links between the Region of Interest and dot enumeration, as well as arithmetical competence, were also heritable.

Implications for mathematics education

The starting point for intervention should be a recognition that some children begin with a disadvantage, and that their disadvantage lies in their capacity to deal with sets and their numerosities. This of course is the basis of arithmetic both from a logical and a developmental point of view. As we show here, low numeracy has heritable component, which confirms recent genetic studies as noted above (e.g. Kovas et al, 2007).

We can use dot enumeration in diagnostic assessments. Because these numerosity-based assessments depend much less on educational experience than tests of arithmetic, they minimise noise from instructional and motivational factors, not to mention family and environmental stressors that can also lead to low math attainment scores. Getting the correct assessment is fundamental to selecting the appropriate intervention.

Early attempts to develop new instructional interventions were based on neuroscience findings and the best practices of skilled teachers (e.g. (Butterworth and Yeo, 2004; Griffin, Case and Siegler, 1994)). An important limitation of these interventions is that they required detailed instructional schemes and one-to-one teaching. It is difficult to implement these interventions in the typical math classroom, which has a whole-class age-related curriculum that makes little allowance for atypically developing children who require more attention and practice. In theory, remediation requires an approach personalised to individual learners. In practice, it is difficult to afford such instruction for even a small proportion of pupils in publicly funded education. In the UK, it has been estimated that effective intervention for 5 to 7 year olds in the lowest 10^{th} percentile, using one-to-one teaching would cost about £2600 per learner.

The result is that many learners are still struggling with basic arithmetic in secondary school (Shalev, Manor, & Gross-Tsur, 2005). And yet effective early remediation is critical for reducing the later impact on poor numeracy skills. Although very expensive, it promises to repay 12 to 19 times the investment (Gross et al., 2009).

As I have argued elsewhere, one approach to the problem of delivering personalised instruction to individual learners is to make use of technology. Personalised adaptive learning technology solutions emulate the guidance of the special educational needs teacher, focusing on manipulation of numerosities (Butterworth & Yeo, 2004; Räsänen et al., 2009; Wilson et al., 2006). These solutions go far beyond the educational software currently in use for numeracy teaching, which mainly targets mainstream learners. Commercial software does little more than rehearse students in what they already know, perhaps building automaticity and efficiency, but it does not foster understanding and it does not address the numerosity processing deficit in many learners, and especially in dyscalculics. Rarely are commercial games founded on good pedagogy.

Of course, there is no clear logical pathway from assessment to educational remedy, so our software seeks to use ideas from the best practitioners, such as Dorian Yeo (Butterworth and Yeo, 2004), and established pedagogical principles, including

1. Constructionism – construct an action to achieve goal (Harel and Papert, 1991)
2. Informative feedback (Dayan and Niv, 2008)
3. Concept learning through contrasting instances, and generalising concepts through attention to invariant properties (Marton and Pong, 2007)
4. Direct attention to salient properties (Frith, 2007). This entails ensuring that everything on the screen is relevant to the task in hand.
5. The zone of proximal development - adapt each task to be just challenging enough (Vygotsky, 1978)
6. Use intrinsic rather than extrinsic reinforcement (Laurillard, 2012)

Examples of the games following these principles have been developed by Diana Laurillard and Baajour Hassan and can be found at http://number-sense.co.uk (see Fig. 2).

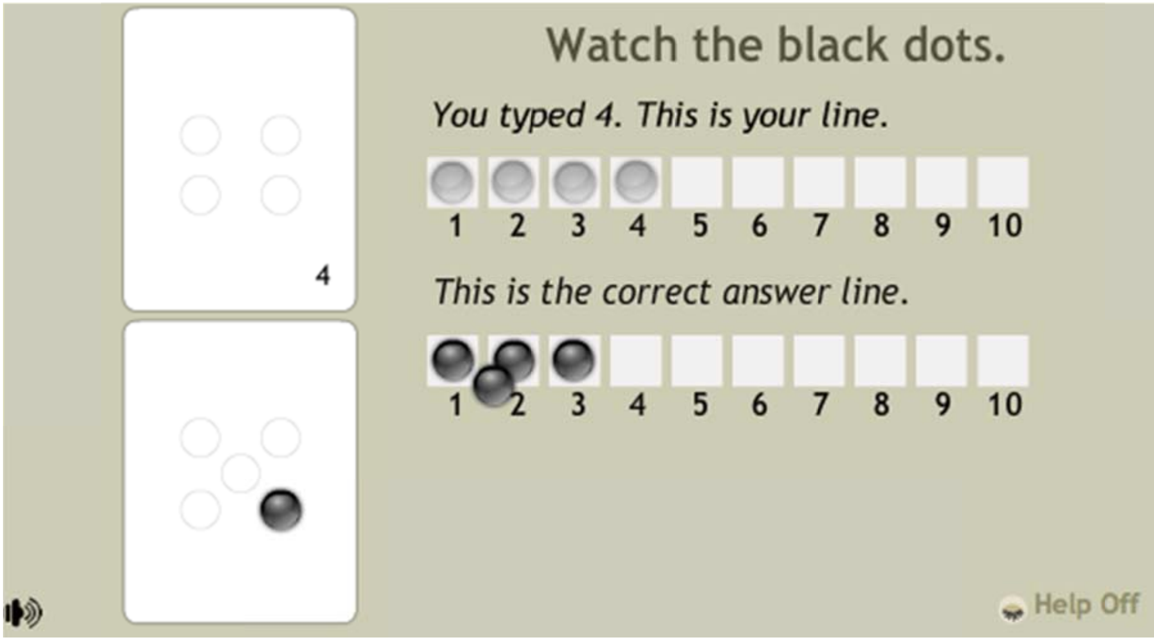

Fig. 2: Dot2Track. For an explanation see text

Their Dots2Track game exemplifies these principles. The task is to type the number of dots in a display. At level 1, these are arranged as in dominoes. In the case of an error, learner's dots are counted onto a line above it and the correct number of dots on the line below it, exploiting principles 2 and 3. There is an opportunity to construct the correct answer by increasing or decreasing the number the learner chose (1). Everything on the screen is relevant (4), and game is adaptive, becoming more difficult depending on the accuracy and speed of the responses (5). The only reward is getting the right answer (6). There is preliminary data on the effectiveness of these games (Butterworth and Laurillard, 2010).

Even if a learner has an inherited deficiency in the number module that is reflected in brain structure and functioning, this does not mean a life sentence of low numeracy. It may be that the right interventions over sufficient time can strengthen the number competence to a typical level, and indeed modify brain to a more typical structure as has been shown in the case of phonological training for dyslexic learners (Eden et al., 2004). This will require a longitudinal study that has not yet been carried out.

Conclusions

I have argued here that the genetic research is supported by neurobehavioural research identifying the representation of numerosities – the number of objects in a set - as a *foundational capacity* in the development of arithmetic. Where this capacity is weak, education should seek to strengthen this capacity using sets of real objects or virtual objects and linking the sets to the spoken and written numbers until the learner can use the numbers fluently and confidently. This will provide a sound basis for developing arithmetic.

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THE THEORETICAL CORE OF WHOLE NUMBER ARITHMETIC

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Abstract

There are at least two different perspectives on whole number arithmetic in primary school. In the US, the tendency is to consider it as only learning to compute the four basic operations with whole numbers. In China, however, whole number arithmetic involves much more than simply learning to carry out the computational algorithms. For example, it is expected that students explore the quantitative relationships among the four operations, and represent these (sometimes quite sophisticated) relationships with (sometimes quite complicated) numerical equations.

In the author's opinion, this exploration of quantitative relationships is made possible by the theoretical core that underlies school arithmetic. In this article, the author will present the central pieces of this theoretical core.

Key words: basic quantitative relationships, four operations, school arithmetic, whole number

Introduction

There are at least two different perspectives on whole number arithmetic in primary school. In the US, the tendency is to consider it as only learning to compute the four basic operations with whole numbers. In China, however, whole number arithmetic involves much more than simply learning to carry out the computational algorithms. For example, it is expected that students explore the quantitative relationships among the four operations, and represent these (sometimes quite sophisticated) relationships with (sometimes quite complicated) numerical equations.

In my article "A Critique of the Structure of U.S. Elementary School Mathematics" in 2013, I pointed out that there is a theory of school mathematics that underlies school arithmetic in China and a few other countries. Although this theory underlies school arithmetic in China, it began to occur in school arithmetic as mass education spread across Europe and the United States around the middle of the nineteenth century. By the beginning of the twentieth century, the theory was almost complete. It gave intellectual power to its predecessor, commercial arithmetic, and turned it into an academic subject.

A complete account of the theory of school arithmetic requires a longer article, or even a monograph. In this short article, I present the central pieces of this theory: the definition of one self-evident concept, unit, and two basic quantitative relationships derived from the definition of unit. The theory built around these central pieces explains all the computational algorithms in arithmetic. Moreover, it can foster primary students' ability to deal with quite sophisticated quantitative relationships.

The central pieces of the theory of school arithmetic

Euclid's *Elements* is known as an exemplar of a logic system. The mathematical scholars who initiated the theory of school arithmetic made a strong attempt to emulate the rigor of Euclid's approach. The central pieces presented in this article are the initial definitions of this system.

The one self-evident concept

Like the *Elements* of Euclid, the theory of school arithmetic starts with several general definitions which form the foundation for the remaining content in the system. Two of these definitions are those from which the two basic quantitative relationships are directly derived.

Definition of *Unit*

One, or a single thing, is called a *unit* or *unit one*.

A group of things, if considered as a single thing or one, is also called a *unit*, a *unit one*, or a *one*.

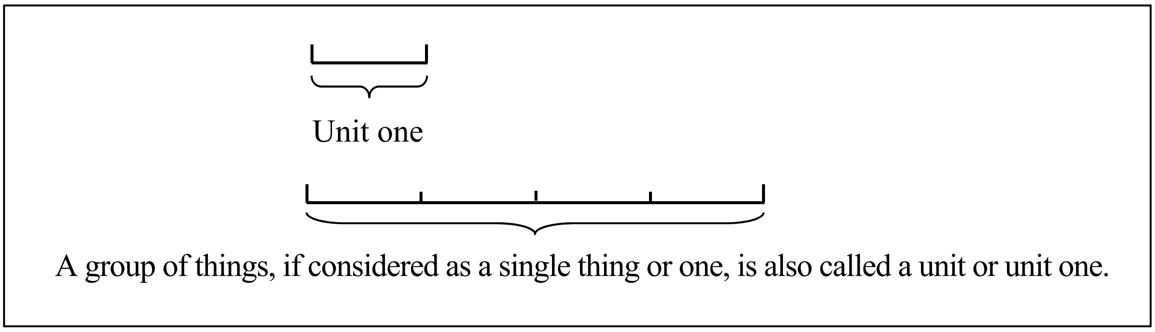

Fig. 1: The definition of unit

(Discussion: "Unit," indeed, is a self-evident concept for us. The definition of "unit" has three levels of abstraction: one thing; one; and a group of things considered as a single thing or one. These three levels of abstraction are the cornerstones from which the two basic quantitative relationships are derived.)

Definition of *Number*

A number is a unit (one) or a collection of units (ones).

(Discussion: The term "number" has several definitions as students progress through school. The definition of number above derived from the definition of unit is consistent with the "natural numbers" (the positive integers) that are part of young children's everyday experiences.[3])

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[3] How many aspects of the number zero should be taught in elementary school is an issue which needs further discussion. Consider Alfred North Whitehead's remark: "The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilised of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought" (1948, p. 43).

One of the two basic quantitative relationships: the sum of two numbers

Based on the above two definitions, the two basic quantitative relationships in school arithmetic are defined. The first one is "the sum of two numbers." Then the operations of addition and subtraction are defined in terms of this quantitative relationship.

Definition of *the sum of two numbers*

The sum of two numbers is a third number which contains as many units as the other two numbers taken together.

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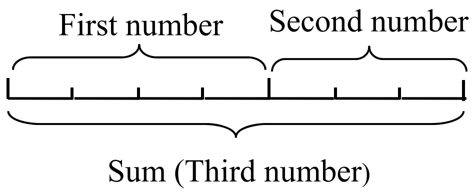

Fig. 2: The sum of two numbers

(Discussion: The definition of the sum of two numbers is derived from the definition of "unit" and that of "number." It, nevertheless, is only related to the first sentence in the definition of "unit: "One, or a single thing, is called a *unit* or *unit one*.")

Definition of *Addition*

The operation of finding the sum of two numbers is called *addition*.

Definition of *Addend*

The two numbers summed are called *addends*.

Definition of *Subtraction*

If a sum and one addend are known, the operation of finding the unknown addend is called *subtraction*.

Definition of *Difference*

The result of the operation of subtraction is called the *difference*.

(Discussion: The quantitative relationship formed by three numbers has the following feature: If two of the three numbers is known, the third is determined. Because of this, it is possible to define addition and subtraction in terms of this quantitative relationship.)

The other basic quantitative relationship: the product of two numbers

The other basic quantitative relationship in school arithmetic is "the product of two numbers." The operations of multiplication and division are defined from this second quantitative relationship in school arithmetic:

<u>Definition of *the product of two numbers*</u>

The *product* of two numbers is a third number which contains as many units as one number being taken as many times as the units in the other.

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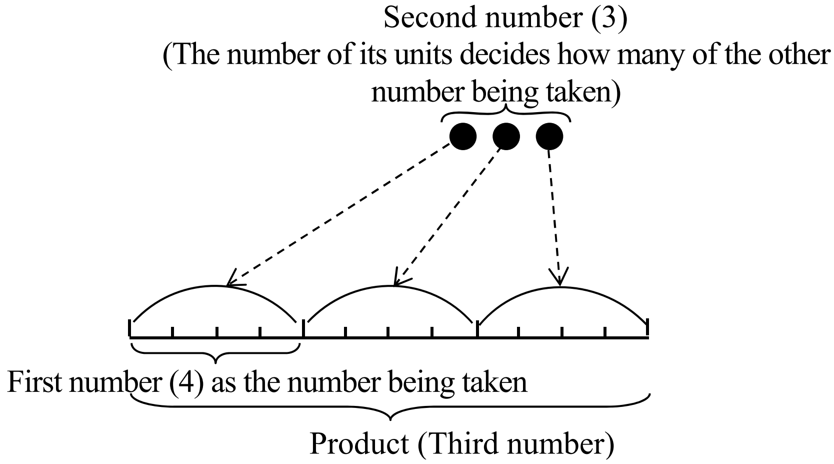

Fig. 3: The product of two numbers

(Discussion: The definition of the product of two numbers is also derived from the definitions of "unit" and of "number." However, the second sentence in the definition of unit, "A group of things, if considered as a single thing or one, is also called a *unit*, a *unit one*, or a *one*," plays a critical role. The third level of abstraction in the definition of "unit" is reached now.)

<u>Definition of *Multiplication*</u>

The operation of finding the product of two numbers is called *multiplication*.

<u>Definition of *Multiplicand*</u>

Multiplicand is the number to be taken.

<u>Definition of *Multiplier*</u>

Multiplier is how many times the multiplicand is taken.

<u>Definition of *Division*</u>

If a product and one of the multiplicand or multiplier are known, the operation of finding the unknown multiplier or, respectively, multiplicand is called *division.*

Definition of *Quotient*

The result of the operation of division is called the *quotient.*

(Discussion: Like the relationship "sum of two numbers," the relationship "product of two numbers" has the following feature: if two of the three numbers is known, the third is determined. Therefore, it is possible to define the operations of multiplication and division in terms of the quantitative relationship "the product of two numbers.")

Quantitative relationships in school arithmetic: from basic to sophisticated

When students, learning whole number arithmetic, tend to consider $3 + 2$ as a sum, $5 – 2$ as a difference, 3×4 as a product and $15 \div 5$ as a quotient, they have attained the ability to analyse quantitative relationships, not only simple ones, but also relatively sophisticated ones. For example, $(2 + 3) + (6 – 5)$, the sum of a sum and a difference, or, $(20 – 2) \times (3 + 1)$, the product of a difference and a sum. This ability, obviously, will prepare them well for moving on to higher-level subjects such as algebra.

Conclusion

There seems to be a gap between the experience of the four operations that young students have in everyday life and the definitions of these operations in terms of the two basic quantitative relationships. However, the gap can be filled by curriculum and instruction that are designed to lead students from concrete experience to abstract thinking. This is a journey during which young students' ability to think abstractly is carefully fostered. It has been realised in the practice of elementary mathematics in some countries, such as China.

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THEME 1: THE WHY AND WHAT OF WHOLE NUMBER ARITHMETIC

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Introduction

Mathematics learning and teaching are deeply embedded in history, language, and culture (Barton, 2008). Different languages have different grammar and emphasise different aspects of number, which may or may not support a deep understanding of number concepts, such as ideas about base ten, place value, and operations. On the one hand, a purpose of education is to support the continuity of the structures and functions that are special to a culture (Leung, Graf and Lopez-Real, 2006). On the other hand, Whole Number Arithmetic (WNA) is a core part of mathematics that all modern cultures require all students to learn in school. A critical question then is, how does a cultural system reflect on its own history, language, and culture, identify disadvantages and advantages of its system, and overcome its disadvantages? What lessons do we learn from these reflections and from interventions that are designed to overcome the disadvantages? These questions are the motivation for Theme 1.

All the fourteen papers accepted for Theme 1 include aspects of how different communities, either historical or contemporary, have conceived of or represented numbers or arithmetic. The papers explore several overlapping aspects of the why and what of WNA: the historic background of WNA, the language foundations of WNA, foundational ideas that underlie WNA, and different expected learning and teaching goals of WNA. By examining variation in WNA across history and language, and across different communities, Working Group 1 can discuss implications of the different views on the why and what of WNA for instruction and for teacher education.

Historic background of WNA: numeration systems and operations

Historically, different cultures have conceived of WNA in different ways, using different symbols, tools, and ideas for representing and calculating with whole numbers. Several authors explore aspects of the history of WNA.

González and Caraballo present excerpts of the use of the Incas' Yupana to do mathematical operations and argue the need of indicate examples of projects applying ethnomathematical approaches that link between traditional artefacts and methods and more formal artefacts and methods to teach the children of these cultures today.

Siu reviews how counting rods and the abacus were gradually replaced with written calculation in China by Tongwen Suanzhi (同文算指). The emphasis Tongwen Suanzhi placed on the learning and teaching of arithmetic exerted influence on the subsequent writing of textbooks in China. Instead of teaching algorithms with the aid of mnemonic poems, the underlying reasoning was brought into calculation as the learning foundation.

Sun discusses how the early Chinese invented the most advanced number name and the most advanced calculation tools (counting rod and Suanpan or Chinese abacus), in which place value is the most overarching principle as the spirit of WNA based on the Chinese linguistic habit. Traces of this influence can be found in contemporary core curriculum practices.

Zou summarises findings from historical investigations into arithmetic in ancient China, including how number units were derived and named and how numbers were represented with rod or bead calculation tools and with symbols.

Language foundation of WNA: regularity, grammar, and cultural identity

Different cultures have different number names. Some number names do not apply the decimal (base ten) principle of the numeration system (e.g., between 10 and 20). Several papers examine issues surrounding the grammar and regularity of number names across cultures or time and how spoken names relate to written representations of numbers.

Azrou reports on the historical and linguistic background of the most widespread languages spoken in Algeria and summarizes some of the issues with spoken and written arithmetic. The findings are an initial step towards developing an intervention in teacher education that will be designed to enhance students' awareness of differences in representing numbers in different languages and promote students' cultural identities.

Chambris studies changes related to place value that were introduced by the *new math* in France and the impact on WNA teaching, from curricular design to teaching practices, and students' learning. The names of the units used in numeration are a key tool to describe and understand changes in teaching and learning.

Houdement and Tempier report on two experiments to strengthen the decimal (base ten) principle of numeration, giving a key role to the use of numeration units (ones, tens, hundreds …) in France.

Foundational ideas underlying WNA

How students can develop fundamental ideas about WNA and what teachers need to know to nurture those ideas are active areas of inquiry. Several papers examine what the foundational ideas of WNA are and how children and teachers might represent and work with those ideas.

Changsri explored first grade students' ideas of addition in two Thai schools in the context of Lesson Study and an Open Approach and found that the students used a variety of representations to express addition ideas.

Dorier gives a short overview of the main stages of the development of numbers in the history of humanity and shows how Brousseau's theory, in accordance with the historical context, can be used to develop the key stages of a teaching sequence on the concept of numbers.

Ejersbo and Misfeldt describes a research project addressing the specific irregularities of the Danish number names by introducing a regular set of number names in primary school in Denmark.

Sayers and Andrews summarise an eight-dimensional framework, which they call foundational number sense that characterises necessary learning experiences for young children. They demonstrate how to use the framework by analysing learning opportunities in first grade across five European contexts.

Thanheiser takes the perspective of variation theory and uses historical number systems as a tool in teacher education, finding that prospective teachers develop a more sophisticated conception of the base 10 place value system by examining, comparing, and contrasting different aspects of historical systems.

Different expected learning and teaching goals for WNA

Just as there are differences historically in ideas about WNA, so too there are differences in contemporary goals for teaching and learning WNA. Several authors address issues concerning the perspectives of different communities.

Cooper discusses how the different perspectives of a university mathematician and a group of elementary school teachers interacted productively, leading to new insights on division with remainder, not just on the part of the elementary school teachers, but also on the part of the mathematician.

Howe discusses how thinking in terms of "base ten pieces" could support well-known properties of the decimal system for development of the attitude of "learning arithmetic with understanding" as a key goal within mathematics learning.

McGarvey and McFeetors identified mutual concerns that the Canadian public has about the goals of mathematics learning and the supports required for students to reach those goals, including teacher expertise and clear teaching resources.

Questions for Discussion in the Working Group

In addition to the background discussion and questions posed in the Discussion Document, the papers for Theme 1 may lead to discussions on the following questions:

(1) How are number concepts represented across language, curriculum, and culture?
(2) How is the place value concept represented across curriculum and culture?
(3) How are number properties (e.g., associativity and community) represented across curriculum and culture?

(4) How are addition/ subtraction concepts represented across curriculum and culture? When should counting be replaced by mental calculation? And how?
(5) How are multiplication/ division concepts represented across curriculum and culture?
(6) How are applications of WNA (word problems) organised across curriculum and culture?
(7) What are characteristics of productive ways to leverage historical perspectives in the teaching and learning of WNA today?
(8) How well do teachers currently understand the base 10 place value system? How do we improve their knowledge of it?
(9) How well do teachers currently understand other foundational ideas about number (in addition to the base 10 place value system) that children must learn in order to make sense of WNA? How do we improve teachers' knowledge of it?
(10) What goals underlie the teaching and learning of WNA? What enables different communities to work together productively to prepare future and current teachers for teaching WNA?
(11) How do we interact productively with a variety of stakeholders (e.g., parents, administrators) when it comes to implementing findings from research about instruction and teacher development in WNA?

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Papers in Theme 1

Azrou, N. Spoken and written arithmetic in post-colonial country: The case of Algeria.

Chambris, C. Mathematical foundations for place value throughout one century of teaching in France.

Changsri, N. First grade students' mathematical ideas of addition in the context of lesson study and open approach.

Cooper, J. Combining mathematical and educational perspectives in professional development.

Dorier, J-L. Key issues for teaching numbers within Brousseau's theory of didactical situations.

Ejersbo, L., & Misfeldt, M. The relationship between number names and number concepts.

González, S., & Caraballo, J. Native American cultures tradition to whole number arithmetic.

Houdement, C., & Tempier, F. Teaching numeration units: Why, how and limits.

Howe, R. The most important thing for your child to learn about arithmetic.

McGarvey, L., & McFeetors, J. Reframing perceptions of arithmetic learning: A Canadian perspective.

Sayers, J., & Andrews, P. Foundational number sense: The basis of whole number arithmetic competence.

Siu, M.K. Pedagogical lessons from Tongwen Suanzhi (同文算指) – transmission of Busan (筆算 written calculation) in China.

Sun, X. Chinese core tradition to whole number arithmetic.

Thanheiser, E. Leveraging historical number systems to build an understanding of base 10.

Zou, D. Whole number in ancient Chinese civilisation: a survey based on the system of counting-units and the expressions.

SPOKEN AND WRITTEN NUMBERS IN A POST-COLONIAL COUNTRY: THE CASE OF ALGERIA

Nadia Azrou, University Yahia Farès, Médéa

Abstract

The aim of this paper is to present some initial steps of a long-term study aimed at intervening in teacher education in a situation of encounter of different cultural influences in a post – colonial country: Algeria. Some preliminary analyses will be reported on how natural numbers are orally represented (spoken numbers) in different ways according to different languages. The long term perspective is to take profit from the existing differences to develop competencies concerning written numbers, and at the same time to enhance students' awareness about the roots of those differences, thus contributing to promote their cultural identities.

Key words: cultural identity, natural language, post-colonial country, spoken arithmetic, written arithmetic

Introduction: Preliminary steps of the research project and their framing

In Algeria, like in other post-colonial countries, the influence of colonial culture still spreads through cultural institutions and shapes many aspects of the dominant culture, in spite of a declared will of autonomy. Moreover, within the local culture, there are power relationships between different ethnic groups, which frequently result in cultural dominance of one group over the others. The educational challenge consists in transforming elements of extraneousness or 'subalternity' of some students to the dominant culture(s), into tools to promote both the development of basic competencies, and personal and group cultural identity for all students. Such purpose requires a preliminary and a careful analysis of the situation, informed by an appropriate theoretical framing.

Being interested in the development of basic mathematical competencies, in particular numerical competencies, I consider, first, the differences between the oral representations of natural numbers in different languages in Algeria and their relationships with writing of numbers according to the decimal- position system. Indeed those differences may intervene in the relationships between street mathematics and school mathematics (cf. Nunes, Schliemann and Carraher, 1993), particularly in the case of those students issued by families less impregnated by the dominant culture. Street mathematics is mainly oral and, at the very beginning of schooling, school mathematics is presented orally by the teacher, in most cases according to the school official language only. We may see here an example of how cultural dominance works through language dominance (cf. Valdés, 1999) and may result not only in potential learning difficulties concerning the subject matter (mathematics) for some students (see Miura and Okamoto, 2003, p. 230), but also in their extraneousness and 'subalternity' (cf. Gorgorio and Planas, 2001). But language, in our case mathematical language (i.e. natural language in the mathematical register: see Boero, Douek and Ferrari, 2008) is also the place where many cultural differences not only surface, but also may be identified and discussed by students with the help of teachers. Thus, acknowledging cultural differences concerning the language of mathematics may contribute to initiate classroom discourses regarding cultural identity and mutual understanding on parity dignity level (see Nieto, 1999; Gorgorio and Planas, 2001). On the other side, linguistic aspects of basic mathematical notions are relevant for their mastery in the teaching and learning of mathematics (like in the case of the relationships between written and oral arithmetic: see Miura and Okamoto, 2003); and linguistic differences may contribute, if conveniently exploited by the teacher, to put into evidence some deep aspects concerning basic notions (in particular, in our case, the structure of the decimal – position system of writing of numbers: see Miura et al., 1994). Thus, the aim of developing fundamental mathematical competencies may be integrated with the aim of promoting cultural identity (intended as conscious take in charge of personal or group past in the perspective of the future in a given cultural context – see (Weinrich, 2003); see also (Holliday, 2010), for the complexity of the problem of identity in a country like Algeria).

I have intentionally used the English word “may” both to put into evidence what frequently happens in post-colonial countries, and what might happen if the potential inherent in cultural differences (in our case, we deal with differences between spoken numbers in different languages in Algeria) would be exploited by the teachers. The project, whose first steps are presented here, aims at providing teachers with knowledge and awareness about the subject and the historical origins and evolution of differences, and also about how to deal with those differences in the classroom. In this paper I will try to briefly report some preliminary work done by me, concerning:

- Some elements of the performed analysis of the differences (rooted in the evolution of the cultural and political situation of my country) concerning the relationships between spoken numbers and written numbers in the main languages spoken in Algeria in different social and institutional contexts;
- Some present difficulties originating in those differences in school;
- Some elements for an intervention study, in which the performed analysis should be used in teacher preparation to enable them to plan and manage teaching experiments aimed at promoting basic numeric competencies and an approach to the development of students' cultural identity with reference to this specific area. In the future, further analyses concerning the first two points will be performed, based on collection of oral data and interviews with teachers in different regions of Algeria.

Historical overview

In spite of being considered as an Arabic country and one of the Maghreb countries (with Tunisia and Morocco) by the political world map, Algeria is a multicultural country. It hosted many civilisations that made its cultural richness. Over years, the history has been shaped there by forces whose roots were in Africa, the Mediterranean region (including the South of Europe), and the Orient. Algeria lived a long period, from 1830 to 1962, under the French colonization. Before that, it was under the Othman occupation from 1515 to 1830. Othman arrived to North Africa to help fighting the Spanish who occupied some cost areas (now Oran, Bejaia and Algiers) till 1555. Many years earlier, around 665, the Arabs came from the East to spread the Islamic religion and power in North Africa, which was populated by Berber people. Arabs stayed there ever since.

Linguistic situation

Algerian population communicate in different languages, according to different regions, ethnic groups, institutions and circumstances. The most spread languages are: Classical Arabic, Dialect, Berber language and French (first foreign language). Classical Arabic is shared by the Arabic countries; the Algerian Dialect, different from the one spoken in other Arabic countries, sounds more similar to the Dialects used in both Tunisia and Morocco.

Classical Arabic

Classical Arabic, the official language, is not the usually spoken language by the Algerian population. It is used only in written and spoken official discourse in media (newspapers, TV news), in books, in information written on all purchased packaging products (food packets, medicines, information on how to use any product…), to write road signs, stores signs, and all written announcements that might appear in the streets. Classical Arabic is the language of the Arabic literature and the language of the Koran (religious Islamic book) as well. This language has an alphabet of 28 letters; it’s written, unlike Latin languages, from right to left. Classical Arabic was not taught during the colonisation period in schools, it was taught in mosques or in small religious schools (zawia). During that period, most of the population was illiterate; the few pupils who could go to school learned all courses in French. We have still some old people with high degrees not able to write and read Arabic. After the independence, for almost ten years the teaching in schools continued to be in French; however classical Arabic was introduced in school as a whole course and as the only one taught in Arabic. This situation lasted till the beginning of the seventies. Then the teaching in schools for all courses shifted to Arabic. At the university, all human sciences are taught in Arabic, while the other sciences (medicine, exact sciences, computer sciences, etc.) are taught in French. People do not master well Arabic even if they have been in school for a long time; it is considered like a foreign language because it is spoken nowhere outside schools, but people understand it and can read it. Most children discover this language for the first time when they go to school, differently from many western countries where children learn at school the same language they speak at home.

Berber language

Berber language (called also Tamazight language) is the language of Berber people who lived in North Africa (including Morocco, Tunisia, one part of Egypt, the grand desert, Mauritania) before the Arabs' arrival. After the Arabs' invasion (665), it has always been spoken by different groups composing Berber population (Kabyle, Chawi, Chenoui, Mzab, Touareg…) since that time so far. Nowadays, in some areas that couldn't be reached by Arabs during the invasion, like in Kabyle area populated only by Kabyle people, it is spoken everywhere. In other areas, where people usually communicate in Dialect, it is spoken only at home. The conflict due to not considering Berber language as a national language by the Algerian government became apparent after the independence, especially during the seventies. The conflict resulted, in particular, in the "bag strike" undertaken by all schools and universities in Kabyle areas: they stopped teaching during one complete year (1994/1995). After that, the government decided to acknowledge Berber language as a national language in Algeria but not as an official one. Now, it is taught in many schools (in parallel with Arabic) and universities, and even many books are edited in Berber language. Its writing system (from left to write), whose origin may be traced back to the third century B.C., has been conserved by the Touareg (people of the desert): it is called Tifinagh, it has 32 letters including vowels.

Dialect

The Dialect is the common language; it is the most used spoken language in daily life in Algeria; it's a mixture of classical Arabic, French and other languages (Spanish, Turkish, Italian…). The Dialect is continuously developing by using new words and dropping some others over years. The use of imported expressions and words has no clear and defined rules; it's common to find a misuse of some words and/or a strange change of a pronunciation. Nowadays, we can see an increasing use of French words in the Dialect, which are pronounced in an Arabic way, whereas the number of individuals mastering French language has strongly decreased over the last ten years. The Dialect is slightly different in the different areas of Algeria (difference of pronunciation, accent and some different words and expressions). In the school, the teaching is done in classical Arabic for all levels, but the Dialect is more and more dominant as an oral language even in the classroom at all levels. Over the last years, this spoken language is getting more and more space even as a written language (written with Arabic or French letters), pushing back the classical Arabic. Some newspapers use it; all TV shows and radio programmes are made in this language except the news; more and more advertisements are made in Dialect; thus, this language is becoming the principal and almost official language in Algeria.

French language

French is the first foreign language, it is taught starting from grade 3 to grade 12. Some disciplines, like exact and technology sciences, are taught at the universities in French, with French books. After the independence, many institutions that already used French, like administration and hospitals, continued to use it. During the seventies, the government undertook an 'arabisation' action to use Arabic language instead of French in all institutions. Some institutions changed their language (e.g. schools), but some other resisted: most administration, banks and hospitals are still working in French, particularly as their written language. In other words, French is a parallel language with classical Arabic; this contributes to the influence of French in the Dialect, especially through administration documents. There are a radio channel, a TV channel and many newspapers in French. Many inscriptions are written both in Arabic and French (e.g. on medicines, panels on roads, streets advertisements, and even purchased imported products). People generally understand French but do not speak it very well (exception in Algiers and in Kabyle areas). The mastery of French has decreased even among people with high degrees graduating in disciplines that have been taught in French at the university.

Spoken and written arithmetic in Algeria, and related students' difficulties

Around ten years ago, there was a big change in teaching mathematics: formulas and symbols are now written from left to right with Latin alphabet, while comments and names are maintained in classical Arabic (from right to left); in the past, mathematics was completely written from right to left (in Arabic). This reform was undertaken to make the transition to university mathematics (taught in French) less difficult. This situation may result in a mess and some difficulties for students. For instance, teachers usually complain about a recurrent difficulty with children when they deal with operations with negative numbers: 6-1 is 5, but it is -5 if it is read from right to left. Other difficulties are worthwhile careful investigation. Here, I will summarise the origins of some of those difficulties according to the differences between the different languages, as concerns the relationships between spoken and written numbers.

- In Arabic: right to left writing of numbers, with a complete correspondence with oral. Numbers (from 11 to 99) are pronounced like they are written from right to left starting by the units and then the tens. But after 99, with hundreds and thousands and more, the system becomes mixed, for instance 234 is pronounced two hundred four and thirty.

In Berber: left to right writing of numbers, with a contradiction with oral that is similar to Arabic.

In French (written-oral): left to right writing of numbers, with a contradiction with the oral for 11 to 16, where we begin by the units and then the tens; and with the "by twenty" oral traces between 60 and 99 (80 is spoken as 4 twenties, 90 is spoken 4 twenties and ten).

In Dialect (oral): the same with Arabic.

- Names of numbers are used in classical Arabic, which are the same as in the Dialect (with a slight change of pronunciation). In Berber language, some digits (from 0 to 9) are different but the other numbers are the same as in Arabic (with a slight change of pronunciation). The French names for numbers are also frequently used in the spoken language.

In French and in Italian, the numbers 11, 12 … 16 are named like in the Arabic system (starting from the units), while for the numbers from 17 on the tens are spoken before the units - which is not the case in Arabic. This might create a problem for children when they learn French (starting at grade 3), similar to the problem met by Italian children when they have to write "tredici" (which sounds like "three-tens") and "diciotto" (which sounds like "ten-eights").

- Names of the arithmetic objects "number" and "digit" are clearly different in both classical Arabic, where we have two words, in French where there are three words (chiffre*, nombre and numéro), but in the Dialect there is just one word for both number and digit, which is the French word (numéro) pronounced in an Arabic way ("numro"). Same situation is in Berber language. More precisely:

- In classical Arabic, 'Adad' means number and 'Rakm' means digit, but also a set of digits used to distinguish a person or an object, an address, etc.
- In French, 'nombre' means number, 'chiffre' means digit and 'numéro' means a set of digits used to distinguish a car number, card number, address…;
- In the Dialect, there is just one word for all; it is the French word 'numéro'.
- In Tamazight, it is the same situation as with the Dialect.

This might create a problem in mathematics and influence students who might not make a difference between number and digit, even though they learn meta-distinction between them in classical Arabic. But as the Dialect is the spoken language, students do not practice this difference outside schools. They might still use one word (number or digit) for both terms with classical Arabic and still make confusions with the three French ones.

Difficulties as educational resources

Usually teachers see the above linguistic phenomena at the origin of big difficulties in the early teaching of arithmetic, and they are right. In the early stages of arithmetic learning, contradictory rules for saying and writing numbers may represent an obstacle and originate disaffection towards mathematics. In the case of Algeria the situation becomes more complex, due to the interference of different criteria of organisation of spoken numbers according to the different languages (particularly in the relationship between French on one side, and Arabic, or Tamazight, and Dialect on the other). In the case of Algeria only parents who manage, not only Dialect, but rather well French and Arabic, may help their children to clarify the differences. But, those difficulties may result in important occasions of students' reflection on basic arithmetic notions: by comparing the French wording of numbers between 11 and 16 with the Arabic or Tamazight (or Dialect), students may better realise the anomalous behavior of the French spoken language. By analysing the structure of French spoken numbers between 70 and 100, they may engage in stimulating exercises of conversion. By comparing the transformation of the naming of the word "numéro" from French to dialect, and the fact that the Arabic concentrates in one word the French "chiffre" and "numéro". Students (but in this case even teachers!) may reflect on the very meaning of three different notions, so relevant in the mastery of natural numbers: digit (the basic signs, like the alphabet of the written arithmetic); ordered sets of digits (as "words" of that written language); and numbers (concepts - like the meaning of words). By considering traditional units of measurement they may reflect on the meaning of related fractions. The learning potential on the mathematical side has a counterpart on the more general educational side: students may realise, under the guide of the teacher, how the French system of spoken numbers reflects some cultural influences coming from other cultures, or ancient traditions (the traces of numeration by twenty). Students may come in touch with easy to understand examples of how mathematics is not a given, out-of-history construction, but an evolving body of knowledge, a "culture" in the anthropological sense. More importantly, every student (independently from her social and cultural origin) may come in touch with important phenomena: the contamination between different cultures; the effects of political domination on culture; but also the survival of dominated cultures, when people resist. These discourses may contribute to a first approach to the development of personal and group identity, given that identity depends on the conscious take in change of the cultural self in a given cultural context.

Educational implications for teachers and students: an outline

Teacher education and development of primary school teachers in Algeria is poor. Due to the increasing demand for teachers, most in-service teachers are not well prepared; a few of them could profit from a period of one to two years in some centers for teacher education, but not at the university. In-service training takes place for some schools few times in a semester, while for other schools it does not exist. The opportunities mentioned before need a suitable teachers' preparation: on the content to be taught (with clear ideas on the technical aspect and differences, and on their historical origins and evolution); on how to use these differences to enhance mastery of the numerical system instead of generating confusions; and on the cultural and educational aspects (concerning the values inherent in acknowledging diversity and contaminations, in the perspective of a plurality of individual and group identities in a multicultural society). Literature provides us with useful tools that should guide teacher preparation and at least partly be shared by teachers themselves to inform their educational intervention. As an example, the work done by Carraher, Nunes and Schliemann on the relationships between street mathematics and school mathematics offers ideas and tools about what to look at, when children from different cultural environments are exposed to the unique culture of the school. The perspective of expansive learning delineated by Engestrom and Sannino and some educational developments derived from it (e. g. Tomaz, 2013) highlight the mediating role of the teacher, who should provide students with suitable knowledge to enrich their cultural horizons. In our case, knowledge should derive from the structured comparison, rooted in the history and in the present social reality, of different wordings of numbers and (later) of meta-knowledge of numbers (digits, numbers, etc.) accessible in the cultural environment. Knowledge should contribute to the mastery of the decimal – position system and contribute to the cultural identity of students (see also Adler, 1997). The project, whose first steps are presented in this paper, should result in some pilot activities of teacher preparation informed by the above considerations with classroom experiments inspired by them.

Notes

* The word 'chiffre' that is digit in French is originated from the Arabic name of zero 'siffre'.

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MATHEMATICAL FOUNDATIONS FOR PLACE VALUE THROUGHOUT ONE CENTURY OF TEACHING IN FRANCE

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Abstract

This paper aims to evidence the existence and the impact of mathematical foundations on WNA teaching, from curricula design to teaching practices, and students' learning. An assumption is that, as its name indicates it, the *new math* is a key period to identify changes in mathematical foundations of WNA teaching. This period is replaced in a longer one. The paper studies the changes related to place value in France: a deeply impacted subject with the introduction of numeration bases other than ten in 1970, and their removal in the 1980s. What the author named *numeration's units* -that is: ones, tens, hundreds, thousands…- appears as a key tool to describe and understand changes in the teaching and learning.

Key words: didactic transposition, France, mathematical foundation, new math reform, numeration's unit, place value

Introduction

This paper aims to evidence the existence and the impact of mathematical foundation (MF) – as understood by the current community of mathematicians – on WNA teaching, from curricula design, to teaching practices, and students' learning.

The new math reform is an international phenomenon which took place in the cold war context (ICMI, 2008; Kilpatrick, 2012). It impacted on the math teaching from the primary school to the university in the 1960s and the early 1970s in several countries; but it has had effects a long time after its end. One of its major concerns was the teaching of some "new" math. This enables to make the assumption that this period may be a key one to identify possible changes in MF. Another concern was to take into account some psychological features related to the learning or to the child development. The famous psychologists Piaget and Bruner contributed at least indirectly but significantly to the implementation of this reform. Two famous subjects were introduced in WNA: set theory, a trace of contemporary math, and the numeration bases other than ten (hereafter called bases) –in order children to understand base ten principles-, a trace of psychology (Kilpatrick, 2012; Bruner, 1966).

Much has been written about MF concerning fractions, taking into account the new math reform period (Steiner, 1969; in Germany: Griesel, 2007; in France: Chambris, 2008), but few about that of decimal numeration. The aim of this paper is to study such MF, surrounding the new math. The case of France will be studied: bases were introduced in 1970 and removed in the 1980s. A main goal of this study is to identify the effects, and even the potential long-term effects, on the teaching of possible changes in the MF. This leads to study MF: before, during, and after the new math. Several types of data from different periods are collected: national syllabi, resources for teacher education, textbooks. For the present period, this set is completed with data from classrooms: students' written test, results from others' research works related to teaching practices and students' learning.

Materials and Methods

How to study the existence and the impact of the MF for decimal numeration on the teaching and learning in France surrounding the new math?

Previous works

Several doctoral dissertations (Harlé, Bronner, Neyret) about numbers' or arithmetic's teaching in the French system highlighted three main periods with two transitional ones before and after the new math: the stable classical period 1870-1950, the new math highly turbulent period 1970-1980, and the continuously evolving contemporary period 1995-2010. So, this study explores the long period: 1900-2014. Treaties by Bezout, then Reynaud (1821) appear as reference books for teacher education, and for textbooks, at the beginning of 20th century. The textbooks' series "math et calcul" was a bestseller in the 1980s.

Theoretical frame

The Theory of Didactic Transposition (TDT) (Chevallard, 1985) (Fig. 1) considers school mathematics as a reconstruction by the educational institutions from the mathematical knowledge produced by academic scholars. The TDT has been often used for secondary school, more rarely for primary school where scholarly knowledge as a reference is not always taken for granted.

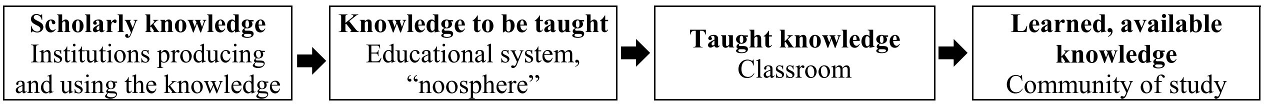

Fig. 1: The process of didactic transposition (Bosch and Gascon, 2006)

The Anthropological Theory of Didactics (ATD) (*ibid.*) extends the TDT. It postulates that practicing math, as any human practice, can be described with the model of *praxeology*. It is constituted by four pieces: a *type of tasks* -a set of similar problems-, a *technique* -a "way of doing" for all the tasks of the type-, a *technology* justifies the technique, and a *theory* legitimates the technology.

The numeration's units

The author (2008) named the units used in numeration: *numeration's units*, NU. That is: the *words* ones, tens, hundreds -O, T, H in the text-, and so on for base ten. It happens that students consider NUs as places (Fuson, 1990; Ma, 1999).

A few French cultural and institutional features

Unlike most East-Asian languages many European languages have strong irregularities in the numbers' names for whole numbers (Fuson, 1990). In English, numbers' names are those of the units, not in French. (Tab. 1) This implies that, in France, the names of the 2nd, 3rd, and 4th decimal NUs for whole numbers are not used in the everyday life language.

French number's name	French NU	English name for both
un (une)	unité	one
dix	dizaine	ten
cent	centaine	hundred
mille	millier	thousand

Tab. 1: Numbers' names and NUs in French and in English

In France, there is a national elementary compulsory syllabus, but no national textbook, and no accreditation of textbooks. The syllabi changed in 1882, 1923, 1945, 1970, 1977-1980, 1985, 1995, 2002, and 2008. Instructions -more or less long texts describing how to teach a subject- have supported some of them.

Data: how to select curricular texts? What to look at in there?

In the French syllabus, roughly, students learn numbers up to one hundred in the 1st grade, to one thousand in the 2nd grade, to ten thousand or one million -depending on the ongoing curriculum- in the 3rd grade. Bigger whole numbers are learned in 4th and 5th grades. The selected data are: the five first grades of the national syllabus and instructions -if any-, and 2nd-and-3rd-grade-textbooks.

Period	2nd grade	3rd grade	Mixed (2nd-3rd-gr.)	Teacher's guide
1900-1970	7	7	6	1
1970-1980	3	3		6
1980-2010		14		12

Tab. 2: Number and type of selected books for the research

A big amount of textbooks exists in France. How to select the most used ones, and the most influential ones in the curriculum's evolution? Ones with several editions and innovative ones are selected. During the new math, many textbooks were published; "math et calcul" is one of the few with further editions. (Tab. 2) Treaties by Bezout and Reynaud, 2nd-and-3rd-grade-teachers' guides by ERMEL from two bestseller-series (old in 1978, new in 1995) are added. ERMEL is the elementary math research team of the national institute of pedagogy. Once a book is selected, what to look at in there? With the ATD, for each period, it is necessary to identify the ongoing praxeologies in order to describe the teaching. The tables of contents of textbooks, and the syllabus are useful to identify the global development of the subject, a part of its logical organization. Instructions and numeration's pages of textbooks give more details. The sentences related to the knowledge to be known are often technologies or parts of the theory. Exercise with model answer is a strong indicator of the knowledge to be taught: such an exercise is often emblematic of a type of task, and the answer shows the technique which is expected. The answer sometimes includes technological features. The set of exercises is useful to identify the types of tasks which exist in an institution at a given moment.

Data: students' knowledge

In early 2012, all 215 students from nine 3rd-to-5th-grade-classes answered 9 exercises in individual written way. Questions were given one after the other. Time was supposed to be enough to answer quietly.

Results

The scholarly knowledge before the reform – the classical theory

During the long period 1900-1970 with hazy changes in some books from 1950, a mathematical numeration's theory -here: *the classical theory*- was displayed in the treaties and adapted in close terms in textbooks. Its basics are the following.

Algorithm to build numbers: 1) The first ten numbers are built one after another, starting with the unit one, and then adding one to the previous number, forming numbers one, two, etc. 2) The set of ten ones forms a new order of units: the ten. 3) The tens are numbered like the ones were numbered before; from one ten to ten tens: one ten, two tens… 4) Then the first nine numbers are added to the nine first tens: one ten, one ten and one one, one ten and two ones… two tens, two tens and one one, and so on, forming the ninety-nine first numbers. 5) The set of ten tens forms a new order of units: the hundred. 6) The hundreds are numbered like the tens, and the ones were; from one hundred to ten hundreds: one hundred, two hundreds… 7) Then the first ninety-nine numbers are added to the nine first hundreds forming the nine-hundreds-ninety-nine first numbers…

Rules for numbers' names: Meanwhile the building of numbers, numbers' names are presented like a literal translation. Rules are stated with exceptions to them. Though it is paradoxical, but in order to lighten this text, exceptions are taken from the English language. After the 3rd step, the correspondence between tens names (ten, twenty…) and the amount of tens (one ten, two tens…) is stated. Then, after the 4th step, the rule to form the numbers' names between two tens is stated. "Say the tens then the ones": as *three tens* is *thirty* and *four ones* is *four*, then *three tens* and *four ones* is *thirty-four*. Then a list of exceptions is given: for instance the usual number's name of *ten-one* is *eleven*.

Positional notation: After building the numbers, the positional notation is stated. To write numbers without writing the units names, it is sufficient to juxtapose the amount of units of each order, the ones on the right side, then each place represents a unit which is ten times bigger as the nearest on its right. Places which are not represented are marked with the sign 0.

Conversely to the 1st and 3rd paragraph, the 2nd one -rules for numbers' names- is useless with regular numbers' names. Due to a cultural difference, this may play a major role in differences in the MF throughout the world.

Changes in the scholarly knowledge: what and when?

During the new math, many things occur due to the bases. Depending on the resource, more or less changes exist in the MF which is used. 1) The classical theory may be adapted for bases: as there is no ten, no hundred in bases, units' names become "unit of the first order", "unit of the second order", and so on. NUs reappear shortly while studying base ten. 2) Generally base-ten-units' names are present in relation with numbers' names, with the classical correspondence. 3) The polynomial decomposition of a whole number n in a given base r -that is: $n = \sum a_i r^i$ $(0 \leq a_i < r)$- always appears while converting from base ten to another one or conversely which is a new type of task of the period. Sometimes the existence and uniqueness of the decomposition is proved in a rather formal way. The Euclidian division is then used. 4) In some teachers' guide, the polynomial decomposition is used to substantiate the column algorithms. In the other ones, the classical theory is used.

From the 1980s, the classical theory disappeared from the textbooks and books for teacher education. In the early 1980s, the theorem for the polynomial decomposition explicitly stands in books. The positional notation is then the juxtaposition of the coefficients of the polynomial, here *the academic theory*. Nowadays, most often, the decomposition remains implicit in teacher's guide.

The arguments used to introduce bases in the new math curriculum were from psychology. And yet, it looks like *after* the turbulent period of the reform, during the so-called "counter-reform", the "new" mathematics –here represented by the academic theory which is in fact not so new– took place as MF for the teaching of place value.

The next step is to understand what kinds of effects have such a change in the teaching, if any. A few examples are selected hereafter.

What role for NUs today?

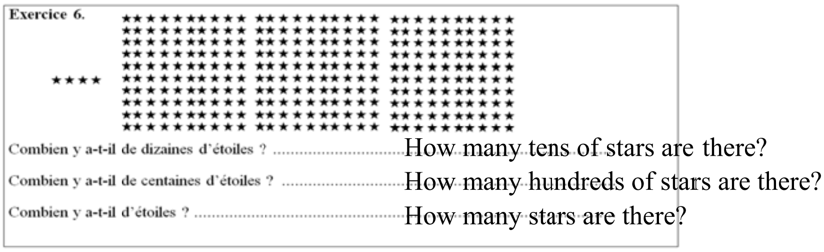

Fig. 2: Tasks developed to number tens and hundreds

As strange as it sounds, it is frequent not to find the relation 1 hundred = 10 tens in present 2nd-and-3rd-grade-textbooks. Might it have to do with the fact that there is no need for the NUs in the academic theory? This never happened before the reform. To convert NU was an ordinary task in the classical period. None of the present textbooks selected displayed conversions' tasks. Tempier (2013) observed three 3rd-grade-teachers, only one of them explicitly calls upon the relations between units; in contrast all of them use the unit's names to describe places. He also investigates mid-3rd-graders' knowledge about conversions (n = 104): 48% succeed to fulfil 1 H = …T, and 31% to fulfil 60 T = …H. None of the observed teachers taught conversions. In 2012, 62% of mid-3rd-to-5th-graders (n = 215) were able to number 304 stars, whereas 35% were able to number tens, and 2% answered 0 tens; 37% were able to number hundreds of stars (Fig. 2).

It is true NUs are few used in French. This shows that NUs are not at the core of the teaching of place value in France: from MF, to curricular content, teachers' practices, and student's knowledge. It sounds contradictory with a deep understanding of place value (Ma, 1999).

How is place value taught with the polynomial decomposition? What do students know about place value? It is useful to have an overview of what happened from the reform before to study these questions.

The appearance and the role of "powers-of-ten written in figures" or: How to teach place value at early grades with the polynomial decomposition?

Two very different texts of the same period evoke nearly the same consequences of the bases in classrooms on students' knowledge. Perret (1985) studies place value within the reform in Switzerland. ERMEL (1978) will be a major source of the French counter-reformed curriculum. Both write basically that students interpret the positional notation as a procedure: grouping, ungrouping. They struggle as soon as they have no manipulatives left, ERMEL adds.

A brief analysis of reform's praxeologies shows that the old numeration's tasks: that is conversions and decompositions –both using the symbolic register of the NU- are no longer taught. It may be due to the lack of most of the units' names with bases other than ten. Whatever, they disappeared as the "bases" appeared.

A major preoccupation of the counter-reformers was to reinsert a work within a symbolic register, so-called the "writings". A huge process of transposition of the academic theory happened; it had been largely achieved or relayed by (ERMEL, 1978); and no change is visible in MF after the counter-reform. In the part devoted to numeration, there is no NU in (ERMEL, 1978), except once in order to evoke rapidly the numbers' name. The process is the following: $\sum r_i a^i$ becomes $\sum r_i 10^i$, then a x 1000 + b x 100 + c x 10 + d or a000 + b00 + c0 + d. An interpretation is the following: as powers of ten's positional notation requires 0 and 1 as multiplicative coefficients which is a predictable difficulty for early graders, a solution seems to be that 10, 100… are given.

Around 1980, "writings"' tasks appear. Among the various "writings", the "powers-of-ten written in figures" (PTWF, named by the author) -1; 10; 100; 1000…- played an increasing role. Within a few years, it looks like the old tasks in NUs would have been translated in PTWF, and replaced by those translations. For instance, "Write in figures: 2 H 4 T 5 O" became: "Compute 2 x 100 + 4 x 10 + 5" or "Compute 200 + 40 + 5". "Convert 4 hundreds in tens" could have reappeared this way: "Fulfil 4 x 100 = … x 10". It didn't reappear.

A major question is: how to get the positional notation with the transposition of the academic theory, especially when a number is given with PTWF? In the academic theory, the juxtaposition of the polynomial coefficients leads to it. From 2x100+4x10+5x1 -a multiplicative decomposition in PTWF- to get 245 -positional notation-, instead of the rule "juxtapose 2, 4, 5" in the academic theory, the present implicit rule seems to use positional rules: 1) To multiply by

100 (10): write two (one) zero on the right, 2) To add numbers: use a “column algorithm”. That is: put them one under the other, aligning from the right side. After a likely transitory period, present textbooks may look like Fig. 3.

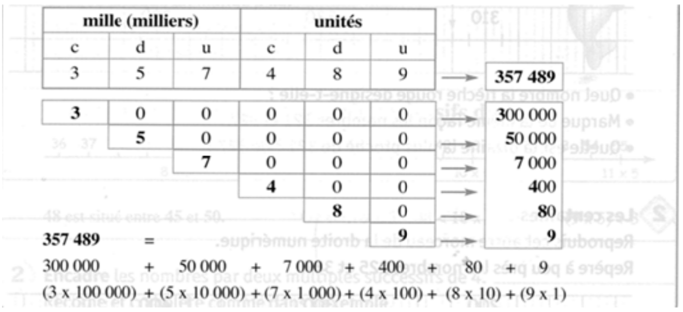

Fig. 3: 3rd-grade-textbook. Nouveau Math Elem, Belin, 2002, p.133

From 1995, decompositions with NU came back. Nevertheless, the way to achieve them seems to be: 3 H = 300 (due to the “hundred’s place”), 4 T = 40, 2 O = 2; then compute 300 + 40 + 2; that is: 342. NUs exist in order to indicate digits’ places, not as units: the only unit “to be taught” is the number “1”.

What does this become in teachers’ practices and in students’ knowledge? Further research is needed. Nevertheless, though there is no causal link proven, several research show that present French 3rd graders fail to solve problems like “how many packs of 10 stamps for 89 envelops?” A recent one evidenced a big decrease between 1999 and 2013.

In the counter-reform, new praxeologies are elaborated, and the old ones disappear more or less rapidly. Hereafter there is an evolution of the “to be taught” praxeologies: this may be partly due to an adaptation to the teaching and learning process, and has effects in taught and learned praxeologies.

Discussion and conclusion

This text highlights MFs for place value in the French curriculum. Important changes in these MFs occurred during the counter-reform which followed the new math. The *classical numeration theory with units* disappeared in favor of the *polynomial decomposition with the exponential notation*. The effects of the changes were probably not anticipated, neither mastered. As units, in the textbooks, NUs -tens, hundreds…- disappeared from the curriculum in the 1970s; they may remain as places’ names. As units, they have been replaced by PTWF -1, 10, 100…-. The nature of what a *mathematical foundation* is should be further investigated. Whatever, these MFs influence clearly the curriculum until nowadays: the number “1” seems to be the only unit to be taught and learned. Is there a lagged impact on fractions’ understanding which requires units (Lamon, 1996)? Does the lack of NU impact on the understanding of place value? In France, understanding of the positional notation seems to be low, but further research is needed to relate it to the lack of NU in the teaching.

Thanheiser (2009) highlights two correct conceptions with US students: 1) with NU, 2) with “writings”. Are there links with the local MF?

How is it abroad? It would be an interesting challenge to identify what part, if any, of the French situation can be transferred in other countries whether they were implied or not in the reform. Is the *classical theory* taught somewhere?

Is it important that teachers know about such phenomena? What would be useful for effective teaching? An interesting feature is to know that teaching subjects are culturally influenced (ICMI 2000). Tempier (2013) aims to reinsert NUs. Teachers change several things in their teaching but they were not able to teach NUs as units. Might it have to do with their own MF, if any? It may be useful that teacher educators know that several math theories may influence teachers' practices -even their own practices as teacher or educator-. As MFs influence even the tasks to be taught, knowing these phenomena seems to be important for curricular developers.

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FIRST GRADE STUDENTS' MATHEMATICAL IDEAS OF ADDITION IN THE CONTEXT OF LESSON STUDY AND OPEN APPROACH

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Abstract

The purpose of this study was to explore the first grade students' mathematical ideas of addition. Target group was 22 first grade students from two schools in 2013 academic year. These two schools have participated in teacher professional development program base on Lesson Study and Open Approach since 2007 and 2009. The processes of this study were: 1. Collaboratively design lesson plan focusing on open-ended problem situations and students' ideas 2. Collaborative, classroom observations focus on four steps of Open Approach and students' ideas 3. Collaborative, reflection on teaching practice focus on students' ideas during the lesson. Data were collected by using 3 lesson plans in the unit of addition, collecting students' written work and video recording during the lessons.

The results found that students' mathematical ideas have been expressed in a variety of representations such as 'together' in the form of a block diagram, increasing by counting one by one on a picture, number sentence, place value and base ten by using a bar chart, composition and decomposition by using a line diagram.

Key words: addition, lesson study, mathematical ideas, open approach

Introduction

Young children have quite different conceptions of addition, subtraction, multiplication and division than adults do. Their conceptions make a great deal of sense and they provide a basis for learning basic mathematical concepts and skills with understanding (Carpenter et al., 1999). As children begin to learn mathematics in elementary school, much of their number activity is designed to help them become proficient with single-digit arithmetic, namely mastery of the sums and products of single-digit numbers and their companion differences and quotients (Verschaffel, Greer and De Corte, 2007). In the earliest grades, children learn to write and manipulate numerals and operation signs (Goldin, 1998). Fuson (1992) claimed that the focus of learning to add and subtract should change from one of children rapidly producing accurate solutions to pages of stereotypical numeral problems to one of children discussing in the classroom alternative solution procedures for a variety of addition and subtraction situations. There are models for children's informal knowledge of counting principles and informal counting strategies and their development into more formal and abstract arithmetic notions and procedures (Beckmann, 2014).

A particular mathematical idea can often be represented in various forms (Hiebert and Carpenter, 1992). The way in which mathematical ideas are represented is fundamental to how people can understand and use these ideas (NCTM, 2000). Because mathematical thinking can be observed mostly in the process of students' problem solving and dialogues among students, the entire process of lesson study is expected to improve mathematical thinking

(Takahashi, 2007). Lesson study attracted the attention of an international audience in the past decade, and in 2002 it was one of the foci for the Ninth ICME (Murata, 2011). Using lesson study as a tool, teachers will be trained to focus on student thinking and questions and investigate how the thinking impacts the learning outcome (Lewis and Hurd, 2011). Murata (2011) claimed that the emphasis on student learning in the lesson study process continually reminds teachers how important it is for them to understand students' ideas and helps bring the visions of reform into their classrooms.

In Thailand, Thai elementary mathematics textbooks consist mainly of routine exercises (Inprasitha, 1997). Teaching and learning focus on remembering; most teachers' teaching role begins by explaining new subject content to the students (Kaewdang, 2000; Inprasitha, 2011). Since 2006, the Center for Research in Mathematics Education (CRME) has been implementing Japanese lesson study incorporating four steps of Open Approach as a teaching approach within the three phases of the lesson study processes (Inprasitha, 2011). In this study, collaboration in each phase of lesson study was comprised of school teachers and the 5th year undergraduate students who are doing their one year teaching practice at school. The project schools have used a Japanese mathematics textbook translated and edited in Thai by Inprasitha et al. (2010). Japanese textbooks introduce the activity of decomposing a number into two numbers of the same amount and of composing two numbers into a number before introducing addition and subtraction. The manipulative activity of composition and decomposition of numbers are a key activity to develop the children to operate on numbers without counting one by one (Hattori, 2010). Thus, to understand students' learning in this context, this study was to explore first grade students' mathematical ideas of addition.

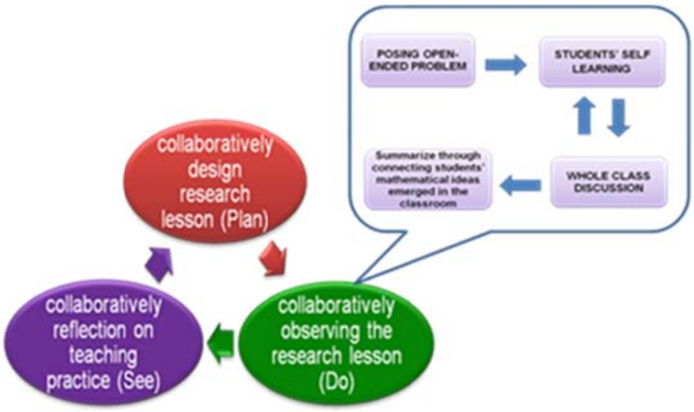

Fig. 1: Teaching practices based on Lesson Study incorporating Open Approach (Inprasitha, 2010; 2011)

Materials and Method

Target group was 22 first grade students from two schools. Each class was taught by a student intern who study in the Mathematics Education Programme, Faculty of Education, Khon Kaen University and had teaching practice for one year at these schools. These two schools have participated in teacher professional development programme in the context of Lesson Study and Open Approach which CRME, Khon Kaen University has been in charge since 2007 and 2009. The weekly process followed Lesson Study and Open Approach (Inprasitha, 2011) presented as showed in Fig. 1.

Data were collected in 2013 academic year by using 3 lesson plans in the unit of addition, collecting students' written work and video recording in three phases of lesson study processes as follows.

Collaboratively design a research lesson: Teachers tried to apply the material and content to be taught as open-ended problems. A first grade Japanese mathematics textbook translated and edited in Thai (Inprasitha et al., 2010) was a major textbook for designing the lesson.

Collaboratively observing the research lesson: During the lesson, teachers taught using four steps of Open Approach by planning with their team as follows: 1) Teachers posed open-ended problem to students. 2) Students were divided into groups and worked to solve a problem by themselves. Teachers observed each group and recorded students' ideas. 3) Each group of students presented their ideas to others. Teacher tried to stimulate students to explain their ideas. 4) Teachers summarized the lesson by connecting students' ideas. During this phase, we used video to record the lessons.

Collaboratively reflecting on teaching practice: Teachers and observing teachers participated in this process. The reflection focused on students' activity and interaction and on students' ideas. Data analysis based on the mathematical concept behind the textbook from the first grade mathematics textbook glossary (Inprasitha and Isoda, 2014).

Results

First grade students' mathematical ideas have been expressed in a variety of representations such as 'together' in the form of a block diagram, increasing by counting one by one on a picture, number sentence, place value and base ten by using a bar chart, and decomposition by using a line diagram as in the following.

Problem Situation 1

Teachers posed problem situation 1 to students by using picture as Fig. 2 and told story to students that "9 children are playing in a sandbox and 4 children are playing on a slide. How many children are there in all?"

Fig. 2: Problem Situation 1 (Japanese Mathematics Textbook translated and edited in Thai by Inprasitha et al., 2010)

According to problem situation 1 (Fig. 2), during the lesson, students could express their mathematical ideas as follows.

1. Students' mathematical idea of 'together' (85.1%)

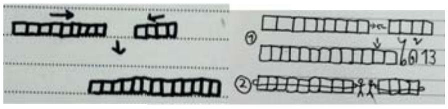

Fig. 3: Students' mathematical idea of together in form of block diagram

Students used 9 block diagrams representing the number of children who played in the sandbox and used 4 block diagrams representing children who played at the slide. Later, students drew the facing arrows that meant moving to meet each other or combined together to have the new number that was 13 block diagrams. Students found the answer was 13 children by using the idea of 'together' in form of block diagram as Fig. 3.

2. Students' mathematical ideas of 'increasing' (100%)

Fig. 4: Students' mathematical ideas of 'increasing' by counting one by one on picture

Students could interpret the problem as addition, thus, they used the idea of 'increasing' by counting and drawing a number on every child in the picture as Fig. 4.

3. Students' mathematical idea of 'number sentence' (100%)

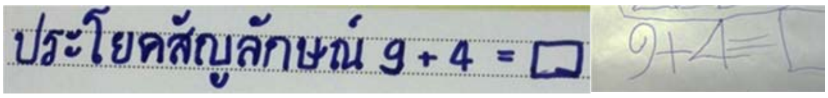

Fig. 5: Students' mathematical idea of number sentence

Students have more sophisticated mathematical idea in terms of 'number sentence'. They knew that they had to combine the number of children using addition and used plus sign (+) in a number sentence as in Fig. 5.

4. Students' mathematical idea of 'place value and base ten' (28.57%)

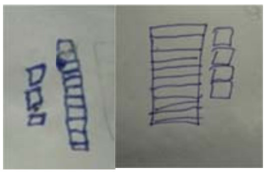

Fig. 6: Students' mathematical idea of 'place value and base ten'

On step of students learning by themselves, they drew blocks and a block track representing a bar chart of the amount of 13 (Fig. 6). But no group explained why they wrote it like that.

In the whole class discussion, the teacher asked students to move 9 blocks like students drew in the work sheet. Students explained more while they were moving blocks such as "We had to move one of those blocks, any of them, just one block (point at 4 blocks and move 1 of 4 to block group of 9). Fill the block to ten, then you know it is 13." Students have more flexible mathematical ideas relevant to 'place value and base ten' by using a bar chart.

Problem Situation 2

Talk about how to calculate 8 + 3

Fig. 7: Problem Situation 2 (Japanese Mathematics Textbook translated and edited in Thai by Inprasitha et al., 2010)

1. Students' mathematical idea of 'together' in form of block diagram (71.43%)

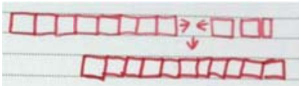

Fig. 8: Using arrow to combine

Students used blocks diagram representing 8 and 3 then combined two numbers together by using arrows, the answer was 11 (Fig. 8).

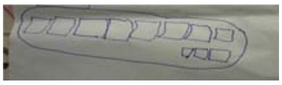

Fig. 9: Using circle to combine

This kind of drawing uses a block diagram and a drawn circle to encircle all blocks to represent the meaning of addition (combination) as Fig. 9.

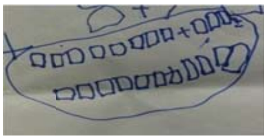

Fig. 10: Using plus sign

Drawing blocks with a plus sign in the middle of two groups of block means combining them together. It was similar to a symbolic sentence but numbers were changed into blocks of the same amount as the numbers (Fig. 10).

2. Students' mathematical idea of 'place value and base ten' (57.14 %)

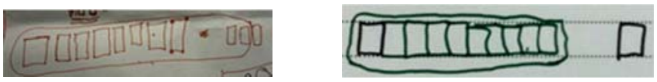

Fig. 11: Using block diagram

Students drew 8 blocks and 3 blocks. Later, students used a circle to encircle blocks to make it to ten (8 blocks and 2 blocks were together) as shown in Fig. 11. Students used blocks and block track to make ten and they drew to show the sum number of the whole blocks as shown in Fig. 12.

Fig. 12: Using bar chart

3. Students' mathematical ideas of 'Composition and Decomposition' (71.43%)

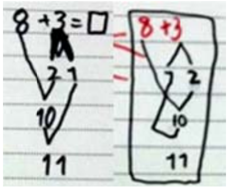

Fig. 13: Using line diagram

Fig. 13 showed that students used their knowledge of composition and decomposition to solve problem in addition. They first decompose 3 into two numbers. After that, they compose separated number to make 10. The last, they combined with another number to make 11.

Problem Situation 3

There were 5 monkeys. 6 more monkeys came.
How many monkeys are there?

Fig. 14: Problem Situation 3 (Japanese Mathematics Textbook translated and edited in Thai by Inprasitha et.al., 2010)

1. Students' mathematical idea of 'together' (42.86%)

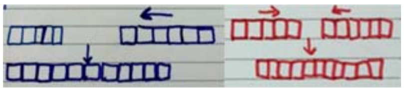

Fig. 15: Using arrows to represent increasing and combination

From the problem situation, meaningful addition was increasing or meaningful addition was combining another number with original number or moving number to add with original number. Students drew 5 blocks then drew another 6 blocks as it was number of the addend. Later, they drew the arrow pointed to expressing the movement of combination (Fig. 15).

2. Students' mathematical idea of 'number sentence' (100%)

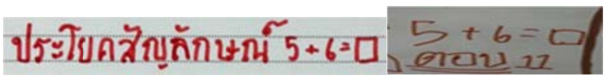

Fig. 16: Using number sentence

When students read problem situation, they wrote number sentence as 5 + 6 = □, they reasoned that 5 was the original number of monkeys and 6 was the added number of monkeys (Fig. 16). The other group of students used the number sentence 5+6 =□, to find the whole number of monkeys. Students explained that 5 was the number of monkeys who were eating fruits and 6 was the number of monkeys who were climbing on the tree or the added number of monkeys. And students used a plus sign representing combination or finding the whole number of monkeys.

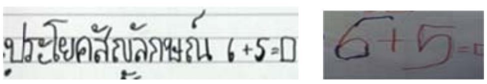

Fig. 17: Using number sentence

There was another group of students, who wrote the symbol sentence 6 + 5 = □. They reasoned that in addition augend and addend could switch positions because the answer was the same (Fig. 17). From Fig. 17 It was seen that a student wrote the symbol sentence 6 + 5 = □, and from the interview, students could explain the meaning of what they wrote. Students had the same understanding that finding the whole number of monkeys was combining two groups of monkeys together and using a plus sign to show combination in number sentence.

3. Students' mathematical idea of 'place value and base ten' (71.43%)

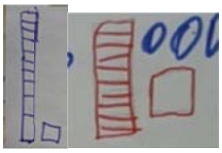

Fig. 18: Using bar chart

Students drew 1 block bar chart and 1 block representing the whole number of monkeys. First, students placed blocks on monkey pictures on a worksheet. Next, they put blocks into the block track to make ten, then, they concluded that there were 11 monkeys. Later, students drew a picture of 1 block bar and 1 block track on a worksheet as shown in Fig. 18.

4. Students' mathematical ideas of 'Composition and Decomposition' (88.89%)

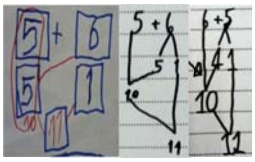

Fig. 19: Using line diagram

Students decompose the addend into two numbers and compose the separated number to make ten. Next, students combined another number with a separated number of the addend to make 11 as showed in Fig. 19.

Discussion and conclusion

As the results were mentioned earlier, concluded that classrooms in the context of Lesson Study and Open Approach were to promote students' ideas of addition in several representations, Particularly, and the idea of ***'place value and base ten'*** and the idea of ***'composition and decomposition'***. They were considered the "How to" for students to learn about the content of number and exceed the process (Inprasitha, 2011). This point of view is consistent with NCTM (2000). Base-ten notation is difficult for young children, and the curriculum should allow many opportunities for making connections between students' emerging understanding of counting numbers and the structure of base-ten representation. The base-ten system is critical for our current sophisticated understanding of whole number arithmetic (Beckmann, 2014). Composing and decomposing are combining and separating operations that allow children to build concept of part and wholes (Clements and Sarama, 2007).

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COMBINING MATHEMATICAL AND EDUCATIONAL PERSPECTIVES IN PROFESSIONAL DEVELOPMENT

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Abstract

University mathematicians and elementary school teachers have their own particular perspectives on whole number arithmetic. The study reported herein investigates how these different perspectives may interact productively in a professional development scenario, observed by a mathematics-education researcher, to achieve new insights. A discursive analysis of a professional development lesson on Division with Remainder, taught by a university mathematician, shows how the differences between the parties' perspectives led to the growth of new insights on the topic, its role in the curriculum, and its teaching. This meeting of communities is offered as a model for achieving rich expertise - an emergent discourse that is more than the sum of its parts.

Key words: commognition, division with remainder, elementary school teachers, mathematicians, professional development.

Introduction

Felix Klein, as quoted in the History of ICMI website (ICMI, 2008), believed that "*the whole sector of Mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organised as an organic whole*". From this perspective it is natural to assume that university mathematicians should have an important role in the professional development of elementary school teachers; however, mathematicians have little or no experience teaching whole number arithmetic (WNA) to young children. Furthermore, the discourse of university mathematics and its teaching may be quite different from its elementary-school counterpart. ICMI study 23 suggests "*taking a mathematical perspective (as practiced by the current community of mathematicians) combined with an educational perspective [to uncover] core mathematical ideas in developing pathways to WNA*". The overarching goal of this paper is to suggest an interpretation for this keyword: combined.

I adopt a Commognitive approach (Sfard, 2008), viewing fields of human knowledge as well defined modes of communication typical of particular communities. More specifically, I draw on the framework of Mathematical Discourse for Teaching – MDT (Cooper, 2014) – in choosing the discourse of mathematics-for-teaching as the discursive unit of analysis. A *combined* discourse can be expected to develop where members of two communities meet and interact. In this paper I describe such a meeting – a professional development (PD) course for in-service elementary school teachers, the initiative of a mathematics professor, which he taught together with a group of mathematics Ph.D. students. The aim of this paper is twofold: to describe the *process* in which a combined MDT developed and the *product* of this process: the nature of this combined MDT as it pertains to a particular mathematical topic – Division with Remainder (DWR). This topic has a university parallel in

Ring theory in the form of Euclidean Domains, however its role in the school curriculum is transient – a way of performing division in grade 4 which will become redundant once students are familiar with the field of Rational numbers. Furthermore, school problems usually focus on the quotient of DWR, whereas in University Algebra the remainders are much more interesting, comprising the elements of a finite cyclic Ring. In view of differences such as these in the two communities' perspectives, my research questions are:

- How can the meeting of a mathematician and in-service elementary school teachers in PD foster the mutual growth of MDT related to WNA?
- What insights regarding Division with Remainder, its teaching, and its role beyond the scope of WNA emerged in this meeting?

This paper is part of a larger project which examines 10 mathematician-instructors teaching 6 groups of elementary school teachers (grades 1-6) in ten 3-hour sessions, covering a variety of mathematical topics. Various aspects of this PD have been reported (Cooper and Arcavi, 2013; Cooper and Touitou, 2013; Pinto and Cooper, Submitted 2013; Cooper, 2014; Cooper, Submitted 2014). The current paper, which builds on methodologies and results established in previous publications, adds a new perspective; in addition to the lesson transcript on DWR, data includes written material (listed below), providing detailed – sometimes explicit – information regarding the nature of the parties' MDT.

Materials and methods

The data for my analysis includes the transcript of a 2-hour lesson on DWR. Fifteen grade 3-6 teachers participated. The instructor was the initiator of the PD program, who wrote a book for teachers on the mathematics of elementary school. The data includes a final draft of his chapter on DWR, which the teachers read prior to the lesson. Data also includes 10 written teacher-responses to this chapter, submitted as an assignment for the course.

The goal of my analysis is to characterise MDTs as they pertain to DWR. The instructor's MDT is revealed in the book chapter. The teachers' MDT is revealed in their written responses. Learning, conceived as shifts in discourse, begins as teachers respond to the mathematician's text. The meeting of the parties in the subsequent lesson further reveals the affordances of this interaction of discourses in the emergence of an enriched MDT.

The framework of MDT organises data analysis in a matrix along three dimensions: 1. There are two communities of discourse – university mathematicians and primary school teacher; 2. MDT consists of six sub-discourses, inherited from MKT's types of knowledge for teaching. In this paper I distinguish only between two broad sub-discourses: Subject Matter Content Discourse – SMCD – pertaining to mathematical content, and Pedagogical Content Discourse – PCD – pertaining to issues of students and teaching interacting with content; 3. Commognitive methodology focuses on four interrelated features of discourse: *keywords* and their usage, *visual mediators*, *narratives* and the rules by which they are endorsed, and *routines* of mathematics and teaching. In my analysis I locate elements of teacher and instructor communication within the 16 cells of this matrix, as a step towards characterising the parties' discourses and the role of DWR therein. For example, the remainder notation $25 : 3 = 8\ (1)$ is a visual mediator in teachers' SMCD.

Results

Results are arranged around themes regarding DWR that emerged in the analysis. For each theme I describe pertaining aspects of university MDT and teachers' MDT, and the process by which an enriched MDT emerged.

Tables 1 and 2 show an outline of the book chapter and of the lesson.

Page	Topic
1-2	What is remainder?
3	Range of remainders
4	Inverse multiplication problems
5-6	Remainder in word problems
7	Calculating DWR
8-14	Remainder arithmetic
15	Problems

Tab. 1: Overview of book chapter

What's going on	Duration
Introduction, rationale	25 min.
The nature of 3 remainder 2	13 min.
Word problems	4 min.
Long division algorithm and DWR	3 min.
Is 0 a remainder?	2 min.
Problems and exercises	16 min.
Remainder arithmetic	31 min.
Signs of divisibility	19 min.

Tab. 2: Overview of the enacted lesson

1. Remainder notation

The instructor was struck by an inadequacy of the standard remainder notation. This is described briefly in the book chapter as a violation of the equals-symbol as an equivalence relationship, namely: If $25{:}\,3 = 8(1)$ and $41{:}\,5 = 8(1)$, then by transitivity we deduce $25{:}\,3 = 41{:}\,5$, which he considers *complete nonsense*. His proposed solution is to change the remainder notation as follows:

Standard notation in Israel: $25 : 3 = 8\ (1)$	Proposed notation: $25 : 3 = 8\ (1 : 3)$

The new notation is read *eight with remainder 1 which needs to be divided by 3*. In the PD discussion the instructor emphasized that (1 : 3) is a whole-number remainder, not a fraction. The new notation no longer implies $25 : 3 = 41 : 5$, since $8(1 : 3) \neq 8(1 : 5)$. The discussion of this notation in the PD raised some interesting issues. Some teachers, unfamiliar with the definition of equivalence (a relationship that is reflexive, symmetric, and transitive), felt that it entails a strong "sameness": "*it's like 2 times 6 equals 12 and 3 times 4 equals 12, but there's really no equivalence, they're not the same*". Of course equivalence does not require this sense of sameness, but interestingly, the new notation actually sets up an *isomorphism* between DWR problems and results. If the result is 8(1 : 3), we can reconstruct the problem as 25 : 3, and not for example 50 : 6, which would yield 8(2 : 6). Note that in this notation we should write remainders that are zero, for example 24 : 3 = 8(0 : 3).

I now return to the instructor's claim that $25 : 3 = 41 : 5$ is *complete nonsense*. He remarked in the lesson: "*What kind of creature is three remainder two? ...Arithmetic expressions have numeric value... the equals symbol tells us that the value of this is the same as the value of that... Is this [8(1)] a numeric expression like I'm accustomed to?*" He was expecting a negative answer, which would further challenge the use of equality, however a teacher responded unexpectedly: "*If we agree that [the term in brackets] is the remainder, then yes, of course [it's a numeric expression]*". For this teacher, each of the two division expressions is evaluated as the *numeric* object 8(1). And indeed there is nothing intrinsically wrong with this equality as an equivalence relationship - the equivalence class of all DWR problems whose result is the numeric object 8(1). This point was raised by another teacher who warranted her claim in the context of a problem of fair sharing of apples: "*But before [the remainder] is divided, it's the same apple that's left over. As far as the [fruit] store is concerned it's the same [remainder]*". The instructor, by his own testimony, missed this mathematical point. Hence, the equality is *complete nonsense* not because it is not a valid equivalence, but rather because it is the *wrong* equivalence, since it is not consistent with fractions, where $\frac{25}{3} = \frac{41}{5}$ is indeed wrong. The instructor's oversight indicates that for him DWR is not a goal of its own, but rather a transitional phase on the way to rational numbers.

The teachers' written reactions to the new notation were generally positive. Five considered it preferable, three had pedagogical reservations drawn from their PCD (e.g. children will prefer the shorter notation), and four saw pedagogical affordances, some quite ingenious, as reported in the following sections.

2. Division with remainder as a precursor to fractions

Division as the inverse of multiplication requires fractions, yet there are reasons to teach it before fractions are introduced: Children face division problems (e.g. fair sharing) before they learn fractions, and such problems call for an arithmetic operation; from an algebraic perspective, division as *multiplication with unknown* ($? \times 3 = 18$) is introduced together with multiplication. Although the motivation for the new remainder notation was mathematical (respecting equality as equivalence), the instructor appreciated its pedagogical merit, offering a smooth transition from WNA to fractions conceived as the result of division. He elaborated on this in the PD meeting, referring to a fair share problem represented by $17 : 5$: "*everyone received 3, and there are two more that need to be divided into 5… I write it in the language of whole numbers… Everything is ripe [for fractions]… what's written is in fact 3 and two fifths*".

3. DWR and its role in the long division algorithm for decimals

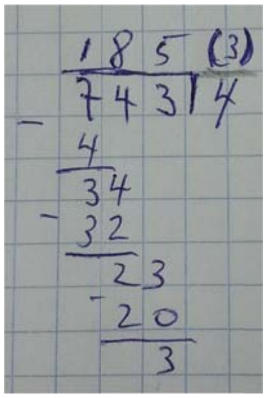

Fig. 1: Remainder as fraction

A teacher appreciated the affordance of the new remainder notation for a topic two or three years down the road (grade 6) - the long division algorithm (LDA) for decimals: "*I saw this in your notation… the remainder [from 743 : 4] is in fact three fourths, and I wanted to show them that they would get … point 75… but they only saw the 3 [in the standard notation], they didn't see the 3 over 4*". This teacher believes that the new notation will help see the remainder as an uncompleted division problem, which students can complete once they have decimals. Another teacher showed her own way of bridging DWR and fractions, by writing the LDA remainder "over" the divisor, suggesting that the remainder 3 is in some sense *three over four*, as illustrated in Fig. 1.

4. Algebraic thinking – connections between DWR and multiplication

In elementary school, division is privileged as one of the four basic operations, yet it is also the inverse of multiplication. When this idea is introduced in the context of whole numbers, teachers and text books are careful to contrive problems without remainder. It is not until children have learned fractions that they will encounter problems such as $5 \times ? = 17$. In his book-chapter, the instructor suggests how DWR can be supported by such algebraic thinking. The problem$17 : 5 = 3(2)$ can be conceived as a problem of multiplication with *two* unknowns: $5 \times ? + ? = 17$, where the second unknown, as a remainder, is restricted to be less than the divisor. This idea, when discussed in the PD, raised the issue of 0 as a legitimate remainder, e.g. $5 \times 3 + 0 = 15$ and its counterpart $15 : 5 = 3(0)$. Many teachers objected to this notation, claiming that in this case there is no remainder and thus there should be no brackets. The instructor accepted this, but one teacher, who agreed with the instructor's preference for writing the zero, added a pedagogical angle to this mathematical issue: "*In 3rd grade remedial class I prepared a table with headings: dividend, divisor, quotient, remainder…we said where there is no remainder it's zero… we kept adding 1 [to the dividend] and we arrived at a rule…*" This teacher saw the productiveness of remainder 0 from the perspective of students making sense of DWR, finding patterns in a sequence of problems where the dividend increases by 1. From the instructor's perspective there were additional reasons, for example in the context of remainder arithmetic, conceived as a finite Cyclic Ring, remainder-0 is the neutral-for-addition member of the Ring.

The multiplication-with-unknown visual mediation did not remain an abstraction; the instructor explicitly connected it to the teaching of LDA. If we take figure 1 as an example, beginning with the hundreds digit, the algorithm begins with "4 into 7 is 1". The remainder (3) is found by subtracting: $7 - 1 \times 4 = 3$, equivalent to the multiplication notation: $1 \times 4 + 3 = 7$.

5. The affordances of DWR for challenging advanced students

The instructor stated his opinion regarding advanced students - they should be challenged, not accelerated. *Systematic* emerged as a keyword in his MDT. When posing the following "routine" problem (using standard DWR notation): $17 :? = 3(2)$ the instructor asked "*what is the systematic way to solve?*" He got the answer he was fishing for: $(17 - 2): 3$. Later he elaborated: "*up until now it was pretty obvious that the problems you solved did not have additional solutions... because there was always a systematic way to find it*". The next two problems were different. First the dividend and the remainder were unknown - $?: 3 = 7\,(?)$ This problem has multiple solutions, but once the remainder is chosen (e.g. 1), the dividend can be found systematically, e.g. $7 \times 3 + 1 = 22$. The next problem: $35 :? = 3(?)$ was considered challenging. A systematic solution, inspired by the commutative rule, is to switch the quotient and the divisor. From $35: 3 = 11(2)$ we deduce the solution $35 : \mathbf{11} = 3(\mathbf{2})$. There are, however, additional solutions, since the larger divisor (11) permits a larger range of remainders, e.g. $35 : \mathbf{10} = 3(\mathbf{5})$. Thus, the challenge lies in overcoming the incompleteness of the systematic solution to the problem.

6. DWR and word problems

Throughout the lesson, DWR was considered in two contexts – the operation grounded in division problems (e.g. fair sharing) and the abstract arithmetic operation. The instructor saw word problems as the major motivation for DWR in elementary school, to which he dedicated much of his chapter and much of the lesson. A move that the teachers particularly liked (based on written responses) was three different word problems that make use of the same dividend and divisor; in one the answer to the problem is the quotient, in another it is the quotient's successor, in another it is the remainder, and in yet another the only plausible answer is a fraction (i.e. it is not appropriate for DWR). Comments, grounded in PCD, included: *they emphasise understanding, often neglected in text books*; *such a question appeared on the standard state test*.

7. A focus on remainders – Remainder arithmetic

The instructor dedicated half the chapter and a full 50 minutes of the lesson to remainder arithmetic (i.e. addition, subtraction and multiplication in the Cyclic Ring of remainders modulo some divisor), a topic beyond even the high school curriculum. He offered two reasons, both related to *understanding* mathematics: 1. In order to properly understand the special case of parity (odd + odd = even, etc.) it should be generalised, and this is done through remainder arithmetic; 2. The elementary school curriculum includes signs of divisibility (e.g. sum of digits for divisibility by 9); understanding why "it works" relies on remainder arithmetic, and as he said: "*my feeling is that it's generally not advisable to teach something without explaining it*".

Three teachers found the chapter on remainder arithmetic difficult to understand on their own; two other teachers considered it unsuitable for the general class, though possibly suitable as enrichment for advanced students. However, during the PD lesson many teachers appeared to be highly engaged, participating in the instructor's mathematical routines in an exploratory manner.

Discussion and conclusion

A striking feature of the results discussed in the previous section is the depth and complexity of the discourse of "elementary" mathematics and its teaching. The instructor, drawing on his university discourse, gained a deep understanding of DWR, and connected it to other elementary topics, some of them beyond the realm of WNA, (e.g. fractions, LDA, signs of divisibility) in ways that the teachers found engaging. But this was only the beginning of the process. To describe how knowledge for teaching developed further, I adopted the Commognitive framework of MDT. In this framework, *knowledge* for teaching in the cognitive sense is evident in the parties' *narratives* (e.g. children don't understand remainder), *visual mediators* (e.g. remainder notation) and *routines* (e.g. LDA). The Commognitive framework extends this conception in drawing attention to the nature of the parties' warrants for endorsing or rejecting particular narratives, visual mediators or routines, grounding them in their communities' practices. Consider, for example, the new DWR notation:

The instructor endorsed the new notation primarily for its mathematical consistency – a warrant from his SMCD – yet he was sensitive to pedagogical issues as well, such as the affordance of his proposed notation for teaching fractions. I speculate that this sensitivity was developed during three years of teaching elementary school teachers in PD. Teachers were respectful of the mathematical warrant, yet in considering whether or not to adopt the new notation, brought their PCD to bear. Some rejected it based on their discourse of students (difficult, cumbersome), others found pedagogical affordances (better understanding of remainder, overcoming difficulties related to decimal LDA). Years of teaching experience did not yield these insights spontaneously; rather it was the teachers' response to the instructor's mathematical ideas that instigated these new insights. In this sense many of the insights that emerged in the PD were a result of the meeting of two MDTs. This meeting began with the teachers' written responses, but developed further in PD discussions. In some cases the teachers' responses challenged the instructor's mathematics, e.g. the question of whether $25 : 3 = 41 : 5$ *makes sense* in the context of DWR. These mathematical challenges were not always appreciated by the instructor at the time. Thus the PD can be conceived as a meeting of *three* communities of discourse, where the third is the mathematics education researcher, who teased out insights that may have been missed by the interlocutors at the time.

I have shown examples of four types of learning opportunities. The teachers extending their SMCD and the mathematician extending his PCD can be conceived as sharing knowledge across communities – a worthy endeavour in its own right – however I have additionally shown opportunities for the teachers to gain new pedagogical insight and for the mathematician to learn new mathematics, where the parties extend aspects of their MDT in which they have already achieved expertise. This, I claim, is the main affordance of this meeting of communities. This is how I understand *combined perspective*. In this paper I have illustrated some examples of insights that emerged through this combined perspective. Many of them warrant deeper analysis, which will be undertaken in a separate publication.

The instructor's decision to teach remainder arithmetic was grounded in his feeling that teaching mathematics entails explaining it (e.g. signs of divisibility). Furthermore, his conception of an adequate explanation was grounded in his university practice (e.g. odd + odd = even should be generalised). Can such "attitudes" to mathematics and its teaching be taught in PD? This question is beyond the scope of the current paper, where teachers' endorsement or rejection of narratives, visual mediators, and routines appear to be strongly grounded in pedagogical concerns. Yet far from being a limitation, these differences between the instructor's and the teachers' MDT are what drove the learning process.

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KEY ISSUES FOR TEACHING NUMBERS WITHIN BROUSSEAU'S THEORY OF DIDACTICAL SITUATIONS

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Abstract

In this paper we present the main key issues of Brousseau's theory of didactical situations and its close interaction with the history of mathematics. We then give a short overview of the main stages of the development of numbers in the history of humanity. We then present the way Brousseau's theory, in accordance with the historical context, develop the key stages of a teaching sequence of the concept of numbers for students from age 4 to 6. In the final stages we reflect on the value of this approach in terms of teachers' training, including the historical aspect of numeration.

Key words: cardinal, equipotent collections, fundamental situation, history of mathematics, quantity, theory of didactical situations

Introduction

This paper does not aim at introducing new material for the teaching of whole numbers, but to present the main lines of a specific approach dating from the 80s, within the Theory of didactical situations in mathematics (Brousseau, 1997). This approach is very popular within French speaking countries, where it is the basis for most curricula in primary school teachers' training. Our aim here is to make this approach more known to an international audience and to show the key issues on which it lies in terms of the modelling of the teaching and learning of whole numbers. Moreover, we will reflect on the value of this approach in terms of teachers' training and its articulation with the history of numeration from prehistory up to recent times.

From early works (e.g. Piaget and Szeminska, 1941), the 70s psychologists made a lot to make us understand essential issues in children's conception of numbers (e.g. Bideaud and Meljac, 1992; Fuson, 1988, Gelman and Gallistel, 1978; or Kamii, 1982). Moreover, the question of designing a school curriculum for the learning of whole number has been, since the 70s, a new challenge in which mathematics education approach developed. In the French context, Guy Brousseau held a central position. His approach is unique in the sense that he developed a new theory based on several sources and in constant interaction with experimentation, in the context of the COREM, a school where teachers and researchers interacted on a daily basis (Brousseau, Brousseau and Warfield, 2014, pp. 168-172). In psychology, experimental devices are built in order to reveal the state of knowledge of a subject. In mathematics education, the purpose is to build some experimental devices for teaching and to control the learning. In this sense, the main goal changes radically compared to psychology, since here the research analysis bears on the relation between the devices, the mathematics and students' behaviour. In this sense, in the Theory of situations, the focus is on analysing the *raison d'être* of a notion creating conditions of uses of elements of knowledge (*connaissances*), which gives the meaning of the notion a radical change compared to the aim of psychology (Brousseau 2012). In French, we have two terms for knowledge: *connaissance* refers to what students put into action or express, in a specific context, while *savoir* refers to a more general decontextualised form recognised by an institution (a theorem for instance). Without developing too much this essential difference, we will use the French words in italic to make the distinction (like Warfield in the previously quoted book).

> A mathematical situation is a set of specific conditions in which a determined set of mathematical *connaissances* (stated or belonging to the *milieu*) permit a subject to realise a declared project by the exercise of appropriate mathematical *connaissances*, known or original. […] A-didactical situations occur in the classroom, and have the goal of reproducing the conditions of a real mathematical activity dealing with a determined concept: i.e., a mathematical situation. In the course of an a-didactical situation the students are supposed to produce a correct and adequate action or mathematical text without receiving any supplementary information or influence. With this definition in hand, a didactical situation can be defined as the actions taken by a teacher to set up and maintain an a-didactical situation designed to allow students to develop some goal concept(s). In particular, the teacher sets up the milieu, which includes the physical surroundings, the instructions, carefully chosen information, etc. (Brousseau, Brousseau and Warfield, 2014, p. 203)

The search of meaning of the notion to be taught is therefore central. This epistemological quest is modelled through the essential concept of Fundamental situation:

> Each item of knowledge can be characterised by a (or some) adidactical situation(s) which preserve(s) meaning; we shall call this a fundamental situation. But the student cannot solve any adidactical situation immediately; the teacher contrives one, which the student can handle. These adidactical situations arranged with didactical purpose determine the knowledge taught at a given moment and the particular meaning that this knowledge is going to have because of the restrictions and deformations thus brought to the fundamental situation. (Brousseau, 1997, p. 30)

Therefore, in order to organise a didactical sequence for teaching a new concept the researcher will search for uses that give the core meaning of a mathematical concept and then organise a scenario in order to organise the meeting of students with the different stages of uses of the *connaissances*. This is based on 4 different types of situations:

> Within the category of didactical situation, there are three notable subcategories, chosen because they correspond to models of completed mathematics or because they have an important place in the genesis of a concept. *Situations of action* reveal and provoke the evolution of models of action without the student's needing to formulate them. The student can, immediately or later, learn to identify them, to formulate them in *situations of formulation* (expression or communication) and to justify them in *situations of proof* (validation or argumentation.) There is a tight correspondence between (a) the composition and

> organisation of the *milieu* (game, communication, debate), (b) the nature of the interactions of the subject with the *milieu* (action, formulation, proof), (c) the type of knowledge these relations call forth (implicit models of action, languages, mathematical *savoirs*.) A fourth type of situation is that of *institutionalisation* [by which the teacher introduced or established known *savoirs*]. (Brousseau, Brousseau and Warfield, 2014, p. 203)

These key notions of Brousseau's theory are now commonly introduced in teachers' training courses and we will show how they can make more sense regarding the question of teaching numbers to students age 4-6. Moreover before doing so, we will rapidly sketch some key historical facts in the development of numeration in humanity. Indeed these are essential to reflect on the *raison d'être* of numbers, which is a central aspect of Brousseau's approach. It is well known that there are at least three fundamental aspects of numbers: cardinal, ordinal and measure. Due to the word limit, we will only investigate here the cardinal aspect of numbers, and only a small part of it. This will be sufficient to illustrate to nature of Brousseau's concept. For a broader presentation, the reader can refer to a very good recent publication (unfortunately only in French) by Margolinas and Wozniak (2013). All that we investigate here concern pupils of age 4-6.

Key historical facts in the development of numeration in humanity

Our goal here is not to give a full account of the vast history of the conceptualisation of number and the evolution of numerations in mankind. We will only refer to a few key elements, based on reference works (mostly, Cousquer, 1998; Crossley, 1994; Guittel, 1975 and Ifrah, 1985). Nevertheless, like it was discussed in the ICMI study 10 on the role of history in mathematics education, we are opposed to: "the naïve recapitulationism that was introduced at the end of the last century following Darwin's writings and the biological paradigmatic view of the evolution of species which assumes that the mental development of the individual (ontogenesis) recapitulates the development of mankind (phylogenesis)." (Radford et al., 2000, p. 144). Rather, for us "the historical analysis is a source of inspiration as well as a means of control. Yet, these activities must not be only a speech of the teacher, nor a reconstruction of the historical development, but they must reconstruct an epistemologically controlled genesis taking into account the specific constraints of the teaching context." (Dorier, 2000, p. 107). Epistemological analysis is central in Brousseau's approach, in order to bring out the characteristics of the fundamental situations, which will give the meaning of the mathematical concept.

Regarding numbers, anthropology tells us that many primitive tribes did not develop the concept of numbers beyond duality, sometimes decoupled as two-two (for four). Indeed, the number three seems more difficult to conceive. Therefore, the first step in conceptualisation is to evaluate and differentiate among several multitudes. In other words, the first difficulty in order to conceive the quantity is to be able to individualise every object and not see the collection only as a whole. The duality "same/different" is essential in this process.

The idea of number is at first attached to the necessity of counting. In the introduction of his famous work on the measure, The French mathematician Henri Lebesgue (1875–1941) gave a very nice overview of this process:

> On imagine volontiers, et les constatations faites chez certaines peuplades primitives semblent confirmer cette hypothèse, que … les hommes en sont arrivés, quand ils veulent comparer deux collections, à compter ; c'est-à-dire à comparer les deux collections à une même collection-type, la collection des mots d'une certaine phrase. Ces mots sont appelés des nombres. Pour compter ou dénombrer, on attache mentalement un objet différent de la collection envisagée à chacun des mots successifs de la phrase (ou suite) des nombres ; le dernier nombre prononcé est le nombre de la collection.
> Ce nombre est considéré comme le résultat de l'opération expérimentale de dénombrement parce qu'il en est le compte-rendu complet.[1] (Lebesgue, 1931, introduction)

At first, shepherds were concerned by about the loss of sheep when these came back to the pen. The two collections to be compared are the collection of sheep leaving the pen in the morning and the collection of sheep coming back in the evening. In this case, a very ancient solution was to build an intermediate collection with stones. The shepherd put a stone on a pile for each sheep going out and in the evening he only needed to take off one stone from the pile for each sheep getting back into the pen. Therefore he creates an idempotent collection through the fundamental action of building a one-to-one correspondence between the sheep and the stone and vice-versa. This is the first step of conceptualisation towards the concept of numbers in its cardinal aspect, the recognition of quantity as a property of collections.

Several other artefacts, some dating back to Paleolithic times (15'000 years B.C.) like notches on a piece of wood, bones or reindeer antlers are the indicators of some certitude of a human activity related to building idempotent collections in order to memorise quantities. Unlike the pile of stones, this collection of notches was transportable and durable in time.

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[1] One can readily imagine, and the findings in some primitive tribes seem to confirm this hypothesis, that… Men came, when they want to compare two collections, to count: that is to compare the two collections to a same collection-type, the collection of the words of a certain sentence. These words are called numbers. To count or enumerate, one mentally attaches a different object from the collection envisaged to each of the successive words in the sentence (or series) of numbers; the last number to be pronounced is the number of the collection.

This number is considered to be the result of the experimental enumeration operation because it is its complete report. (our translation)

Around 3'500 B.C. Sumerians are known to have developed the idea that instead of using one object, a small cone of clay, for each sheep, it was easier to replace every ten cones by an other object, a small ball, then every 60 small cones again by an other object, a bigger cone, 600 small cones by a bigger cone with a round print, etc… This is the beginning of the idea of basis, in an additive structure. Indeed, each piece of clay has an invariant value and one has to add the values of each to get the total. These were used to keep a trace of the quantity of sheep during transhumance. Two identical collections were built and enclosed in a clay sphere with distinctive marks from both the owner of the sheep and the shepherd. In order to simplify this, Sumerians got the idea to represent this with marks on a clay tablet. This is the beginning of cuneiform writing.

The way men were using words to give account of these activities is not easy to reconstitute so there are big lack of information, in order to understand the process that brought the idea of using a structure collection of words in a certain order to give account of the enumeration.

As announced, due to the limitation of space, we will not pursue any further the history of numerations and come now to the way, in Brousseau's theory, in which these first element of epistemology were used to build the first teaching sequences on number.

Brousseau's didactic engineering on numbers

The first fundamental situation for numbers has to do with quantity (not yet cardinality). It can be expressed as "building an idempotent collection to a given collection". Typically, one has to bring exactly as many[2] (no more no less) glasses as there are plates on the table.

Following Margolinas and Wozniak (2013) we will illustrate this with eggs and cups. So the fundamental situation will be "Bringing exactly as many eggs as cups". The cups are on a table and many eggs are in a container, while the pupil has to put his eggs in a basket. The task can be validated by the pupil himself, by putting one egg per cup. He wins if there is one egg per cup and no egg is left in his basket. There is no need for a didactical intervention by the teacher, as it is said that there is a validation by the milieu, which is a necessary guarantee for an a-didactical situation.

As such the task is still very open and several things need to be specified regarding the activity, they are choices that have to be made by a teacher in order to build a scenario for presenting the situation to his pupils. These choices (made against others) are modelled as choices of values for didactical variables.

[2] One can't say the same number! With 4-6 years old kids the teacher will have to work on vocabulary to be sure that the task is understood.

Here for instance, the number of cups is a didactical variable. If there are only two to five or six cups, we know that pupils can use directly evaluate the quantity by *subitizing* and therefore easily succeed, on the other hand, if there are more than 10-12 cups, the task will be unnecessarily complicated for pupils of age 4-6. Therefore there are three significant sets of values for this didactical variable, namely [2, 6] [6, 12] and [12, +∝[, where each set corresponds to choices that favour or hinder some specific strategies. The consistent sets of values for a didactical variable are therefore those that change the hierarchy in the strategies in terms of accessibility, validity or difficulty… there are essentially two types of didactical variables, depending whether it bears on the mathematical activity (number of cups, of eggs in the container, distance between container and table with cups, etc.) or the scenario (work in small groups, present results on a poster, etc…).

Each choice of a set of values for the variables corresponds to a different situation. A didactical engineering for the recognition of quantity as an essential feature of collection (a first step to the idea of cardinality) consists in organising a progression made of different variations of the same fundamental situation.

The first step is crucial for the process of devolution. The *connaissance* has to be put into action only and recognised as a way to solve the problem. The container with the eggs is close to the table with cups (between 6 and 12). The task is to put in the basket as many eggs as cups. Some pupils will just roughly evaluate the quantity, if not just pick some eggs at random, and fail. They will realise this, when putting the eggs from the basket in the cups. An easy way to succeed is to take the eggs one by one out of the container and dispose them in front of the cups before putting them in the basket (the choice of values of the didactical variables does not block this strategy). This is typically a *situation of action*, not just because the pupil is active, but essentially because the *connaissance* that permits to solve the problem is directly put into action without a need of explicitation. Moreover, even as simple as this could seem, the fact that the number of eggs do not change when putting them in the basket after aligning them in front of the cups has been pointed by Piaget and Kamii as a difficulty in the well-known test of conservation. Even if, in the act, the *connaissance* here is a means of anticipation that only the final step (when one put the eggs in the cups) validates.

A second stage in the progression will bring the necessity of *formulation*. This does not just mean that the pupils will formulate things. Pupils will formulate for instance their action when sharing what they have done in the first phase, but this is not a situation of formulation. On the contrary in a situation of formulation, pupils may not exactly formulate their *connaissance* in the sense that they may not express it verbally by oral or written means. What is at stake in a situation of formulation is the fact that the *connaissance* needs to be postponed either in space or in time and therefore needs to be transferred, hence "formulated". Nevertheless the transfer can be to oneself (auto-communication) or to somebody else (communication). The other person can be familiar or not and it can be a group or an individual. There might be some issue regarding rapidity, readability, etc… The choices of values for didactical variables consists in putting the container far from the cups, in a place where the pupil cannot see cups (otherwise he could still make the one-to-one correspondence in action). The eggs have to be brought back in one go. This is the easiest stage of formulation, in a situation of auto-communication. In order to succeed, not by chance, the pupil will have to build an intermediate idempotent collection. There, the teacher can choose to give, let take if ask or forbid, paper and pencil (one can make one mark per cup, like the notches on a piece of wood, bones or reindeer antlers), some tokens (that can be used like the shepherd used stones). Pupils can use their fingers, even if this might be more difficult, if there are more than 10 cups. In this phase of auto-communication, the pupil builds his *connaissances* on the validity of intermediate idempotent collection. A new stage would be to postpone the action in time: the pupil can see the cup one day, but will have to bring the egg the next day, when the cups will be hidden. This makes the construction of the code more difficult. Then comes the proper communication, when one pupil has to buy some eggs from a seller: the order can be made orally or in written form, with various constraints on the type of message or the rapidity, if some competition is added. The situations of proof concern phases where pupils in a collective task have to agree on way to succeed and *situation of institutionalisation* where the teacher will point out officially what has been learnt and refer it to a social practice that is recognise outside the class (decontextualisation). The next stage concerns the cardinal, recognised as the equivalence class of quantities regarding the relation of *idempotence*, but we do not have space to develop this aspect.

All these phases are essential in the process of conceptualisation of the number, and it is important that pupils get some opportunity to give sense to essential ideas related to numbers before they acknowledge numbers and counting as the socially recognised consistent ways of solving the task.

Conclusions

In France and French speaking Switzerland, Brousseau's theory of didactical situations is an essential part of primary school teachers' training in *didactique des mathématiques*. It is particularly adapted to the teaching of whole numbers, where the key concepts of the theory can be illustrated in relation to the historical context of the conceptualisation of number and numeration. This gives some essential cultural background in mathematics for use by teachers, as well as useful tools for teaching, even if a lot more needs to be done. Moreover it illustrates some aspects of research in didactics of mathematics.

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THE RELATIONSHIP BETWEEN NUMBER NAMES AND NUMBER CONCEPTS

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Abstract

Different countries have different names for numbers. These names are often related in a regular way to the base-10 place value system used for writing numbers as digits. However, in several languages, this regularity breaks down (e.g., between 10 and 20), and there is limited knowledge of how the regularity or irregularity of number naming affects children's formation of number concepts and arithmetic performance. We investigate this issue by reviewing relevant literature and undertaking a design research project addressing the specific irregularities of the Danish number names. In this project, a second, regular set of number names is introduced in primary school. The study's findings suggest that the regularity of number names influences the development of number concepts and creates a positive impact on the understanding of the base-10 system.

Key words: base-10 value system, grade K-2, new number names from 11-99

Introduction

"The Danish number names are very old and reflect a number concept that is primitive in relation to mathematical thinking," (Gyldendal, 2009–2010, our translation) such as *treoghalvfjerds*, 73 (three and half-four), *toogtredive*, 32, (two and thirty), and *seksten*, 16 (sixteen). These old roots are unknown to most students; furthermore, the number names are abbreviated. As an example, 70 (half-four) in Danish was once named half-four-times-twenty, but the *times-twenty* has been lost in the counting numbers yet retained in the ordinal numbers.

Frankly, Danish numbers seem rather peculiar with their mystical names; compared to other countries, this appears to be the case. Specifically, the irregularity involves the number names between 10 and 20, where 11 and 12 have their own names, while 13 to 19 each ends with a ten. The two-digit numbers from 13 to 99 have an inversion property (the ones are said before the tens) and the tens have names inspired by a 20-base system. However, most European languages break away from a clear regularity with respect to the base-10 place value system, especially with the numbers from 11 to 19. Understanding the effects of such irregularities on mathematics teaching remains an unresolved question. The present paper aims to contribute to the knowledge of this issue; it combines the literature on the subject with a description of a research project that is currently refining the hypothesis empirically.

Number names and arithmetic performance

There are major differences in the kind of system a language uses to describe numbers. Most European countries have an irregular naming of the teens; both German and Dutch systems feature an inversion property of the numbers between 13 and 99, similar to the Danish one. These inversion effects had been studied by Moeller, Pixner, Zuber, Kaufman, and Nuerk (2011) for two-digit numbers, showing how inversion-related difficulties predict later arithmetic performance.

Different studies (Miura and Okamoto, 1989; Miura et al., 1993; Miura et al., 1999) compared Japanese, Chinese, Korean, and English-speaking American first graders' (6–7 years old on average) cognitive representations and understanding of place value. The findings confirmed that the Asian-language speakers showed a preference for using base-10 representations to construct numbers, whereas English speakers favoured using a collection of units. Note that a significant difference between American and Asian number names appears between 11 and 19, exactly when the base-10 system starts to use two digits. In Miura and Okamoto's (1989) study, children were asked to construct the numbers 11, 13, 28, 30, and 42 from sets of ten and unit wooden blocks. The results showed that 91% of the American first graders used unit blocks to represent the numbers on their first attempt. In contrast, about 80% of the Asian children used sets of ten blocks when representing the numbers on their initial attempt. These differences in cognitive representation were mainly ascribed to language (Miura et al., 1993). Nonetheless, the validity of this conclusion is challenged by many other cultural and educational differences between Asian and Western children. On the other hand, research has also been conducted among children who share similar cultures and belong to similar school systems but have different mother tongues. For example, Dowker, Bala, and Lloyd (2008) compared English and Welsh students, who have similar cultural conditions, although Welsh names for numbers are as regular as those of the Japanese. Dowker and colleagues (2008) found no statistically significant difference between the two groups' overall arithmetic performance test but:

Welsh-speaking children find it easier than English-speaking children to read and compare two-digit numbers, suggesting that they are better at using the principles of place value (p.531).

This issue raises the question of why and how different languages influence number concepts and perhaps even the ability to learn simple arithmetic.

Another research project concerning numbers and names similarly concluded that the names matter (Xenidou-Dervou et al., 2014). They compared Dutch and English students in kindergarten and grade 1 in terms of the development of nonsymbolic and symbolic, approximate arithmetic. The Dutch system follows the same inversion as that of the Danish in two-digit number names. Some conclusions of the project are that Dutch-speaking kindergarteners have delayed developmental onset, lagging behind their English-speaking peers in symbolic (but not in nonsymbolic) arithmetic, demonstrate a working memory (WM) overload, and are significantly worse in naming large numbers.

Xenidou-Dervou and colleagues (2014) interpreted their findings as a “first evidence for the effect of the inversion property on the onset of symbolic approximation”.

Number concepts

Learning to count and understand the base-10 system are cognitive challenges involving many small steps. We have chosen to focus on oral counting; the cardinal principle of combining a name with a cardinal value; and the combination of words for a number, its cardinal value, and the digit sign.

Developing familiarity with the symbolic number system begins with oral counting. Although children start oral counting quite early, it is not clear if they understand what they are doing. Counting appears to be learned first as a linguistic routine through which the number names are perceived as ‘sign systems’ or cultural semiotic systems that enable the symbolic representation of knowledge (Goswami, 2008).

At 3–5 years of age, children more or less understand the counting principles, at least until the number 10, even when they err in their counting (Siegler, 2003).

Children typically learn the names of numbers as a long list of words and demonstrate knowledge of the stable order principle by almost always saying the number words in a constant order and emphasising the last number (Goswami, 2008). The names are developed as sounds connected to the number of objects in the sets.

The developmental shift to understanding the number name as a cardinal value requires a qualitative shift in children’s representation of numbers. The cardinal principle requires comprehension of the logic behind counting (Goswami, 2008) and the ability to judge the size of a set. It relies on a representation of quantitative information in which the coding of smaller quantities is different from that of larger quantities (Goswami, 2008). Children’s conceptual understanding of numeration depends on their ability to make a connection between a number name and its cardinal value, which they learn to do by grouping and quantifying sets of objects (Thomas, Mulligan and Goldin, 2002).

Learning how to connect the number name, its cardinal value, and the digit sign is another challenge. As discussed, two different systems must be combined with different representations. Becoming an expert at combining these two systems means developing rapid access to an automatic use of written numbers and simultaneously being able to multitask to solve other problems in parallel. If the two systems are iconic and support each other, the child encounters less difficulty in learning this skill, as is the case for Japanese-speaking children. If the two systems are irregular and therefore conflict with each other, it is more problematic for the child to understand and remember the connection among the name, the cardinal value, and the sign. Duval (2006) described this situation as a conversion between registers and observed that the congruent conversions seem the easiest for students, meaning that the representation in the starting register is transparent to the target register.

Materials and Methods: Project intervention

In order to address the question of the influence of number names on number concepts, we have initiated a design research project that works on improving the teaching of numbers in the Danish primary school. The project aims to improve students' understanding of the base-10 system by using mathematical names for the numbers. For example, one-ten-one for 11 and five-ten-six for 56 follow the words for the numbers in the same order as the written numbers. The project involves 10 classes and 9 teachers at a suburban, low-income area school in Copenhagen. The project combines the renaming of numbers with supporting the teachers in instructing the students in kindergarten and grade 1 accordingly. In each class there are between 15-30% migrating children, still all the children speak Danish and all the teaching is in Danish. The research is planned to last for three years; we currently have data from the first year. The data consists of students' performance in classroom observations, a number understanding test, teachers' portfolios, and notes from collaboration with teachers.

Using a method inspired by design research (Cobb and Gravemeijer, 2008) we have formulated our hypotheses for empirical investigation. The hypotheses is grounded in the case that Danish children experience difficulties in understanding and learning Danish number names; consequently, they face challenges in working with numbers due to the complicated and irregular number names. The project builds on the following two hypotheses:

1. The number names function as cognitive artefacts; hence, concordance between spoken and written language is sensible.
2. Language constitutes concepts, which is why clear terminology seems effective in developing lucid concepts.

These hypotheses/explanations are grounded in learning theories, semiotics, and cognitive science, as well as empirical comparisons of students' skills in countries with different levels of transparency between written numerals and spoken number names.

Our research aims to provide an understanding of the extent to which it is possible to influence students' arithmetic skills, including strategies that offer the use of mathematical number words in teaching. The intervention consists of agreeing on and discussing the introduction of a second set of words for two-digit numbers. We do not provide special teaching materials or change the textbooks that the teachers normally use. Rather, we support the teachers in how they can use their existing materials together with this project's tools and aims. In a similar fashion, we have tried to build on the existing practices and organization in the school; hence, the preschool teachers have already developed simple, daily routines for working with numbers. They start by talking about what day it is (e.g., Monday or Wednesday), which month it is, how many days the students have been in school so far, and if there are other particular occasions to mention—is it anybody's birthday or is there any other celebration? In this sense, we have sought to build on the existing routines, materials, and organisation. We have also held regular meetings with the teachers to discuss and learn about their experiences, as inspirations for everybody involved in the project.

Results: Examples from the learning environment

We have just completed the first year of our project and our results are more tentative than finalised. Therefore, we present some cases from the observations, the test, teachers' portfolios, and some topics from the meetings with the teachers. Next, we discuss and interpret our observations.

We observed a grade 1 class in which the children were working with number tiles. The goal was to arrange numbers from 0 to 99 in an organised pattern (Figs.1 and 2):

Figures 1 and 2: Grade 1 students work with placing tiles in the right order

The task entailed both finding each correct number from the pile and putting it in the right place in the number pattern. As shown in Figure 2, the group chose to start with 0 in the upper left corner and to build the rows up to the nines. It progressed easily and the students were very engaged. When a girl (E) was looking for a tile in Figure 1, the interviewer (I) asked her some questions:

I: What number are you looking for?

E: Four-and-threes (64).

I: Which digits are in the number four-and-threes?

E: There is a four and … (she starts to count on her fingers, ten, twenty, thirty, etc.).

Another student said, "There's a six in four-and-three (64)."

I: In what order do the two digits come?

E: Four and six.

Another student replied, "No, it is six and four."

E: OK, six and four.

I: And you are allowed to call the number just six tens and four.

After this communication, the student easily found the number 64.

This scene showed an example of communication between the interviewer and the students. It occurred at the beginning of the school year when the students were still at the initial phase of renaming the numbers with mathematical names.

The case also illustrated how the focus on naming numbers interacted with the students' conceptualisation of numbers. The conversation between the interviewer and the girl's classmates helped her realize how to approach the task of finding the number 64. In this sense, the increased awareness of numbers and their names might provide an important explanation for the results we would obtain in this project.

When they finished the pattern, the interviewer asked the students about what kind of patterns they could find within the large pattern. The session developed into a discussion about how the pattern could be seen in both vertical and horizontal directions. The interviewer asked whether the same number occupied two columns. Several students immediately answered that it was impossible and one continued that it was similar to the school, where you could not attend two different classes at the same time. The student furthermore pointed at each column as representing a class. This viewpoint is interesting from the perspective of learning, in which she could see class divisions in the numbers. The ways that the students are asked questions are crucial to how they develop mathematical thinking.

Besides the observations, we also conducted a test in the middle of the school year. We made the students take the test twice, once with the mathematical numbers read aloud and the other with the traditional numbers read aloud. Half of the classes started with the mathematical numbers. We noted the effect of repeating the test and compared both results.

Furthermore, we took the minutes of the meetings with the teachers and collected their written portfolios that described their collaboration and considerations in the project.

From the observations, we recognised that especially in the kindergarten classes, the students gained confidence in learning the numbers between 10 and 20 and placing numbers on an empty number line. The grade 1 pupils demonstrated an improved understanding of the strategies for adding across tens and a generally better comprehension of the base-10 system. They also showed improvement in writing the correct number when it was spoken.

Discussion and conclusion

In this article, we have described the empirical experiences from the first of a three-year research project, which theoretically relies on our hypothesis that

Danish number names are very complicated; therefore, children encounter more difficulties in learning and working with Danish numbers.

We have shown how the differences in naming numbers may give rise to linguistically determined differences in how children learn number concepts, as well as in the cognitive load of arithmetic processes. In the intervention, we have created an easier relationship between spoken and written numbers. To combine a set with spoken numbers is one process; to combine it again with written numbers is another step. In our project—in both the classroom observations and the test—we have already found out how the mathematical number names help the students recognise numbers and write them correctly in an easier way.

Our observations of the students in grade 1 reveal that if the task is to say the name of a written number, 63 for example, children often repeat the rhyme ten, twenty, thirty, etc., and use their fingers. The children stop upon reaching the sixth finger and then they know the word. This case may be perceived as a type of interfering process; in cognitive terms, it means that two parallel processes are in conflict with each other. The semantic treatment demands too much attention; therefore, it is not possible to multitask and complete both processes at the same time (Baddeley, Eysenck and Anderson, 2009). This interfering effect means that Danish children take a much longer time to automate the learning process of combining spoken and written numbers. The logic in the base-10 system disappears in the Danish language; thus, the combination of the names and the written digits has to be learned somehow by rote.

The first year has produced both positive results and challenges. Working in school as the research field raises the issues of cooperating with the teachers, monitoring what happens in the classroom, and coping with all the unpredictable circumstances. Nevertheless, after the first year, we realise that our hypothesis seems to show results in an almost positive direction. Over the next two years, we expect to find some strength, especially in developing arithmetic strategies, which often emerge from an in-depth understanding and knowledge about facts.

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NATIVE AMERICAN CULTURES TRADITION TO WHOLE NUMBER ARITHMETIC

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Abstract

In Latin America there are more than 23,000,000 natives that even today speak their own language and many are marginalised because they do not speak the Spanish language. They have their own conceptualisation of whole numbers. Many studies have been conducted on the Mathematics of these cultures. It is highly important for teachers to be able to understand their Mathematical approach of whole number arithmetic (WNA) to be able to teach these children. In this paper, a summary of some of the WNA of Incas are presented, and how an Ethnomathematics approach, as the theoretical base to teach Mathematics in this context, is used in order to diminish the exclusion in the mathematics education of native children of these cultures.

Key words: ethnomathematics, Inca's mathematics, intercultural mathematics education, native American number system, Whole Number Arithmetic, Yupana

Background

The situation of the exclusion in the natives education in Latin American countries has been documented. Gaete and Jiménez indicated that "Indigenous schools have the same achievement problems than the other schools, but there are worsen by the absence of intercultural education"; they pointed out that "there is a linguistic gap that affect students education because their Spanish literacy is diminished". Also, they indicated "the low preparation of teachers to be able to understand and give response to their students from an intercultural perspective" (2011, p. 113).

White (1988) claims that mathematical concepts have their origin in the cultures and traditional ways of thinking of human beings. From this anthropological perspective, the place where inventions and discoveries are created is related to the cultural zone in which people live. Today, we continue to find mathematical concepts that were developed in primitive times, for example in the many different counting systems of ethnic groups in different locations. In native cultures, the Mathematical knowledge is transmitted from generation to generation in written, graphically or by oral tradition.

In order to work towards greater equity and inclusion rather than exclusion of diversity, Healy and Powell (2013) discussed the new approach between characterising "disadvantage as an individual or social condition that somehow impedes mathematics learning, which has resulted in the further marginalisation of individuals whose physical, racial, ethnic, linguistic and social identities are different from normative identities constructed by dominant social groups." They also pointed out that "recent studies have begun to avoid equating difference with deficiency and instead seek to understand mathematics learning from the perspective of those whose identities contrast the construction of normal by dominant social groups"; they propose that "understanding" disadvantage can be discussed as understanding social processes that disadvantage individuals. And, "overcoming" disadvantage can be explored by analysing how learning scenarios and teaching practices can be more finely tuned to the needs of particular groups of learners, empowering them to demonstrate abilities beyond what is generally expected by dominant discourses." (p. 69)

With respect to teacher education in Mathematics education for equity, Healy and Powell (2013) indicated that "A consensus among mathematics education researchers concerned with preparing teachers to work with diversity and for equity is that any attempt to understand disadvantage brings into play questions of social justice." (p. 90)

"Gutstein (2006), proposes three essential knowledge bases for teaching mathematics for social justice and diversity: "classical mathematical knowledge, community knowledge, and critical knowledge". Similarly, in considering the question of what teachers need to know to support learners in bilingual and multilingual classrooms, Moschkovich and Nelson-Barber (2009) stressed the importance of addressing issues related to cultural content, social organisation and cognitive resources". (cited by Healy and Powell (2013), p. 90)

In the framework of the discourse of the relations between Mathematics and education, culture and politics emerged the concept of Ethnomathematics. The term was coined by Ubiratan D'Ambrosio in 1985 (see D'Ambrosio, 1985).

In the same vein, Villavicencio (2011) indicates that the research of Zalavsky, D'Ambrosio and Bishop have helped in the construction of a generalised conceptualisation of the capacity for Mathematical expression of a cultural group as part of its identity as their language capacity. Villavicencio also points out that in order to make Mathematics Education with an intercultural focus operational, taking into account the research mentioned and her own experiences and reflections in the experimental Project of Bilingual Math Education in Puno, Perú, she assumed this concept of Ethnomathematics: The knowledge of an identifiable sociocultural group, that implies the processes of counting, measurement, locate, design, play and explain; these processes have been identified by Bishop as the six activities that gave place to the Mathematics development in the different cultures.

Ethnomathematics approaches include four areas: in-context cognition, cultural knowledge, education and mathematical production. Bishop (2000) indicated that: Ethnomathematics refers both to the study of the relations between Mathematics and culture as to the concrete mathematical practices that are conducted in the communities where is located the school". (p. 40).

D'Ambrosio (2007) affirms that the ethnomathematics may promote a humanistic Mathematics that can be viewed as a discipline that preserve diversity and eliminate the discriminatory inequality between different types of knowledge. Also, D' Ambrosio (2005) indicates that Ethnomathematics means "a set of arts, techniques to explain, understand and manage reality of different cultural groups in their social cultural and natural environment". (p. 58)

Methodological approach

To identify the relation of native American cultures tradition on the curriculum of WNA, a qualitative approach was followed through locating, review and classify core documentation of the WNA in the Incas culture, identifying the numerical systems they constructed, the main concepts they developed, numerical representations they used, operations they did and the typical procedures to conduct them, the devices they used and the type of problems they solve in their daily lifes. Also, to identify examples of actual good practices to offer Mathematics education to children of the native American cultures from the perspective of Ethnomatematics and in the framework of intercultural education.

Results

Number systems in the Native cultures of the Latin American Countries

Many researchers have identified important developments in Mathematics in the native cultures of Latin American countries with different characteristics. For example, Mayans and Aztecs developed a base 20 numerical system while most of the cultures in the Andean region developed a base 10 system. In this paper, the focus will be on the Andean region cultures and in particular the Incas' Whole Number Arithmetic.

Number systems in Amazonic and Andean Ethnomathematics

Even though each one of the numerical systems of Quechua, Aimara, Shipibo Konibo and Asháninka cultures has their own history, it is known that all are base 10. In the table, shown in Figure 1, are presented the names of the first natural numbers until twenty and some multiples of ten that are used in these cultures.

Número	Quechua Collao	Quechua Incahuasi Cañaris	Aimara	Shipibo Konibo	Asháninka
1	*huk*	*uk*	*maya*	*westiora*	*aparoni*
2	*iskay*	*iskay*	*paya*	*rabe*	*apite*
3	*kimsa*	*kimsa*	*kimsa*	*kimisha*	*maba*
4	*tawa*	*ĉusku*	*pusi*	*chosko*	*otsi*
5	*pichqa*	*pichqa*	*qallqu*	*pichika*	*koni*
6	*suqta*	*suqta*	*suxta*	*sokota*	*iko*
7	*qanchis*	*qanĉis*	*paqallqu*	*kanchis*	*tson*
8	*pusaq*	*pusaq*	*kimsa qallqu*	*posaka*	*soti*
9	*isqun*	*isqun*	*llätunka*	*iskon*	*tin*
10	*chunka*	*ĉunka*	*tunka*	*chonka*	*tsa*
11	*chunka hukniyuq*	*ĉunka uk*	*tunka mayani*	*chonka westiora*	*tsapani*
12	*chunka iskayniyuq*	*ĉunka iskay*	*tunka payani*	*chonka rabe*	*tsapite*
13	*chunka kimsayuq*	*ĉunka kimsa*	*tunka kimsani*	*chonka Kimisha*	*tsa maba*
14	*chunka tawayuq*	*ĉunka ĉusku*	*tunka pusini*	*chonka chosko*	*tsa otsi*
15	*chunka pichqayuq*	*ĉunka pichqa*	*tunka qallquni*	*chonka pichika*	*tsa koni*

16	*chunka suqtayuq*	*ĉunka suqta*	*tunka suxtani*	*chonka sokota*	*tsa iko*
17	*chunka qanchisniyuq*	*ĉunka qanĉis*	*tunka paqallquni*	*chonka Kanchis*	*tsatson*
18	*chunka pusaqniyuq*	*ĉunka pusaq*	*tunka kimsa qallquni*	*chonka posaka*	*tsasoti*
19	*chunka isqunniyuq*	*ĉunka isqun*	*tunka llätunkani*	*chonka Iskon*	*tsatin*
20	*iskay chunka*	*iskay ĉunka*	*paya tunka*	*rabe chonka*	*pitetsa*
30	*kimsa chunka*	*kimsa ĉunka*	*kimsa tunka*	*kimisha chonka*	*mabatsa*
40	*tawa chunka*	*ĉusku ĉunka*	*pusi tunka*	*chosko chonka*	*otsitsa*
50	*pichqa chunka*	*pichqa ĉunka*	*qallqu tunka*	*pichika chonka*	*konitsa*
60	*suqta chunka*	*suqta ĉunka*	*suxta tunka*	*sokota chonka*	*ikotsa*
70	*qanchis chunka*	*qanĉis ĉunka*	*paqallqu tunka*	*kanchis chonka*	*tsontsa*

Fig. 1: The Quechua, Aimara, Shipibo Konibo and Asháninka numbers
(Source: Intercultural Basic Education. Ministry of Education Perú)

It is interesting to note in Fig. 1, for Quechua Collao, the repeated patterns in the name of the numbers making references to the base ten system.

WNA in Incas' culture

The Incas, a civilisation that existed even before the arrival of Columbus in America, had advanced engineering techniques. They had a structured organisation of their state and, in the 15th and 16th centuries, they used a cords system to make alphanumeric records to code information and solve numerical problems. These systems of cords were called "quipus". Quipus are a system of strings with different colors and different knots. The analysis of the position of the cords, the type of knots and the color of the strings are elements of logical-numerical nature. (Ascher and Ascher, 1981)

The Incas used the "Yupana" to make calculations, they were highly competent doing calculations of every type with this artefact. Most of the authors that have studied Mathematics issues of the Incas indicate that the calculation with the Yupana is done using a base 10 system. Fig. 2 shows a picture of a Yupana.

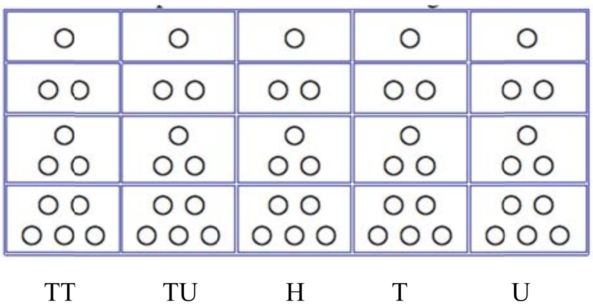


Fig. 2: A picture of a Yupana

The Yupana is used as an abacus and was utilised by accountants (quipucamayos) in the Inca Empire. The name is a Quechua word that means "What is used to count". The Yupana is a rectangular table, around 20 cm x 30 cm with five rows and four columns (in the picture, the Yupana is presented with a 90 degrees turn); its base is one of the shorter side. It has white and black circles distributed in columns; in the first column it has five circles, in the second three circles, in the third, two circles and, in the last, one circle.

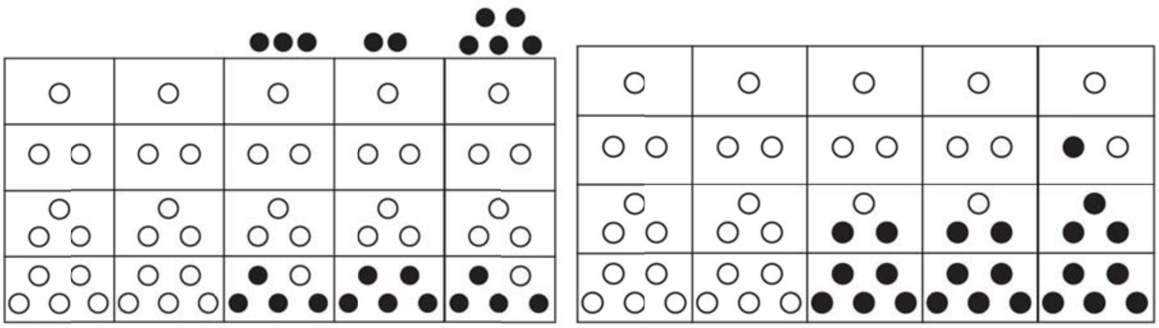
As an educational material, the Yupana is an abacus in which the units, U, are placed in the first column, the tens, T, are placed in the second column, the hundreds, H, in the third column, units of thousands, TU, in the fourth column, and the ten of thousands, TT, in the fifth column. It is used to represent numbers and also to do arithmetical operation of addition, subtraction, multiplication and division and to solve problems involving these operations.

It should be pointed out that an Italian engineer Nicolino di Pasquale, cited in Università degli Studi di Milano - Dipartimento di Tecnologie dell'Informazione (2000), proposed a hypothesis indicating that the Incas used a numerical system of base 40. This Italian mathematician explained that the system has a base of 40 and that the calculation was done from right to left and that Incas did not used the cero. He noted that "Incas based their calculation system in 40 (40, 80, 120, etc.)" but always using the same exponential criteria when they used big numbers.

Operations with the Yupana

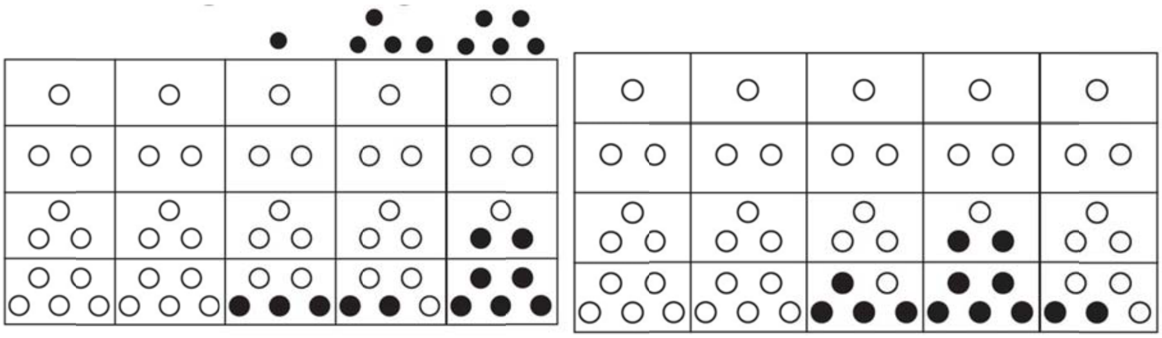
<u>Addition</u>

Example 1: Addition without regrouping, 454 + 325:



One of the addends is placed inside the Yupana as represented in the graph above using counters and the other above the Yupana. Then, the counters are grouped.

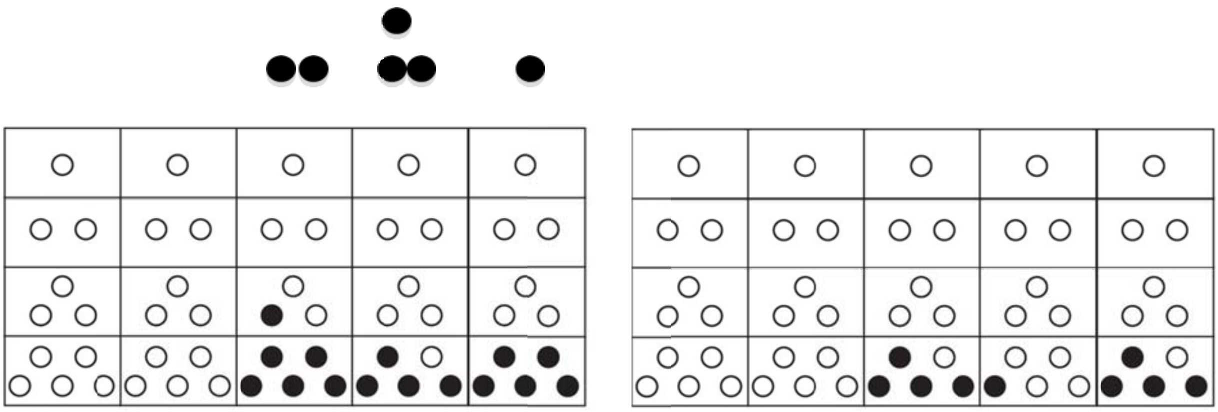
Example 2: Addition with regrouping, 327 + 145:



It should be noted that the Incas knew the way of regrouping units from one positional value to the next. In this case, twelve units were regrouped in one ten and 2 units to obtain 472.

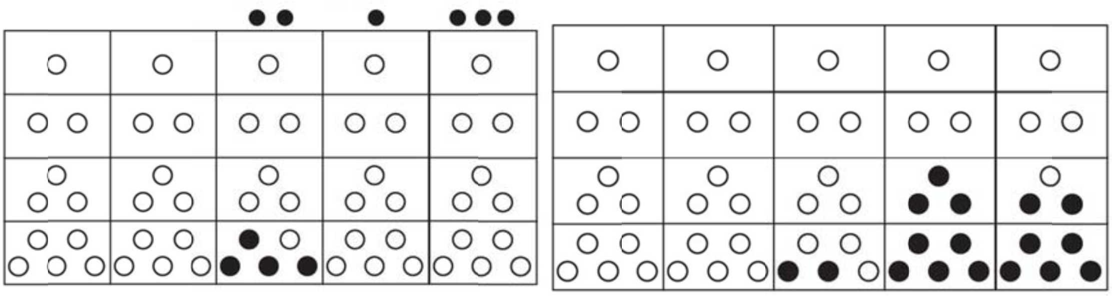
<u>Subtraction</u>

Example 3: Subtraction without decomposing, 645 – 231:



In this case, the minuend is represented inside the Yupana and the subtrahend is placed above the Yupana. The procedure is to take out the counters from each column inside the Yupana that correspond to the numbers above the Yupana.

Example 4: Subtraction with decomposing, 500 – 213:



The decomposing from a higher order unit to the next order unit is done when appropriate. In this case, previous to do the subtraction, one hundred unit was decomposed into ten units of tens and then one unit of ten was regroup as a ten units; then the subtraction was completed.

Mathematics Education in native cultures today

Today, the mathematics approach of native cultures is widely used in Peruvian schools. For example, Villavicencio has worked for more than 30 years in the region of Puno and the Ministry of Education in Perú has developed an Intercultural Education Program that is a good example of an adequate approach. Another example is the Program developed with the support of the Spanish Agency for International Cooperation (AECI) in which more than 600 teachers in Perú use the Yupana to teach in more than 200 schools in Loreto; in this program, around 14,000 children of native culture origin have benefit and teachers report that the self-esteem of children is strengthen since they identified themselves with this device that is part of their own culture.

Discussion and conclusion

This paper has discussed WNA approach of Incas Mathematics and their relation to themes on contemporary curriculum. The mediation through the use of the Incas' Yupana to do mathematical operations and the need of an

Ethnomathematical approach to teach the children of these cultures today. It is important to have instructional environments that encourage the participation and performance of multi-linguals through the use of their rich resources for mathematical sense making.

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TEACHING NUMERATION UNITS: WHY, HOW AND LIMITS

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Abstract

In French teaching practices there is currently a lack of consideration for the decimal (base ten) principle of numeration system for whole numbers. This is the reason why we implemented two experiments to strengthen this principle, giving a key role to the use of numeration units (ones, tens, hundreds …). The first one was led in the context of designing a resource for teachers in grade 3, the second one involved training teachers in grades 1 and 2. We analyze how students and teachers take into account numeration units and put them in relation with standard representations of numbers. Both exemplify the complexity of the teaching-learning process of numeration.

Key words: Base-ten, numeration units, place value, teaching, whole numbers

Introduction

As many scholars (among them Bednarz and Janvier, 1988; Kamii and Joseph 2004) we conceptualize the numeration as a network of skills about counting, grouping, representing the quantity in various ways and understanding the meaning of place-value. The place-value numeration system for whole numbers is based on two inseparable principles (Ross, 1989):

- The position of each digit in a written number corresponds to a unit (for example hundreds stand in the third place): this is the "positional principle";
- Each unit is equal to ten units of the immediately lower order (for example one hundred = ten tens): this is the "decimal principle".

Understanding this numeration system (and its relationship with oral numeration) is part of the school curriculum in all countries. Detailing what such understanding means and organizing it at primary school levels is probably less uniform: for instance Ma (1999) stresses the different ways used by US and Chinese teachers respectively to quote the decimal principle about the subtraction algorithm; in France (grades 1 to 5) the current curriculum (2008) does not refer to *place-value principles* before grade 3, and with no further details (Houdement and Chambris, 2013).

Supported with our study of the curriculum as well as some textbooks and some teachers' practices, we hypothesize an illusion of transparency of base-ten number concepts in the current French teaching (Tempier 2013). Of course it deals with position and associated value of a digit in the written number and the terms *ones, tens, hundreds …* - what we name *numeration units* (**NU**) as Chambris (2008) - are visible, principally used as names of position of each digit (positional principle). An indication of this illusion of transparency is the low percentage of success of 104 French 3-graders (8-9 year-old) in tasks involving relations between units (Tempier, 2013): "1 hundred = … tens" (48% success), "60 tens = … hundreds" (31% success) and "in 764 ones there are ... tens" (39% success). Yet, ones, tens, hundreds... are organized as a system of units: there are units of all orders and two units are always in a 10^n-to-1 ratio. The numeration units system is a named-values system (Fuson and Briars, 1990) which explains the positional base-ten system of whole numbers. This system has vector space properties (computation on numeration units) and allows different ways of writing: 5 ones 6 tens 3 hundreds or 2 hundreds 16 tens 5 units, with a number of units bigger than nine, which gives it great instrumental potential.

We are convinced of the interest of finely connecting three representation systems of numbers (Van de Walle, 2010): the two standard ways, *i.e.* the written numbers (**WN** as 56) and the spoken numbers (**SN** as fifty-six), and a third one, the in-numeration-units numbers (**NUN**) which are written as they are spoken. Teaching specifically and gradually the third system (NUN) in relation to the two others might facilitate understanding (1) of base-ten-place-value system (Tempier, 2013), (2) of computation algorithms (Ma, 1999), (3) of the decimal form of rational numbers, especially of decimal fractions. (4) It would resonate with the teaching of measurement units: length units, mass units ... (Chambris, 2008). And (5) this system provides an alternative to saying a WN (72 : 7 tens, 2 ones) without using the SN system which often doesn't reflect the way the numbers are written in Western languages (72 = soixante-douze in French, *i.e.* sixty-twelve). (6) It can also help to bridge WN and SN (65 = 6 tens 5 ones = sixty five).

The questions addressed in this paper are: how do teachers incorporate experiments supported with this proposal into their practice? What can be perceived of the acceptances or resistances of students and teachers?

Materials and methods

Two experiments supported with problems made us progress on these points: the first one in the context of a PhD thesis (Tempier, 2013), studying the use of a resource by four teachers in a design-based research (with a methodology of didactical engineering for the development of a resource) at grade 3 (8-9 year-old); the second one in a context of teacher education about six teachers of grades 1 and 2 (6-8 year-old) in the same school in a difficult area.

In both experiments we choose to introduce numeration units early in the mathematical organization of the year, so that they are available for working on computational techniques. We rely on two types of problems:

Type A. Write in WN a quantity from a collection of objects (or from a NUN): we call these problems "counting problems";

Type B. And the inverse problem that consists in producing a collection of objects (or a NUN) from a WN: we call these "ordering problems" (in reference to the ordering of a collection by a "shopkeeper").

In grades 1 & 2 the collection of objects is still present (either initially given or to build), whereas in grade 3, the work consists mainly in translating NUN in WN and *vice versa* (collection possibly used for validation). The collection consists of wooden sticks, a groupable ten base model (Van de Walle, 2010): this manipulative provides access to a first meaning of the NU, possibilities for organizing a collection that fits the various ways of writing with numeration-units. We use these possibilities of organization in the problems given to students.

Experimentation 1

Here are three problems of the resource proposed to the teachers:

	Problem A1	**Problem A2**	**Problem B3**
Type of problem	Counting a collection totally organized (in groups of tens, hundreds...)	Counting a collection partially organized (NUN→WN) resulting of the union of two collections.	Ordering a collection (WN→NUN) taking into account constraints (*e.g.* "there are no more thousands of sticks")
Mathema-tical issues	Explaining the position principle	Converting units into higher order units	Converting units into lower order units
Examples	The 3 thousands are written in the fourth position of WN: 3024.	3 thousands 12 hundreds 1 ten 5 ones = 4 thousands 2 hundreds 1 ten 5 ones (because 10 h = 1 th)	2615 = 2 thousands 6 hundreds 1 ten 5 ones = 26 hundreds 1 ten 5 ones (because 1 th = 10 h).

Tab. 1: Three problems of the resource

We indicate thereafter the various uses of the numeration units in the implementations of the resource by the teachers.

Experiment 2

In experiment 2, the eleven teachers of the whole school reported difficulties in teaching and learning place-value system (for whole number in grade 1 and 2, for decimals in grades 3, 4 and 5 and for numbers beyond 1000) and the feeling of a lack of coherence in the practices. In the continuity of experiment 1, one of the authors developed a brief training in numeration units as an essential element for understanding written numbers and spoken numbers, supported by manipulative: wooden sticks but also students' fingers. Teachers agreed with the idea of types of problems from grade 1 to grade 5, with a game on the variables (size of numbers, organizing collections and various writings used). We will only talk about grades 1 and 2 (5 teachers) which have been more thoroughly surveyed.

Teachers implemented lessons where students were to produce a number of fingers or of a given collection of wooden sticks (less or more organized) in WN or NUN form (type A problem), or a collection of fingers or sticks corresponding to a WN or a NUN (type B problem).

Results

Experiment 1

In implementing Problem A1 (Tab. 1), all the teachers involved use the NU for highlighting the link between group and position in WN. The NU are mainly used to designate these positions; this is done in the "place value chart" with NU written on the top line. In implementing Problems A2 and B3 (Tab. 1), we could observe three types of resistances to an appropriate use of NU by teachers: avoidance of the use of units to describe the material groups, predominant reference to materials when conversions are involved and failure to use NU for writing conversions.

The first resistance was observed in only one class, Mrs. A.'s. Despite the proposed resources, the teacher only used the expressions "boxes", "bags", etc. to speak of the groups. Yet, she always used the NU to describe the positions in WN. See below how she came back to the addition carrying in the A2 problem.

Mrs. A: eight plus four, what is… what is happening here in the hundreds?
A student: twelve
Mrs. A: we find our twelve bags but what is happening ?
A student: we carry 1
Mrs. A *finishes writing the algorithm of addition on the blackboard:*

Th	*H*	*T*	*O*
1			
1	*4*	*2*	*4*
+ 1	*8*	*1*	*0*
3	*2*	*3*	*4*

Mrs. A: well, that's why we carry 1, because we will keep only two bags for hundreds and here's ten that will make us a box of thousand more. [...] We now understand better the carrying: this is our small box of thousand more.

In Mrs. A.'s class, the NU are only used to label the positions of WN but not for material groups, which prevents the students from making sense of NU as quantities.

In the other classes, the NU are used to describe material groups, as proposed in the resource. However when there are more than ten units at an order, the teacher refers systematically to material groups (real or drawn). For example, in the implementation of Problem 2 (Mrs. B's class), a student (Marc) is asked to draw the union of two collections on the blackboard (a box for thousand, a bag for hundred…). Faced with Marc's difficulties with 12 bags, Mrs. B asks another student (Joris) to help him:

Joris: in fact when you have twelve, you have more hundreds [...] If you got ten hundreds, what does it correspond to?

Mrs. B: what can you do? If you got ten bags what can you do?

Marc: ah yes a thousand. *The student draws a new box on the blackboard.*

Mrs. B: The ten [hundreds] you'll put them in a box. [...] When we have ten bags we can put them in a box. [...] So as soon as it exceeds ten we can put them in a thousand box.

While students use NU orally, the teacher reformulates by reducing to material issues (put ten bags in a box). This effective reference to the material is never questioned even if it becomes an obstacle. Mrs. C explains her confusion after a lesson during which students were asked to order 8004 sticks (problem B3) from a "shopkeeper" who didn't get thousands. She feels helpless with the difficulties encountered by students because she has no alternative but to use materials: " I had no way to help them because I have not eighty hundreds." The use of materials seems the only recourse available for teachers to explain and justify transformations of different stick organisations.

In the four classes is observed a third type of resistance, the failure to use the NU for writing conversions, even if NU are orally used. For example Mrs. C never writes a conversion (such as 12 hundreds = 1 thousand 2 hundreds), except basic conversions as 1 Th = 10 H, even if conversions are mentioned. At the blackboard Mrs. B reasons on the drawings of the groups without ever writing conversions. The previous excerpt shows it: she doesn't transpose the idea suggested by Joris into writing: 12 hundreds = 1 thousand 2 hundreds. When another student explains what he did to find the total number of sticks ("seven hundred and three hundred it's doing ten"), Mrs. B does not write the associated conversion (7H + 3H = 10H = 1Th) but continues to refer to materials ("I transformed my bags, I put it in a box because I have ten"). It prevents students from taking over the writing task of conversions between units and so from strengthening their constructive abstraction (Chandler and Kamii, 2009).

Experiment 2

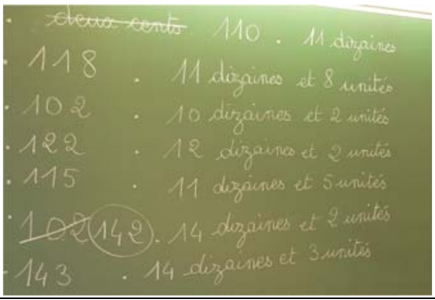
Consider two examples of students' and teacher' relation to numeration units.

In grade 2 (7-8 year-old), to count a collection (type A problem) the teacher gets her students to organizing collections in groups of ten and to using numeration-units to name bundles of sticks (tens, hundreds) and the ten fingers of a child (a ten). The teachers ask for a response without specifying its form, then the students give a standard spoken number resulting of a relatively easy counting by tens and ones (ten, twenty, thirty, thirty one, thirty two). In Tab. 2, first line, the two students of the group first counted ten, twenty…, ninety, a hundred, *two hundreds* and after the teacher noticed their error they announced to their peers *one hundred and ten*. Then the WN is collectively deduced from SN by touching the number strip and simultaneously counting one by one on the number strip, starting from a known number name:

95	96	97	98	99	100	101	102

Fig. 1: An extract of the usual number strip

When asked to count the tens (1, 2, 3 tens and 2 ones), it is more difficult, as was expected: students often waver for example between 110 tens and 11 tens on the second line. This is a sign of the difficulty of the twofold point of view on the NU: 1 ten must be understandable both as a multiplicity (ten ones) and a whole (one ten) ("composite unit of ten", Steffe, 2004; Kamii and Joseph, 2004; Houdement and Chambris, 2013*)*.



~~Two hundreds~~ 110 11 tens
118 11 tens and 8 ones
102 10 tens and 2 ones
122.........12 tens and 2 ones

Fig. 2: The table of responses (grade 2, January)

With the help of the teacher the lines are progressively written on the blackboard. Sometimes the WN is obtained first, sometimes it is the NUN, as a description of a totally organized collection. For these students there are two ways to represent collections that are not yet articulated: the WN is deduced from the written translation of the SN by using the number strip, it is never deduced from the NUN by students, or even asked for by the teacher under this form.

Despite the training the teacher may think that the WN can only be deduced from the SN. Yet spoken numbers are rather part of an ordinal logic that consists in moving forward in the counting song, number-name by number-name, relying on specific numbers (ten, twenty, thirty…named *beacon numbers* by Mounier, 2010). This logic can be an obstacle in the cardinal logic in which a hundred is considered as ten tens, not only as a successor of ninety-nine, or ten more than ninety ("Numerical Composite of Ten" *versus* "Composite Unit of Ten", Steffe, 2004).

In grade 1 (6-7-year-old), the teacher writes 47 and asks students to try to show 47 fingers, using a minimum of children with fingers up (type B problem).

The group is presenting their response. One student of the group remained in the back of the room.

The peers are protesting and saying: it does not work: *there are thirty four fingers* or *there are thirty seven fingers*! Probably the girl on the right tries to show the 4 of 47 (beside the 7). Some students suggest the girl raise ten fingers.

Fig. 3: Students producing a fingers collection for 47 (grade 1, January)

The students globally accept this suggestion (change four fingers up into ten fingers up): after an effective control by counting the problem is solved.

On the other hand when the teacher asks for the number of fingers up on the picture (Problem A2), nobody notices that there are 41 fingers up. Students seem to think that isolated ones (4 and 7) by different students cannot be put together to compose 11. Even if this work requires some flexibility –*i.e.* students being able to conceive simultaneously a ten and 10 units− this example shows how young students can rigidify their vision of quantity depending on the material: it is difficult for them to see a fourth ten in 4 fingers here and 7 fingers there (Chandler and Kamii, 2009).

Discussion and conclusion

Both experiments highlight the complexity of the teaching-learning process of the numeration. Grade 3 students seem quite ready to use numeration-units to refer to the collections; it is the teacher who checks this use by bringing them back to a description language of the materials in boxes, bundles.... Their use of the numeration units remains mostly oral or if written associated to the place-value chart. In this way they prevent students from conceptualising the relations between numeration units that are yet the goal of the session. To avoid the learning of two types of words, for materials and for numeration units (bundles and tens), we decided to introduce in grades 1-2 only the word ten (resp. hundred) to describe a bundle of ten (resp. of hundred). Teachers and a large part of students appropriate orally numeration units (tens, tens of fingers, tens of sticks) more easily than grade 3 students, and the same goes for the simple relations: 1 T=10 O, 1 H=10 T... But grade-2 students cannot yet deduce the relation between NUN and WN, despite juxtaposing the two writings on the blackboard (Fig. 2). Although informed, the teachers are surprised that students cannot deduce NUN from WN or *vice versa*, while the relation seems evident to them. But for these young students the only link between the two representations (NUN and WN) is the material and the different ways to organize it. Grade-1 students notice the presence of tens in a collection, first using the counting song ten-by-ten (ten, twenty, thirty: then 3 tens), then counting directly *the* tens. But they can also rigidify the representation of a number as full tens and isolated ones and no longer see 1 ten of fingers included in 7 and 4 fingers. This lack of cognitive flexibility is known by scholars (Chandler and Kamii, 2009), but is always difficult for teachers to understand.

Globally the in-service teachers are not fitted to manage this complexity:
- Due to a lack of institutional indicators: in French curriculum the decimal principle is only referred to as “grouping and exchanging tasks”“. The interest of numeration-units as a representation halfway between materials and WN remains hidden, the conversions seeming to be limited to manipulative.
- Due to a strong belief in some manipulative containing and directly displaying mathematical structures. Reference to the material often becomes the favourite way for teachers to help students not to forget the quantity associated with each numeration unit.
- Due to a lack of Mathematical Knowledge for Teaching (Ma, 1999; Ball et al., 2008) about the WN sense (the role of numeration units) and the differences

between WN and SN (particularly the obstacle that SN can represent for the understanding of WN).

This study might possibly indicate a need for further teacher education to enrich the teaching of WN, about the necessity (and the ways) of teaching various number representations (among them NUN) and their mutual links. We would be interested in knowing how other countries deal with this challenge.

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THE MOST IMPORTANT THING FOR YOUR CHILD TO LEARN ABOUT ARITHMETIC

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Abstract

The paper argues for a specific ingredient in "learning arithmetic with understanding": thinking in terms of "base ten pieces". In support of the argument, well-known properties of the decimal system are collected for review.

Key words: base ten/decimal system, base ten pieces, estimation, order of magnitude

Introduction

Cultural variation is one of the most prominent features of modern global life: people do things differently in different places. Language is perhaps the most obvious example of this variability, but difference can be found in almost all spheres of activity. However, there is a cultural artefact that transcends language and is almost universally used in the civilised world: the decimal system, or base ten place value system, based on the Hindu-Arabic derived symbols for the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. (The actual symbols for the digits are not quite as universal as the system itself, with variants being especially prevalent in Arabic countries and India!) Recent scholarship (Lam and Ang, 2004) suggests that the decimal system as we know it today is a written adaptation of the ancient Chinese system of calculation using counting rods. The translation of this system into written form required invention of the zero to maintain uniqueness of interpretation; this is seen as a watershed event in the development of mathematics.

The decimal system is so widely used because it is wonderful. It is a tool of remarkable sophistication and efficiency. It provides a unique way to represent each whole number, no matter how large, in compact form. Moreover, it supports computation with efficient and learnable algorithms for the arithmetic operations. It also makes comparison and estimation easy. Thus, it supports all our efforts of quantitative reasoning. In schools throughout the world, learning whole number arithmetic means studying the decimal system. The goal of this paper is to help make this study more robust by offering a suggestion regarding an ingredient of "learning arithmetic with understanding". To provide a rationale for the suggestion, basic and well-known features of the decimal system will be reviewed.

Basics of the Decimal System

How does the decimal system accomplish its marvels? The secret is in the structure behind how it represents numbers. First, each base ten number is implicitly a sum:

$$352 = 300 + 50 + 2.$$

Each number is represented as a sum of very special numbers. There seems to be no standard short name for these numbers[3]. In this article, we will just call them the “base ten pieces" of the number, or just “pieces" for short.

Furthermore, and critically, the base ten pieces have multiplicative structure. The first part of this structure is, the digit for each base ten piece records the number of copies of a *base ten unit* that is used to create that number. Thus,

$$300 = 3 \times 100, \quad 50 = 5 \times 10, \quad 2 = 2 \times 1.$$

The numbers 1, 10 and 100 are the base ten units, and the digit multiplying each unit tells how many copies of that unit are used to compose the number. The symbol for the number, 352, indicates which unit is multiplied by which digit by the position of the digit. The rightmost digit, 2, tells the number of 1s needed; the next digit to the left, 5, tells the number of 10s needed, and the digit next to the left of that, the 3, tells how many 100s are needed. Thus, the size of the base ten unit corresponding to each digit is revealed by the position of the digit in the base ten representation. This of course is why the decimal system is called a *positional*, or *place value* system.

To make this system work, it is essential to have a 0 -- a symbol representing nothing, so that if no copies of a particular base ten unit of a given size are needed to represent a given number, one can still write something in the corresponding place, which allows the digits representing larger base ten units to appear in the place corresponding to the correct unit. This principle is seen especially in the representation of the base ten pieces: 300 is made of 3 hundreds, 0 tens and 0 ones, and similarly, 50 is made of 5 tens and 0 ones. It is the 0s in the tens and ones places that signal to us that the number means what it does.

Knowing that each special number is a digit times a base ten unit does not exhaust the multiplicative structure of the special numbers. The base ten units themselves stand in a multiplicative relationship to each other. The unit “ten” consists of 10 ones. It is 10×1. Likewise, the unit “hundred” consists of 10 tens: it is 10×10. And the next base ten unit, “thousand”, consists of 10 hundreds: it is $1{,}000 = 10 \times 100$. And so on, and so on, and so on.

We can notice that the number of times we have multiplied by 10 to get a certain unit is the number of zeroes used in representing the unit. This is a happy side-effect of using positional notation, and it suggests the even more compact representation of larger numbers, by simply recording the number of factors of 10 that have gone into creating a given unit. Thus (not in elementary grades, but as part of the study of algebra), it is common to write

[3] To this author, this is a remarkable lacuna in the mathematics education literature.

$10 = 10^1, \quad 100 = 10 \times 10 = 10^2, \quad 1{,}000 = 10 \times 100 = 10 \times 10 \times 10 = 10^3,$

and so forth. We will call the power of ten that designates a given base ten unit the *order of magnitude* of that unit, or sometimes just the *magnitude*.

Thus, there is a huge amount of structure built into the conventions of the decimal system. First, each number is a sum of special numbers. Each special number is a digit times a base ten unit. Which base ten unit a digit is multiplying is shown by the location of the digit in the base ten expression. Moreover, the base ten units themselves embody a multiplicative principle, which is called forming *powers* of the base of the system, i.e., 10.

Making all this structure explicit reveals five stages of place value:

$$\begin{aligned} 352 &= 300 + 50 + 2 \\ &= 3 \times 100 + 5 \times 10 + 2 \times 1 \\ &= 3 \times (10 \times 10) + 5 \times 10 + 2 \times 1 \\ &= 3 \times 10^2 + 5 \times 10^1 + 2. \end{aligned}$$

The first stage is of course the standard form of writing numbers. The second is often mentioned in the early grades, under the name *expanded form*. It identifies and isolates the base ten pieces of the number. The next two stages make more explicit the multiplicative structure of the base 10 pieces. The third stage factors each piece into its digit times its base ten unit. The fourth stage exhibits the base ten units as products of several factors of 10, or as powers of 10. These might be called the *second expanded form* and the *third expanded form*. The fifth and final stage makes a connection with algebra: it reveals that base ten notation is a very compact way of representing numbers as “polynomials in 10". This point of view sheds light on the secret of the power of base ten representation: it is using all the structure of algebra – addition, multiplication, and exponentiation – simply to represent numbers.

Arithmetic with Base Ten Numbers

The structure of the base ten pieces and the base ten units has a remarkable consequence for computation:

> the sum or the product of an arbitrary pair of whole numbers can be found by combining calculations involving only two base ten pieces.

Moreover, the calculations with the base ten pieces reduce to single digit calculations, combined with order of magnitude considerations. Space limitations do not allow a full description here. We will make a few basic observations, and refer to (Epp-Howe, 2008) for details.

The basis for performing addition with base ten numbers is the fact that, since every base ten unit is the same multiple (i.e., 10) times the next smaller unit (and because of the Rules of Arithmetic) addition of single digit multiples of any base ten unit behaves in the same way as addition of single digit numbers themselves:

$$2 + 5 = 7, \quad \text{and} \quad 20 + 50 = 70, \quad \text{and} \quad 200 + 500 = 700.$$

And this remains true when the single digit sum is more than 10:

$$7 + 8 = 15, \quad 7{,}000 + 8{,}000 = 15{,}000, \quad \text{and} \quad 700{,}000 + 800{,}000 = 1{,}500{,}000.$$

This parallel structure of addition at all orders of magnitude gives rise to a simple method for finding the sum of any two base ten numbers. The key steps:

i) Break each number into its base ten pieces.
ii) Add each pair of pieces of the same order of magnitude.
iii) Recombine the sums into a base ten number.

Here is an example (with no regrouping; for that, see (Epp-Howe, 2008)):

$$\begin{aligned} 352 + 416 &= 300 + 50 + 2 + 400 + 10 + 6 \\ &= 300 + 400 + 50 + 10 + 2 + 6 \;=\; 700 + 60 + 8 \;=\; 768. \end{aligned}$$

The recombinations here may seem complicated, but that is only on paper. From a mental point of view, this is very close to the standard algorithm. In fact, the standard algorithm can (and should!) be seen simply as an efficient way to juxtapose the digits of the same-size base ten pieces of the addends for convenient (columnwise) addition:

$$\begin{array}{r} 352 \\ +\ \underline{416} \\ 768 \end{array}$$

Multiplication also can be handled in a similar spirit: multiplication of any two numbers can be accomplished by suitable combinations of multiplications of their base ten pieces, which amount to single-digit multiplications, combined with products of base ten units. Here the decimal system provides huge value. Multiplication was a task left to experts before the introduction of the decimal system, but afterward, anyone could do it.

The overall process is governed by the principle of Each With Each (EWE) (see (Epp-Howe, 2008)): to multiply two sums, multiply each addend of one sum with each addend of the other, and sum all the products. For example:

$$\begin{aligned} 352 \times 416 &= (300 + 50 + 2) \times (400 + 10 + 6) \\ &= 300 \times 400 + 300 \times 10 + 300 \times 6 \\ &\quad + 50 \times 400 + 50 \times 10 + 50 \times 6 \\ &\quad + 2 \times 400 + 2 \times 10 + 2 \times 6 \\ &= (3 \times 4) \times (100 \times 100) + (3 \times 1) \times (100 \times 10) + (3 \times 6) \times (100 \times 1) \\ &\quad + (5 \times 4) \times (10 \times 100) + (5 \times 1) \times (10 \times 10) + (5 \times 6) \times (10 \times 1) \\ &\quad + (2 \times 4) \times (1 \times 100) + (2 \times 1) \times (1 \times 10) + (2 \times 6) \times (1 \times 1) \\ &= 120{,}000 + 3{,}000 + 1{,}800 \\ &\quad + 20{,}000 + 500 + 300 \\ &\quad + 800 + 20 + 12. \end{aligned}$$

The product is found by adding all these 9 products of base ten pieces. The fine points of multiplication algorithms are devoted to organising the sums.

Estimation and Approximation

Comparison and estimation can likewise be handled in terms of base ten pieces. It is important to note that comparison and estimation involve considerations of size, which is a very different matter from the operations of arithmetic. Although computation usually gets more attention in the curriculum, size is the most important aspect of numbers for most applications. So the fact that the decimal system handles size comparisons as cleanly, if not more so, than the arithmetic operations, is very valuable.

For this discussion, we need to extend the term "order of magnitude" from base ten units to all numbers. We say that the order of magnitude of a base ten piece is the same as the order of magnitude of the base ten unit of which it is a (single-digit) multiple. And the order of magnitude of any whole number is the order of magnitude of its largest non-zero base ten piece. In other words, the order of magnitude is one less than the number of digits.

The base ten units increase very rapidly in size as their order of magnitude increases: each unit is 10 times as large as the next smaller one! It is hard to keep this in mind without some effort. Many people do not distinguish strongly between a million and a billion, and think of them both as "very large numbers". Of course they are, but there is a huge difference: a million is puny compared to a billion. And these days, to understand the U.S. budget, you have to deal with trillions. It may be helpful to think in terms of time. A thousand seconds is enough time to have a cup of coffee or a short lunch. A million seconds ago is 11 to 12 days -- the middle of last week. A billion seconds is over 30 years – a about half a lifetime. And a trillion seconds ago was the old Stone Age – the pyramids of Egypt were far, far in the future, and Neanderthal people were roaming Europe.

Since a base ten unit is 10 times as large as the next smaller one, and since in a base ten number, only multiples up to 9 are allowed of any unit, a single base ten unit is larger than any base ten number of smaller magnitude. This means that, for any base ten number and any order of magnitude, if we just delete all the base ten pieces smaller than that magnitude, we will of course get a smaller number; but if we delete all those smaller pieces, and add <u>just one</u> base ten unit of the given magnitude, we will get a larger number. The first procedure is called *rounding down*, and the second procedure is *rounding up*. Here are some examples:

$$350 < 352 < 360, \quad 300 < 352 < 400.$$

$$37{,}340{,}000 < 37{,}344{,}192 < 37{,}350{,}000$$

It follows from this "sandwiching" type of relationship that there is a very simple criterion for comparing two base ten numbers. Given two of them, compare the base ten pieces, starting from the largest order of magnitude. Find the largest order of magnitude for which their base ten pieces are different. Then the number with the larger piece of this order of magnitude is the larger number. Here again are some examples.

$$416 > 400 > 352; \quad 37{,}344{,}192 > 37{,}340{,}000 > 37{,}330{,}000 > 37{,}328{,}793.$$

We can regard rounded versions of a base ten number as approximations to the number. There are many reasons for working with rounded numbers. First, they are simpler to deal with – to write, to remember, to calculate with – since they have fewer pieces. Second, since each base ten unit is only $\frac{1}{10}$ as large as the next larger one, the smaller base ten pieces contribute a rapidly smaller and smaller share of the whole number. For example, if we take the number 37,344,192, and round it down to various places, we find that the rounded number captures large percentages of the whole number, and that these percentages rapidly approach 100%.

30,000,000	>	80%	of	37,344,192,	
37,000,000	>	99%	of	37,344,192,	
37,300,000	>	99.8%	of	37,344,192,	and
37,340,000	>	99.98%	of	37,344,192.	

You do not need very many of the larger base ten pieces before you have all of the number you can make practical use of. In many situations, just the leading piece will be enough. In most situations, the two leading places will be enough. In almost all situations, the three largest places will be all you need. In fact, for many quantities that you may work with, if they are reported to more than 3 base ten pieces, you should be suspicious, because the smaller pieces may well not have any basis in reality. This applies especially to figures derived using statistical methods. Consider population. It may seem a simple thing to count, but when the counting is of people, who are dying and being born and moving around, it is not. The population of even a medium sized city probably changes daily, in ways that can be difficult to keep track of. So any statement of population of a city or a state that claims to know the exact number of people there is a lie.

Thus, it is both happy for us that the decimal system provides accurate approximations with so little effort, and it is incumbent on us, in this data driven, quantitative world, to understand both the limitations of accuracy with which we can know numbers, and to be sensitive to the accuracy that base ten numbers provide. We should not use more digits than our knowledge supports. Reporting numbers to too many digits is one of the more pervasive forms of innumeracy.

The Moral

What should the student of mathematics take away from this discussion? We have seen that the basic principle of base ten notation is to decompose any whole number into special numbers, its base ten pieces. We have also seen that this structure is highly compatible with computation, and with estimation and approximation. Moreover, an arbitrary number can be expressed with high relative accuracy using only a few of its largest base ten pieces. In particular, the overall size of a number is well captured by just its largest base ten piece. Moreover, for many measurements, expressing the results with more than 2 or 3 pieces does not make sense.

The moral of this story is: the most important thing a student of arithmetic can learn is to think in terms of the pieces.

It is to be distinguished from thinking in terms of the digits. Indeed, it is possible to deal with arithmetic in terms of formal manipulations of the digits, and many students, including prospective teachers, end up doing this (Kamii, 1986, Thanheiser, 2009). However, the digits do not convey the size information that is an essential aspect of the pieces, neither the absolute size nor the relative size, and size is the main information we are seeking when we work with numbers. Understanding the base ten pieces as quantities of particular (and quite different) sizes, and working with those quantities in terms of their size, is key to understanding arithmetic. It would include routinely mentally breaking up numbers into their expanded form, understanding the nature of each piece, and doing estimation of sums and products using mental math with the two largest pieces. Also, by relating the units of various sizes to powers of 10, students should be much more ready to deal with decimal fractions and polynomial arithmetic.

Unfortunately, space does not permit discussing teaching for this way of thinking. See (Howe, 2010) for some ideas for first grade. It would also include systematically discussing the 5 stages of place value at appropriate places throughout the elementary curriculum. The study (Yang and Cobb, 1995) is consistent with the claim that thinking in terms of the pieces promotes learning, and provides examples of real-world implementation.

Your children should learn to think in terms of the pieces.

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REFRAMING PERCEPTIONS OF ARITHMETIC LEARNING: A CANADIAN PERSPECTIVE

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Abstract

Flexible computation approaches are emphasised in curriculum and teaching resources in Canada. Yet, these approaches are often unfamiliar to educational stakeholders, including parents and some teachers. The unfamiliarity potentially alienates and disempowers the very people needed to make change. While previous literature tended to dichotomise the arguments (e.g., procedural vs. conceptual; memorisation vs. understanding), we sought to reframe criticism into mutual concerns as a starting place for conversation. Through a phenomenographic analysis of online responses to newspaper articles reporting Canada's faltering PISA scores, we identified two categories of mutual concerns: (1) Expected goals of mathematics learning, and (2) Essential supports for students to reach expected goals.

Key words: Canadian Math Wars, computation, curriculum, public perception

Introduction

Our kids do not learn the basics anymore. I read through my sons [sic] grade 3 math lessons and was appalled at the method he uses for basic addition and subtraction methods. It took me some time to see where he was getting his answers. Although they were correct answers he could have saved time with the old methods. We are creating lazy minds with the methods that are taught today. (Gitersos Nazarali in Sands, 2014)

What a bunch of garbage! They are not teaching our kids the basics. There is no memorisation of the times tables. Ask a kid what 6x8 is and it will take them five minutes to come with an answer which may or may not be correct. It is a constant frustration to see them decline in ability and understanding. (Bolduc Czyz in Sands, 2014)

The two quotes above are responses to media coverage of a Canadian panel on the results of the 2012 Programme for International Student Assessment (PISA) (McGarvey, Reid, Savard and Wagner, 2014; OECD, 2013). Panel members independently emphasised that "Slipping math scores don't equal a crisis" (Sands, 2014); yet, the two scathing comments above are representative of thousands of similar remarks posted in response to Canadian newspaper reports on the 2012 PISA.

Current curricula in Canada emphasise the development of flexible and mental mathematics computation strategies through concrete materials, visual representations, and number sense knowledge. For example, rather than solving equations solely using standard algorithms, students learn and use compensation methods (e.g., $54 – 37 = 54 – 40 + 3$) and properties of numbers and operations (e.g., $8 \times 6 = 8 \times 3 \times 2$) to compute. As illustrated in the initial quotes, the unfamiliarity with these alternative strategies potentially alienate and disempower the very stakeholders needed to ensure students are successful. Yet, there is limited research on parents' perceptions of reform (e.g., Bartlo and Sitomer, 2008; Civil and Bernier, 2006; Remillard and Jackson, 2005), with no research available in Canada. Understanding perceptions of curriculum is a necessary starting point to learn how best to communicate to the public, gain support, and re-engage parents in their children's learning of mathematics.

In order to gain insight into stakeholder perceptions, we used a phenomenographic approach to analyse public responses to newspaper coverage of Canadian results of PISA 2012. We asked: What are the public's perceptions of curriculum change in mathematics? More specifically, what is the nature of their concerns? As reported elsewhere (McGarvey and McFeetors, accepted), the predominant issue raised was with regard to whole number computation skills. To offer a context for the present study we provide a brief background on the current state of mathematics education in Canada.

State of Mathematics Education in Canada

The "Canadian Math Wars" (see Chernoff, n.d.) have been in full force since the December 2013 release of the 2012 PISA results (Brochu et al., 2013). Canadian students ranked thirteenth in the world in mathematics, but media focused on Canada dropping out of the top ten (e.g., Alphonso, 2013). Although PISA is a problem-solving test for 15-year-olds, the public, including many parents, teachers, and mathematicians, targeted curriculum outcomes and pedagogical approaches related to whole number computation as the direct. Public outcry has been unprecedented. Petitions were launched in three provinces[4] and thousands of people petitioned for a "back to the basics" approach to teaching (Houle, 2013; Murray, 2013; Tran-Davies, 2013). One petition initiated by a concerned parent of a Grade 4 student claims that,

> the new strategy-based curriculum [leads to] weak understanding and poor grasp of basic mathematical concepts in addition/subtractions and multiplication/divisions, which in effect ill-equip our children to reconfigure equations in their own minds, problem-solve, and think critically… the system has clearly failed the first wave of children subjected to their grand experiment. (Tran-Davies, 2013)

While the media and public referred to the current approach as "new math," a curriculum emphasis on flexible and mental mathematics computation or "strategy-based" approach stems back to research in the 1930s. Brownell and Chazal (1935) describe how children frequently used "indirect" strategies for computation. That is, rather than memorizing that $3 + 4 = 7$ children reported thinking of it as $3 + 3 + 1 = 7$. This early research resulted in changes to

[4] In Canada, each of its 10 provinces and 3 territories has an Educational Ministry with independent responsibility for determining curricula in its respective province. Although decisions are made independently, there is significant overlap in content. At present, eight of the ten provinces and all three territories use the Western and Northern Canadian Protocol in Mathematics (Alberta Education, 2006) as the basis for its curriculum framework.

provincial curriculum documents, including the Alberta *Programme of Studies for the Elementary School* in the 1940s:

> Many of the chronic difficulties of arithmetic learning in elementary schools have their origins in early arithmetical experiences that tend to emphasise wholesale memorisation of abstract computations rather than meaningful understandings of the number system and of the fundamental processes. (Department of Education, 1947)

Current research continues to demonstrate that number sense and mental mathematics strategies have significant advantages over traditional algorithmic approaches to computation in nearly every area including problem solving, mathematical modelling, disposition towards mathematics, and equity for potentially disadvantaged students (e.g., Baroody, 1999; Boaler, 2002; Erlwanger, 1973; Russell and Chernoff, 2013). Despite research on the teaching and learning of arithmetic, attempts to communicate the advantages to parents have largely failed. Unless public concerns are more clearly understood and addressed, curricular expectations are susceptible to political lobbying and public pressure, leaving children caught in the middle.

Materials and Methods

In this section we describe a phenomenographic approach used to explicate public perceptions of teaching whole number computation. Phenomenography "investigates the qualitatively different ways in which people experience or think about various phenomena" (Marton, 1986, p. 31; Marton and Booth, 1997). Grounded in a nondualist ontology, phenomenography relies on participants' expressions of and reflections on experience. "Selected quotes are grouped and regrouped according to perceived similarities and differences along varying criteria" to map variation in the ways people experience a phenomena (Akerlind, 2012, p. 118). *Categories of description* of collective experiences are generated and further substantiated through rich description of data excerpts.

A premise of phenomenography is that there is a limited number of ways of experiencing a phenomenon. Therefore, in this study, the results of the approach provide a useful framework of public's concerns around learning arithmetic. The framework can then be used to foster future conversations with parents while keeping their concerns in mind.

Our data consists of online comments posted by readers to national newspaper articles in Canada from June 2013 to June 2014. The 62 articles published, with 5062 reader comments, addressed issues related to mathematics education in schools, discovery methods of learning, curriculum change, provincial standards testing results, and/or the PISA results. A sample of 15 articles was selected that addressed Canadian parents' perspectives on arithmetic learning. A total of 2442 online reader comments comprised the data pool.

Each post was treated as a unit of analysis and typically included one claim with a supporting example or reason as evidence for the perspective. Initially, we sorted subsets of the data set independently by noting qualitatively different concerns expressed. We then compared our sorting processes, noted commonalities, and resolved differences in categories generated through illustrative responses. A second sorting process ensued to exhaust the range of different perspectives expressed by readers. In both phases of sorting, we used a constant comparative approach (Glaser and Strauss, 1967), moving among the reader comments and emerging categories fluidly. Through our joint comparative analysis, two categories of description, each with subcategories, were generated and are reported below (see Tab. 1).

Results

Our results describe what we see as *mutual collective concerns* with regard to teaching and learning mathematics. The two categories of description include expected goals and essential supports (see Table 1) point overwhelmingly to concerns regarding the teaching and learning of whole number computation in elementary schools. Below we briefly highlight results from within each of the two categories, followed by unresolved tensions.

Categories of Description	**Subcategories:** *Reframing Criticism into Mutual Concerns*
Students need the opportunity to reach **expected goals** of mathematics learning.	*1. Need to master basic computational skills.* *2. Need to be able to problem solve.* *3. Need to be functionally numerate citizens.* *4. Need to understand mathematical principles.* *5. Need to develop discipline and intellectually through mathematics.*
Essential supports must be in place for students to reach goals of mathematics learning.	*6. Need teachers who can teach according to curriculum expectations.* *7. Need teaching resources that are clear to parents and teachers.* *8. Need to ensure success without substantial extracurricular support.*

Tab. 1: Categories of description and mutual collective concerns

Expected goals of mathematics learning

The first category of description emphasised what the public perceived as the necessary and expected goals of mathematics learning. The following quotes[5] are representative of the comments relevant to this category of description.

[5] To reference the online responses using 'in response to' (i.r.t.) along with the newspaper article author and date of publication. Given that hundreds of responses were often provided in the articles, a reference may appear more than once, but reflects different responders. Online quotes are printed as is with no changes or corrections. Links to newspaper articles are

> As a teacher in this province, I can state with experience that the biggest problems in education today [include:] We're not teaching the basics to mastery at the elementary level; as a result, kids can't apply knowledge or effectively engage in discovery or progressive learning because they have weak basic skills to start with. (i.r.t. Anderssen, 2014, Mar. 1)

> We were made to chant multiplication tables through most of grade 3. We were picked on and belittled in class if we could not instantly answered simple mental arithmetic questions. I hated it, but over 5 decades later I can do basic mental arithmetic quickly in my head. There is some room for creativity in math, but only after a rigorous foundation is established. (i.r.t. Alphonso, 2014, Mar. 25)

> I'm a computer scientist and I did not memorise the multiplication table as a kid. I think in this day and age it's much more important to teach kids to look at how they can solve these problems rather than memorising the answer. Calculators are cheap, thinkers are not. (i.r.t. Kay, 2014, Mar. 25)

> My child (grade 4) had a multiplication math test a few weeks ago and each multiplication problem had to be solved using a different strategy. It was very cool to see him write out logical answers for each strategy. (i.r.t. Alphonso, 2014, Mar. 25)

Within the category of expected learning goals we identified five subcategories of mutual concerns, including the *need to master basic computational skills* and *problem solve* being the most common concerns. For many readers, learning mathematics in elementary school is synonymous with whole number arithmetic. Readers insisted that computational efficiency is a foundation to other opportunities to learn in mathematics class. In fact, readers saw a deficiency of basic facts as closing the door to problem solving proficiency and advanced mathematics. Certainly, few would argue that both mastery of arithmetic and problem solving are important outcomes, but the process by which to achieve them and the order in which they should be achieved is often debated. The dichotomising of these skills has been a primary source of tension and yet, achieving both are seen as essential.

Essential supports for learning mathematics

In order to achieve the mutually agreed upon learning expectations stated above, our analysis revealed a second category of description referring to the essential supports needed to ensure student success with arithmetic. The following quotes again illustrate the concerns:

> Too many elementary school teachers not only don't understand math, they actually abhor it. Do teachers have to pass a math test to gain their credentials? Any teacher should be able to competently teach K to 8 mathematics. If not, then

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accessible alphabetically by author in the following dataset repository: https://sites.google.com/a/ualberta.ca/mathnewsrepository/

> they shouldn't be teachers. All teachers should have to have at least 5 yrs of 'real world' work experience. (No more go to school, go to university, go to teacher's college, and then go to work at a school.) (i.r.t. Tran-Davies, 2014, Jan. 1)
>
> I have 3 school aged children and the textbooks they bring home are badly written and the math baffling. "Nelson Math" and "Math Makes Sense" textbooks are useless. My husband and I have spend long hours pouring over convoluted world problems from these texts trying to figure them out and both of us are University grads. (i.r.t. Alphonso, 2013, Dec. 3b)
>
> I've heard from many parents who say they spend about a couple hours a night teaching their kids, and have had to give up other extracurriculars like sports or music. Other parents spend over $1000 a month on Kumon and special phonics tutors. This is not an option open to all. (i.r.t. Tran-Davies, 2014, Jan. 1)

The three quotes provide the three common themes expressed by online responders. Readers were critical of elementary school teachers' effectiveness to teach mathematics due to the likelihood that many of them had poor understanding, limited confidence, and minimal education to overcome these issues. The current strategy-based arithmetic approaches require even more of teachers than a rule-based approach. The perceived mismatch between teacher expertise and curriculum expectations is a concern acknowledged by educators and other stakeholders in education.

Criticisms of the quality and clarity of teaching resources featured in many of the responses. Readers commented frequently on the need for a sequenced and structured approach not found in their children's textbooks, but available in other programmes (e.g., JumpMath, Singapore Math, Kumon). Resources that teachers can implement systematically and parents can use to support their children's learning at home would alleviate concerns.

Given that these resources are not available or used in schools, readers expressed frustration with what they perceived of as unacceptable levels of support needed from parents and tutors to help their children attain essential computational skills. Parents complained that the current system puts more pressure on parents to do the job of teaching mathematics. More disconcerting was the potential for a two-tier system created by parents who paid for tutoring for their children.

The perceptions regarding expectations for learning and the supports needed to ensure student success revealed that the public concerns were not in opposition to what we might want as mathematics educators. Emphasising the alignment in these areas offers the potential for communicating to parents in ways that acknowledge their concerns. Yet, we also identified perceptions that appear to diverge from these commonly shared goals.

Unresolved differences

While the vast majority of responses could be placed in one of the two categories of description identified above, we revealed several perceptions that could not be resolved as mutually held concerns. The following quotes provide a few examples of these divergent perceptions.

> The thing is that not everyone is going to be engineers or scientists. Thus, not everyone needs to have the complex deep understand of the underlying nature of the equations used. This is why wrote learning works so well for the vast majority of people. In order to get by in society, a basic knowledge of wrote learned equations will suffice for most situations (eg. memorising times tables or learning long division). (i.r.t. Wente, 2014, Mar. 4)

> The number of mathematically challenged youth is staggering – incapable of anything beyond the most rudimentary arithmetic, if that! It is an absolute disaster, brought on by teaching for self-esteem, instead of teaching for results. (i.r.t. Hopper 2014, Feb. 28)

> Let's stop exposing our children to these unproven experiments in the classroom and get back in the business of teaching them math. (i.r.t. Wente, 2014, Mar. 4)

Although the number of remarks that didn't fit into one of the two categories was minimal, the three examples above and others make us pause. How is it possible to communicate the purpose of a strategies-based approach to computation when perceptions by some people are that there is no need to understand mathematics or that mathematics educators are not providing the evidence needed to demonstrate that the current approach is appropriate? Knowing that there are divergent approaches to teaching and learning whole number arithmetic, is it possible to offer sufficient evidence to these individuals?

Discussion and conclusion

Numerous national newspaper articles with many online reader comments indicate a Canadian public interested and invested in how children learn arithmetic in elementary school. We often, as mathematics educators, read comments by the public as explicit criticisms of our carefully constructed research and thoughtfully implemented changes in mathematics curricula. At the same time, we take seriously their concerns, and have found them to be informative as we consider how to better communicate meaningful approaches to teaching and learning arithmetic.

Through our analysis, we were able to lay aside the rhetoric of oppositional stances in approaches to learning computation to identify and describe commonalities. Our findings highlight our shared goals for children's mathematical learning and particular supports necessary. The public's expressed perceptions provide a starting place for us, as mathematics educators, to begin engaging in purposeful conversation with the public, and parents in particular. Rather than beginning with the processes of learning computations, we can see how collectively identifying with parents what we expect for children's learning is a more productive approach to eliciting support.

We suggest that similar studies of the public's perception of mathematics learning be taken up in other countries that participate in the PISA test in order to substantiate and broaden the categories of description presented in this paper. An international comparison has the possibility of demonstrating commonalities beyond countries' borders. And while the public's perception provides a backdrop, we recommend that parents' perceptions of mathematics curriculum change and arithmetic learning be solicited in order to develop approaches to re-engaging parents with their children's mathematical learning in schools.

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FOUNDATIONAL NUMBER SENSE: THE BASIS FOR WHOLE NUMBER ARITHMETIC COMPETENCE

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Abstract

Children begin school with different number-related competences, typically due to variation in geographical location and familial circumstances. This variation, which necessarily creates inequity of opportunity, prompts the question, what number-related experiences are necessary if the first year of school is to prepare children equally and adequately for their learning of the mathematics? To address this question we summarise recent work on the development of an eight dimensional framework, which we have called foundational number sense (FoNS) that characterises those necessary learning experiences. We then show how FoNS can be simply operationalised for analysing the learning opportunities offered grade one students in five European contexts. The results indicate that FoNS, as an analytical tool, is not only cross-culturally sensitive but has the propensity to inform developments in curriculum, assessment and teacher education.

Key words: England, foundational number sense, Hungary, Poland, Russia, Sweden

Introduction

The quality of a child's basic number understanding is known, internationally, to be a strong predictor of later arithmetical competence (Aubrey and Godfrey, 2003; Aunola et al., 2004). For example, simple counting skills have been implicated in the arithmetical competence of students in Canada, England, Finland, Flanders, Taiwan and the USA respectively (LeFevre et al., 2006; Aubrey and Godfrey, 2003; Aunola et al., 2004; Desoete et al., 2009; Yang and Li, 2008; Jordan et al., 2007). Also, children who start school with a poorly developed understanding of number remain low achievers throughout school (Geary, 2013). In this paper, focusing on the first year of schooling and from an international perspective, we examine the competences research has shown will both avoid later difficulties and ensure later success.

Underpinning much research in this area has been the concept of number sense, which has rarely been defined adequately (Griffin, 2004), due to psychologists and educators working with different conceptualisations (Berch, 2005). This problem has been exacerbated by psychologists, typically working in general cognition or learning disabilities, also employing different definitions. That said, our reading reveals three distinct but related perspectives on number sense, which here we label preverbal, applied and foundational.

Preverbal number sense reflects those number insights that are innate to all humans and comprises an understanding of small quantities in ways that allow for comparison (Ivrendi, 2011; Lipton and Spelke, 2005). For example, young babies can discern 1:2 but not 2:3 ratios (Feigenson et al., 2004). This, numerical discrimination, which "becomes more precise during infancy" (Lipton and Spelke, 2005, p.978), is independent of formal instruction and develops innately as a consequence of human, and other species' evolution (Dehaene, 2001; Feigenson et al., 2004).

Applied number sense concerns those number-related competences that make mathematics sensible and prepare learners for an adult world (McIntosh et al., 1992). It underpins many curricular specifications and much of the material written on number sense (See, for example, Anghileri, 2000).

Foundational number sense (FoNS), the primary focus of this paper, comprises those understandings that require instruction during the first year of school (Ivrendi, 2011; Jordan and Levine, 2009). Unlike preverbal number sense, it is a "construct that children acquire or attain, rather than simply possess" (Robinson et al. 2002, p. 85). Unlike applied number sense, it does not seek to facilitate a world beyond school but underpins later arithmetical and mathematical competence. It has been argued that these basic number competences are to the development of mathematical competence what phonic awareness is to reading (Gersten and Chard, 1999).

Materials and Methods

In this paper we summarise the key components of FoNS. Our intention was not to construct an extensive list of characteristic learning outcomes, as found in Berch (2005) or Howell and Kemp (2006), but a concise conceptualisation that would support a range of activities, including developments in curriculum, teacher education or assessment, as well as cross-cultural classroom analyses. In so doing we aimed to see beyond how different curricular traditions frame such matters (Aunio and Räsänen, 2015). To achieve these objectives we exploited, in an atypical manner, the constant comparison analysis advocated by grounded theorists. Peer-reviewed research articles and book chapters typically addressing grade one students' acquisition of number-related competence were identified. These were read and FoNS-related categories identified. With each new category, previous articles were re-examined for evidence of the new. This approach, placed, for example, *rote counting to five* and *rote counting to ten*, two narrow categories discussed by Howell and Kemp (2005), within the same broad category of systematic counting. Among the works examined in this process were Aubrey and Godfrey (2003), Aunola et al., (2004), Berch, (2005), Booth and Siegler (2006), Clarke and Shinn (2004), Dehaene (2001), Desoete et al. (2009), Gersten and Chard (1999), Gersten et al. (2005), Griffin (2004), Howell and Kemp (2005), Hunting (2003), Ivrendi (2011), Jordan et al. (2007), Jordan and Levine, (2009), Lipton and Spelke (2005), LeFevre et al. (2006), Lembke and Foegen (2009), Malofeeva et al. (2004), Noël (2005), Thomas et al. (2002), Van de Rijt et al. (1999), and Yang and Li (2008). In the following, to avoid repetition and save space, each component is summarised independently of the literature on which it is based. However, no single reference underpins all components.

Number recognition: Children recognise number symbols and know their vocabulary and meaning. They can identify a particular number symbol from a collection of number symbols and name a number when shown that symbol;

Systematic counting: Children are able to count systematically and understand notions of ordinality and cardinality. They count to twenty and back, or count upwards and backwards from arbitrary starting points, knowing that each number occupies a fixed position in the sequence of all numbers.

Awareness of the relationship between number and quantity: Children understand not only the one-to-one correspondence between a number's name and the quantity it represents but also that the last number in a count represents the total number of objects.

Quantity discrimination: Children understand magnitude and can compare different magnitudes. They use language like bigger than or smaller than. They know that eight represents a quantity that is bigger than six but smaller than ten.

An understanding of different representations of number: Children understand that numbers can be represented differently, including the number line, different partitions, various manipulatives and fingers.

Estimation: Children can estimate, whether it be the size of a set or an object. Estimation involves moving between representations of number; for example, placing a number on an empty number line.

Simple arithmetic competence: Children perform simple arithmetical operations, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction.

Awareness of number patterns: Children extend and are able to identify a missing number in a simple.

In sum, our literature review identified eight distinct but not unrelated FoNS categories. The fact that they are not unrelated is important as number sense "relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures" (Gersten et al. 2005, p. 297). In other words, without the encouragement of such links there remains the possibility that children may be able to count but not understand, for example, that four is bigger than two.

Having derived an eight component FoNS-related entitlement for grade one students, it was important to examine whether the framework would identify FoNS-related opportunities in grade one classrooms in varying cultural contexts. Piloting in this way would allow for instrument refinement and an evaluation of its sensitivity to cultural nuances. In the following we summarise recent case studies, each undertaken in different European contexts and with different mathematical topics, in which we have evaluated the efficacy of the FoNS framework. The first examined the teaching of sequences in English and Hungarian lessons (Back et al., 2014), the second the development of students' conceptual subitising in Hungary and Sweden (Sayers et al., 2014), while the third teachers' use of the number line in Poland and Russia (Andrews et al., 2015).

None of the examined lessons, which were typical of their teachers' practice, were captured with a FoNS-related analysis on mind, but all proved amenable to one. Each lesson, derived from previously collected and unrelated data sets, had been video-recorded for research or teacher education purposes, and all teacher utterances recorded. Each teacher, against local criteria, was construed as effective. All recordings were supplemented by transcripts of all the utterances made by teachers themselves and as much students talk as the recordings allowed. Each lesson was viewed repeatedly by two researchers, and, in the manner of the Mathematics Education Traditions of Europe (METE) project (Andrews, 2007), lesson episodes were scrutinised for evidence of FoNS components. This approach allows for episodes to be multiply-coded according to which components were observed. Thus, as found by the METE project team, uncomplicated analyses based on frequencies (Andrews, 2009) and complex analyses based on the interactions of codes in each episode (Andrews and Sayers, 2013) are made possible.

As data derived from different projects in five different countries, ethical procedures and permissions were managed according to local norms. In all countries permission from school principals and participating teachers was obtained by means of letters confirming the right of teachers to withdraw without notice or reason and anonymity. With respect to the Hungarian, Polish and Russian students, all parents, at the point at which their children entered their school, had signed to agree their child’s participation in ethically conducted classroom based research. In England and Sweden, parental permission letters explained the projects and, alongside the promise of minimal classroom disruption, guaranteed the same protective principles as above.

Results

In the following we present the pilot case studies introduced above. Space prevents a detailed summary, not least because it would normally be necessary to offer images to show how classroom tasks played out. That said, we believe that sufficient can be gleaned to demonstrate the sensitivity of the FoNS framework to both cultural context and mathematical context.

In the first study (Back et al., 2014), episodes focused on number sequences were analysed. In addition to examining the functionality of the FoNS framework an aim was to examine how teaching, focused explicitly on one FoNS component, would yield other components. The analyses, based on three episodes from each lesson sequence, indicated that Klara in Hungary addressed six of the eight FoNS components while Sarah in England addressed four. Both encouraged, throughout their respective excerpts, students’ recognition of number symbols, vocabulary and meaning. Both encouraged the awareness of number patterns and the identification of missing numbers and both exploited simple arithmetical operations, typically to facilitate finding the next values in a sequence. In respect of differences Klara addressed three categories, the relationship between numbers and quantities, comparisons of magnitude and different representations of number that Sarah did not, while Sarah was seen to address systematic counting when Klara did not.

However, while both teachers encouraged various FoNS components, Klara’s teaching was more didactically complex, with an average of four components per episode, than Sarah's, with an average of barely two. Moreover, Klara encouraged mathematical reasoning, while Sarah seemed to subordinate such reasoning to an examination of the coloured patterns on her interactive whiteboard. Leading us to conclude that if number sense develops gradually as a result of exploring and visualising numbers in different contexts (Sood and Jitendra, 2007) then Klara’s practice seemed more richly focused than Sarah’s.

In the second study (Sayers et al., 2014), analyses focused on conceptual subitising in grade one lesson sequences taught by Klara, again, in Hungary, and Kerstin, in Sweden. Conceptual subitising, which relates to how an individual identifies “a whole quantity as the result of recognizing smaller quantities... that make up the whole” (Conderman et al., 2014, p.29), has been promoted as a key component of early number learning (Clements, 1999). It can be summarised as the systematic management of perceptually subitised numerosities to facilitate the management of larger numerosities. In both cases, an average of five FoNS components were identified in each of the teacher's three analysed episodes, indicating that claims for the efficacy of teaching focused on conceptual subitising (Clements, 1999; Conderman et al., 2014), are not without warrant.

It was also interesting to note that in neither case was conceptual subitising an explicit intention, nor were teachers expecting to address FoNS categories of learning. It is also interesting to note that despite substantial differences in the management of their lessons - Klara spent all her lesson orchestrating whole class activity with only occasional expectations of students working individually, while Kerstin spent the great majority of her time managing and supporting students working in pairs - the FoNS components addressed in their respective excerpts were remarkably similar.

Finally, the third pilot study (Andrews et al., 2015) examined episodes drawn from lesson sequences focused on the introduction and exploitation of the number line taught by Olga, in Russia, and Maria, in Poland. Here the analyses, as in the first case study, showed that such a didactical emphasis on one FoNS component does not necessarily restrict opportunities for other FoNS outcomes. For example, Olga's episodes addressed an average of almost five components, while Maria's almost four. Not surprisingly, bearing in mind the number line emphasis, all analysed episodes addressed number recognition and systematic counting, while all but one showed evidence of children being asked to work with a different representation of number.

With respect to differences, whenever Olga asked her students to represent a number on the number line, she insisted on their pointing simultaneously to zero with their left hand and the desired number with their right. In this manner her teaching focused on the relationship between number and quantity. By way of contrast, Maria presented simultaneously three distinct number lines, each showing zero to eight but with different sized intervals. In so doing she highlighted the arbitrary size of the interval alongside the need for a consistent interval size. Both teachers also used the number line to facilitate simple arithmetical operations, including tasks involving several operations simultaneously. Finally, Maria also used the number line in relation to number patterns, particularly even numbers and the identification of missing numbers.

Discussion and conclusion

Acknowledging that all classrooms are constructed by the cultures in which they are found, the FoNS framework has proved amenable to the identification of FoNS-related opportunities in typical grade one classrooms in five different European contexts. Thus, as in the METE project, a simple to operationalise, episode-focused coding scheme has the potential to yield a variety of analyses, particularly when undertaken comparatively. Indeed, our next step is to undertake a more extensive comparative study of the FoNS-related opportunities teachers offer their students. Finally, the framework's simple structure should facilitate initial teacher education interventions. For example, the eight components, each amenable to an in-depth examination, provides a manageable framework for university-based teaching and the assessment of students during their school placements. Similar arguments apply to curriculum development.

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PEDAGOGICAL LESSONS FROM *TONGWEN SUANZHI* (同文算指) – TRANSMISSION OF *BISUAN* (筆算 WRITTEN CALCULATION) IN CHINA

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Abstract

In 1613 the official-scholar LI Zhi-zao (李之藻) of the Ming Dynasty, in collaboration with the Italian Jesuit Matteo RICCI (利瑪竇), compiled the treatise *Tongwen Suanzhi* (同文算指). This is the first book which transmitted into China in a systematic and comprehensive way the art of written calculation that had been in common practice in Europe since the sixteenth century. This paper tries to see what pedagogical lessons can be gleaned from the book, in particular on the basic operations in arithmetic and related applications in various types of problems which form the content of modern day mathematics in elementary school education.

Key words: basic operations in arithmetic, problems in arithmetic, *Tongwen Suanzhi*, written calculation.

Introduction

In the early part of the seventeenth century the official-scholar LI Zhi-zao (李之藻 1565-1630) of the Ming Dynasty, in collaboration with the Italian Jesuit Matteo RICCI (利瑪竇 1552-1610), compiled the treatise *Tongwen Suanzhi* (同文算指, literally meaning "rules of arithmetic common to cultures") (Li and Ricci, 1613/1993), which first transmitted into China in a systematic and comprehensive way the art of written calculation that had been in common practice in Europe since the sixteenth century. This treatise, accomplished in 1613, was a compilation based on the 1583 European text *Epitome Arithmeticae Practicae* (literally meaning "abridgement of arithmetic in practice") of Christopher CLAVIUS (1538-1612) and the 1592 Chinese mathematical classic *Suanfa Tongzong* (算法統宗, literally meaning "unified source of computational methods") of CHENG Da-wei (程大位 1533-1606) (Cheng, 1592/1993). This work is also an attempt of LI Zhi-zao to integrate European mathematics with traditional Chinese mathematics, which was a prevalent intellectual trend of the time known as *zhongxi huitong* (中西會通, literally meaning "integration of Chinese and Western [learning]), started by the dedicated work of another official-scholar XU Guang-qi (徐光啟 1562-1633) who translated the first six books of Euclid's *Elements* (from a fifteen-book version compiled by Christopher CLAVIUS in the latter part of the sixteenth century) also in collaboration with Matteo RICCI and published it as *Jihe Yuanben* (幾何原本, literally meaning "source of quantity") in 1607 (Siu, 2011).

The aim of the present paper is not to present a historical study of the book — its content, its historical context and its influence on Chinese mathematics in the eighteenth-century Qing Dynasty. For that historical aspect interested readers are invited to consult some more scholarly works (Takeda, 1954; Chen, 2002). Rather, we try to see what pedagogical lessons can be gleaned from the book, in particular on the basic operations in arithmetic and related applications in various types of problems which form the content of modern day mathematics in elementary school education.

Basic operations in arithmetic

The first book in the first part (Preliminary Part) of *Tongwen Suanzhi* explains the notation in positional system and the four basic operations in arithmetic. Apart from division the other three operations – addition, subtraction, multiplication – are done in the way a schoolboy of today is familiar with. Division is done by the so-called *galley* method, which will be illustrated below. The second book deals with the arithmetic of fractions ending with a collection of miscellaneous problems to consolidate the skill in written calculation that has just been learnt.

It would be instructive to compare the transmitted algorithms with the methods in traditional Chinese mathematics. We will look at how multiplication and division were done in ancient China as explained in *Sunzi Suanjing* (孫子算經, literally meaning "Master Sun's mathematical manual") of the fourth/fifth century (Lam, 1966; Lam and Ang, 1992/2004; Guo, 1993). The following two examples are taken from (Lam and Ang, 1992/2004) (see Fig. 1 and Fig. 2, with the last item in modern notation inserted for comparison and with rod numerals replaced by what we are familiar with today).

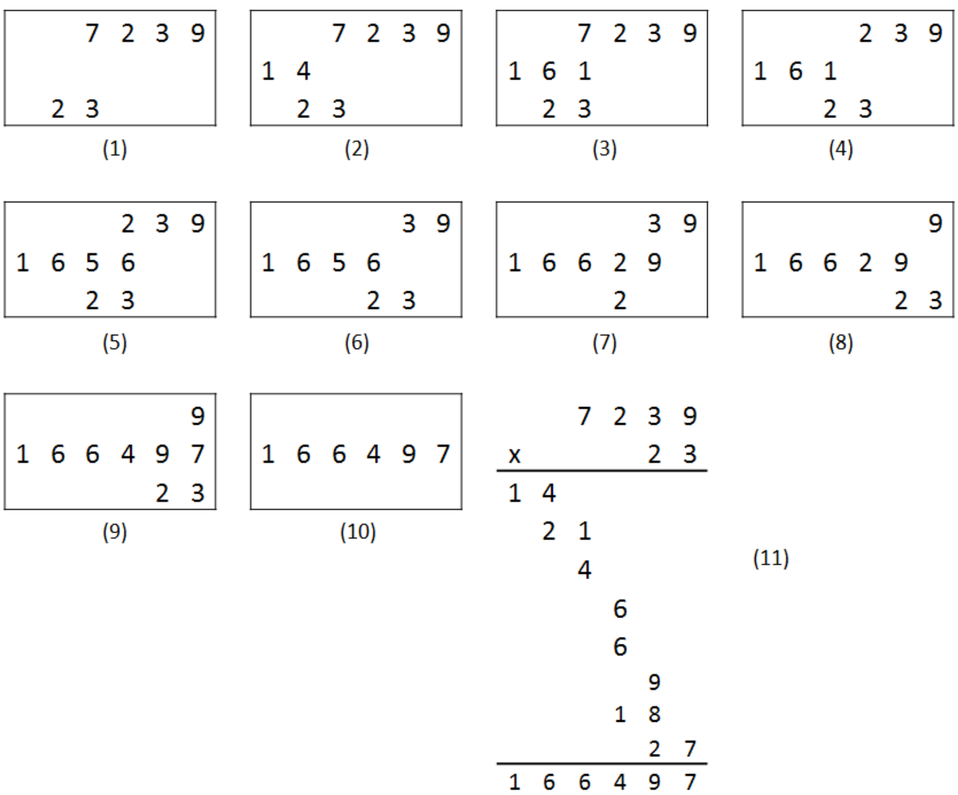

Fig. 1: Multiplication in ancient China

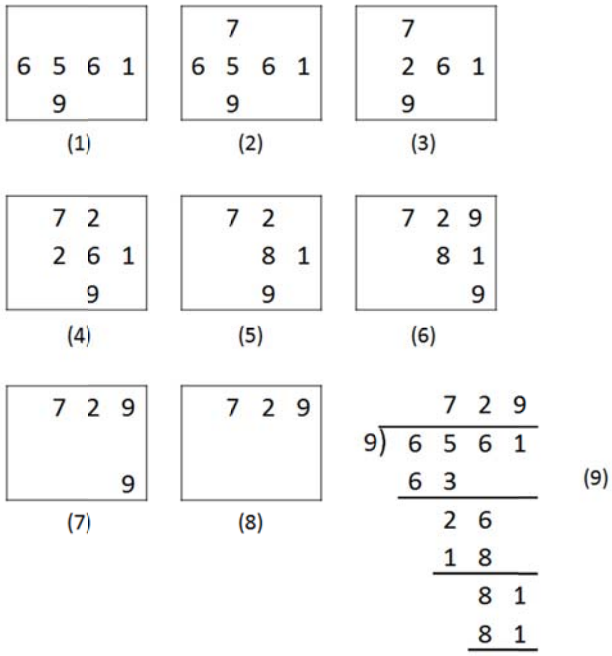

Fig. 2: Division in ancient China

In the Western world there was a movement of contest in efficiency of reckoning between the so-called "abacists" and "algorist" towards the latter part of the Middle Ages (Swetz, 1987). In particular, a method known as the *gelosia* method, coming from the Islamic world, was commonly used at the time. Written calculation did appeared in Chinese texts before *Tongwen Suanzhi*, for instance, in *Jiuzhang Suanfa Bilei Daquan* (九章算法比類大全, literally meaning "comprehensive collection of computational methods in *Nine Chapters* devised by analogy [with ancient problems and rules] ") of 1450 by WU Jing (吳敬 15th century) (Wu, 1450/1993) and in *Suanfa Tongzong* of 1592 by CHENG Da-wei (Cheng, 1592/1993), but not in a way as systematic and as comprehesive as in *Tongwen Suanzhi*. In both of these texts the *gelosia* method was introduced into China, called by CHENG Da-wei by the picturesque name of *pudijin* (鋪地錦, literally meaning "covering the floor with a glamorous carpet"). LI Zhi-zao seemed to prefer the more modern method to this pictureque *pudijin*. However, the latter may provide interesting exercise for a modern classroom (see Fig. 3, with the last item in modern notation inserted for comparison).

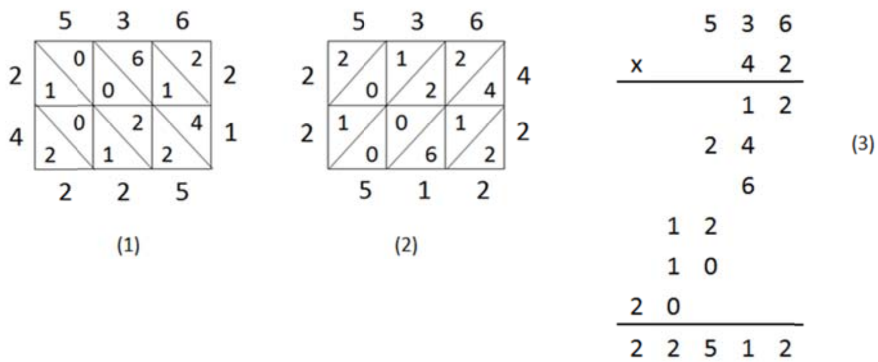

Fig. 3: *Gelosia* method of multiplication

In *Tongwen Suanzhi* division is performed by the *galley* method. Let us illustrate this method by an example taken from (Lam, 1966), which in turn appeared in the *Treviso Arithmetic* of 1478 (Swetz, 1987) (see Fig. 4, with the last item in modern notation inserted for comparison) .

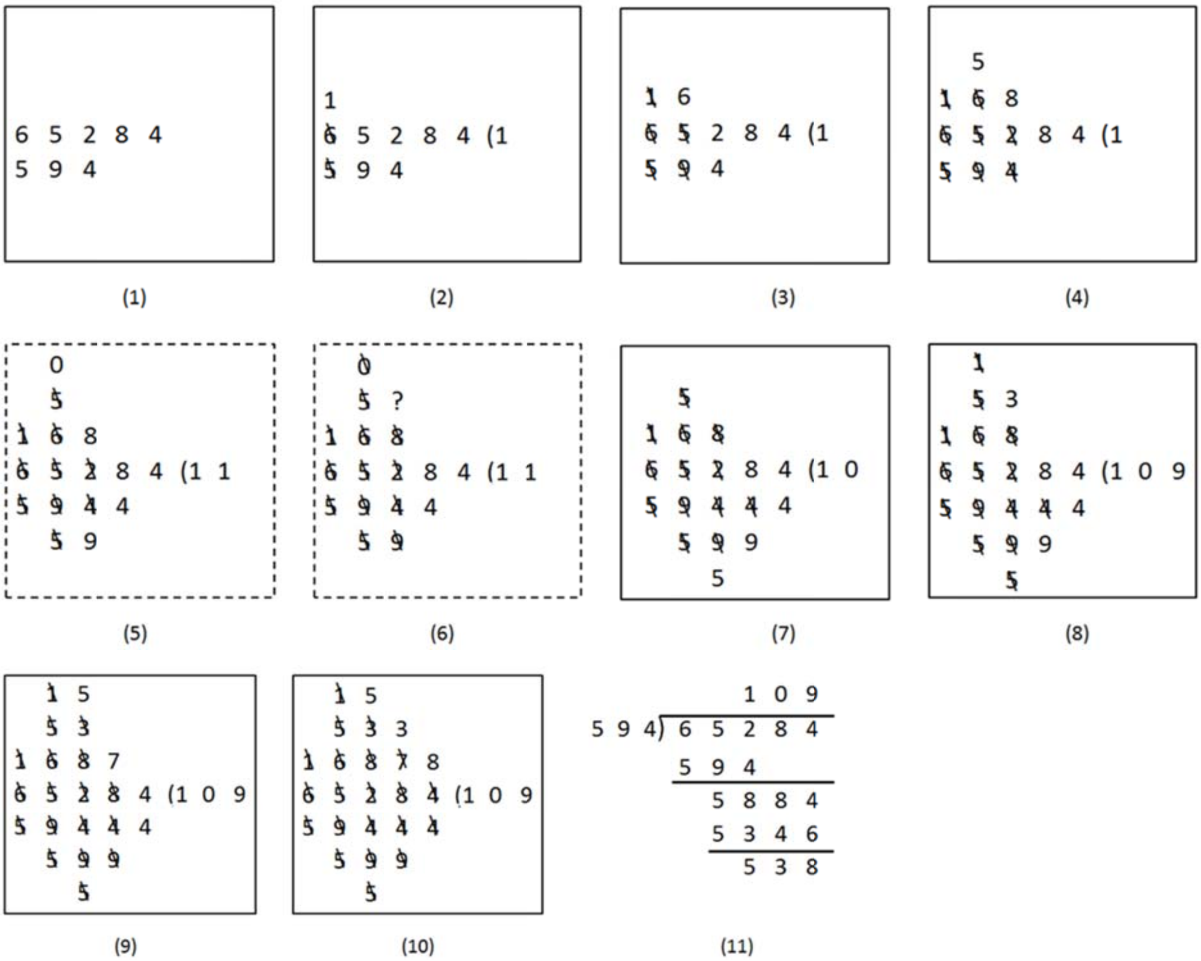

Fig. 4: *Galley* method of division

All the examples given above can be suitably utilized to prepare worksheets to enhance the understanding of the reasoning underlying the basic operations in arithmetic. To keep the paper within prescribed length the discussion on pros and cons of various methods is not included, but its importance to teachers and mathematics educators is certainly recognized (Fuson and Beckmann, 2013).

Another interesting feature in the first book of *Tongwen Suanzhi* is the discussion on methods of checking the answer. For each basic operation several methods are given. One is applying the reversed operation to the answer, which is logically questionable (as far as the presentation in a textbook is concerned) because before subtraction (respectively division) is explained one should not make use of it to check the answer obtained by addition (respectively multiplication)! On the other hand it indicates the awareness of the mutual reversibility of the operations involved. A second method is to carry out the operation with the given numbers in a different order, which indicates the awareness of the commutative law of the operation. The most interesting method is casting out nines (or sevens), which indicates an awareness of modulo arithmetic. As expected, it was assumed that the checking works without bothering about the mathematical fact that the method using modulo arithmetic tells when the working is wrong but cannot guarantee that the working is correct. Compared to the practical usefulness of these methods, such logical slip is a minor blemish.

More problems in arithmetic

The second part (General Part) of *Tongwen Suanzhi* contains a large collection of various problems, which appeared in mathematical texts in traditional Chinese mathematics such as *Suanfa Tongzong* of CHENG Da-wei, which in turn were handed down from the famous ancient Chinese mathematical classic *Jiuzhang Suanshu* (九章算術, literally meaning "nine chapters on the mathematical art") compiled between the second century B.C.E. and the first century C.E. (Guo, 1993). These problems are treated in the newly introduced method of written calculation, thus amplifying the attempt of LI Zhi-zao in integrating Chinese mathematics with European mathematics.

Let us look at just two examples out of the many. The first example is on proportion treated by the method known in the Western world as "Rule of Three", or the so-called "Golden Rule". The second example is on "extraction of square root with an accompanying term", which is solving a quadratic equation in the Western term.

Example 1 (in Section 1) . "Suppose 4 *guan* (貫) of money can purchase 12 *jin* (斤) of goods, how many *jin* of goods can 20 *guan* purchase?"

Example 2 (in Section 14) . "Suppose a rectangular field has area 864 [square] *bu* (步) and its width is less than its length by 12 *bu*, what is its width?"

Some pedagogical lessons

(1) The techniques developed at the time fit in with the historical development of the time. One example is how the high cost of paper at the time explains why the *galley* method of multiplication was preferred before long division familiar to us today was developed and adopted later in history. Another example is the very detailed explanation in the calculation using fractions that arose as a necessity in commercial activities involving a diversity of European currencies. Indeed, the rise of written calculation has a lot to do with the upsurge of commercial activities since the sixteenth century in Europe (Swetz, 1987). The lesson for us is that the design of curriculum has to take into account contemporary need (or diminishing need) so as to ride with time.

(2) Old techniques may provide instructive exercises for the benefit of teaching and learning. One example is multiplication by the *gelosia* method, also known as the *grid* method. Another example is division by the *galley* method. Still another example is the method of casting out nines. These techniques are no longer necessary skills to be learnt nowadays but may offer good ways to understand and to consolidate understanding if employed in a thoughtful manner. It may also add a humanistic touch to mathematical lessons by letting students see how people did things in the past.

(3) The first part (Preliminary Part) of *Tongwen Suanzhi* explains the four basic operations in arithmetic and the calculation with fractions. The second part (General Part) is a comprehensive account on various problems treating proportion, extraction of square and cube roots, method of double false position, and solving linear and quadratic equations. The third part (Special Part), which was undated and short, introduces basic knowledge in trigonometry. The second part plays a central role in which the authors tried to make use of these problems to consolidate the skill in written calculation, at the same time indicating the prowess of written calculation. This textbook design based on such a pedagogical objective is far superior to that of a heavy load of straightforward but boring drilling exercises in some modern day textbooks!

The emphasis *Tongwen Suanzhi* placed on the learning and teaching of arithmetic exerted influence on the subsequent writing of textbooks in China. Instead of paying attention to teaching algorithms through the aid of mnemonic poems the underlying reasoning was brought into the study. The use of counting rods and the abacus was gradually replaced by the use of written calculation.

(4) In ancient times people calculated by using manipulatives such as pebbles, sticks, counting rods, abacus, etc. By today's standard one may see these as clumsy and inefficient. However, with sufficient practice this needs not present an obstacle to efficient calculation. For an expert who had acquired the skill it can even mean a quick and convenient method. Likewise, the adoption of an ancient recording system of numerals by the grouping method may seem cumbersome to a modern day schoolboy but not so for an ancient Egyptian scribe well versed in the art of calculation. Hence, what is so good about positional system in numeration and what is so good about written calculation?

The main advantage of written calculation lies in keeping intact a record of the intermediate steps which affords easy checking afterwards. It also allows one to see the procedure and to gain understanding of the underlying reasoning without having to memorize what is going on during the calculation. This is difficult to attain in calculation using manipulatives (although nowadays calculation by using manipulatives can gain its own pedagogical advantage in learning). Along with this benefit the advantage of positional system is revealed. Without the invention of a positional system written calculation as we know it cannot be invented.

But then this leads us to the next question in this age of computers. Ironically we are turning back the wheel of history in some sense in that we erase the intermediate steps when we calculate by punching a few keys on an electronic calculator! For all practical purposes it is definitely much more efficient to calculate by using an electronic calculator than to calculate by hand, just like one would not like to cook by setting up a fire instead of using a kitchen stove. It is true that because of that the emphasis in learning calculation would be shifted to skill and knowledge in estimation in order to guard against careless manipulation or errors in the calculating machine. But do we still need to pay so much attention to written calculation in schools? For instance, in some places there is a suggestion for de-emphasis of long division.

The rationale for learning written calculation, at least once in a person's lifetime, seems to be the acquirement of understanding of the underlying principle of the basic operations in arithmetic. For some learners this kind of understanding is essential in future endeavour. Let us just cite one example about a commonplace operation as multiplication. For the computation with very large numbers various algorithms had been developed to speed up the time by reducing the number of steps of simple multiplication of one-digit numbers, for instance the Karatsuba algorithm in the early 1960s, the Schönhage-Strassen algorithm in the early 1970s and the Fürer algorithm in the late 2000s (Gathen and Gerhard, 1999/2003/2013). In order to devise such algorithms one has to understand the underlying principle of multiplication. Admittedly, only a fairly small percentage of the population of all school pupils will need to have that kind of understanding in their future career. But it does not seem advisable to teach it only to these selected few after they reach a more advanced and specialized level. If it is going to be taught at all it would be advisable to teach it to all at the elementary school level. To include this topic in the elementary school level we can regard the art of calculation through the basic operations in arithmetic as a cultural heritage handed down to us by our ancestors and had undergone improvement with time, and is therefore something worth knowing even though tedious drilling in the past practice is no longer needed nor desirable in this computer age. Viewed in this light, written calculation still has its value in modern day education, but with a different emphasis. In this respect, looking at it through a historical perspective, supplemented with exercises suitably designed and based on historical material (as mentioned in (2)), maybe a good alternative.

(5) In the preface as well as in two forewords to *Tongwen Suanzhi* LI Zhi-zao and his friends and fellow official-scholars XU Guang-qi and YANG Ting Jun (楊廷筠, 1557-1627) stressed the meaning of *tongwen* (literally meaning "common cultures") adopted as part of the title of the book (Li and Ricci, 1613/1993), which exhibits their open mind and receptive attitude to foreign learning, at the same time indicating a deep appreciation of the common cultural root of mathematics despite different mathematical traditions.

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CHINESE CORE TRADITION TO WHOLE NUMBER ARITHMETIC

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Abstract

This paper discusses the ancient Chinese tradition of whole number arithmetic (WNA) and its influence on current curriculum practices. The ancient Chinese tradition is reviewed from both linguistic and historic-epistemological perspectives. Based on the Chinese linguistic habit, the early Chinese invented the most advanced number name and the most advanced calculation tools (counting rod and Suanpan or Chinese abacus), in which place value is the most overarching principle as the spirit of WNA. Traces of this influence can be found in contemporary curriculum practices. These include: i) number, addition and subtraction are closely connected; ii) place value is the dominant principle; and iii) composition and decomposition of numbers and problem variation are central idea. Their implications are discussed.

Key words: addition and subtraction, Chinese tradition, number name, number system, place value, rod calculation, SuanPan

Introduction

For the past half century, Chinese students have repeatedly performed better than their peers in cross-national studies. Explanations for this have been sought in numerous cross-cultural studies. One well-known advantage of Chinese education is its foundation, known as "two-basics" (Zhang, 2006). Chinese education as part of Eastern civilisations may provide a resource for new thinking for global development. Whole number arithmetic (WNA) is the most important schooling stage, which could be a place to illustrate the value of the Confucian educational heritage. However, Chinese tradition and practice of WNA are rarely examined systematically, a missing paradigm of the international mathematics education community that has been underrepresented in the mathematics history and education community because of linguistic, geographical, and political issues. In this paper, we examine ancient Chinese tradition to WNA and its influence on the current curriculum practice, which affects number knowing, operations, applications, and further learning. We will start by describing early historical evidence for Chinese numerical practice, and then consider how this tradition continues in today's instruction.

Linguistic and historic-epistemological perspective

The Hindu-Arabic system of numerals is more effective computation than any other known system of base ten numeration, and consequently has been adopted by countries all over the world during the last century, so that it is now essentially universal. However, it is interesting to note that Lam and Ang (2004) argue that the idea for the Hindu-Arabic system of writing and calculating derived from the old Chinese rod numeral system. This system, which was systematically presented in the Sunzi Suanjing (Lam and Ang, 2004), was transmitted to India during the 5-9th centuries, to the Arabic empire in the 10th

century, and then to Europe in the 13th century by the Silk Road (see Guo, 2012). This indicates that the Chinese have long history and tradition in arithmetic. In fact, before the 14th century, ancient Chinese used the decimal place value number system. Based on the advanced number system, advanced arithmetic theories (e.g., the first fraction theory in 九章算术 Jiuzhang suanshu (Guo, 2012)) have been developed to solve application problems, wherein application mathematics traditions, not western theoretical mathematics tradition, were built. Notably, mathematics is called the academics of calculation in ancient times, which is an art of computation (算術) in the Chinese-spoken community before 1977, and could be related the foundation of the Chinese language as well.

Chinese foundation of WNA: Chinese linguistic habit

Chinese is one of the most widely used languages in the world. However, written Chinese is considered one of the most difficult languages to master due to its use of characters (Marton, Tse and Cheung, 2011). Unlike English and most Indo-European languages, written Chinese is logographic rather than alphabetic, and uses the radical ("section header") as the basic writing unit. Most (80-90%) of characters are phono-semantic compounds, combining a semantic radical with a phonetic radical. Thus, the large majority of words have a compound, or part-part-whole structure. This differs from the phonetically based structure of writing in most Western languages, in which order is more important than the combination of parts. The difference can be seen in the structure of Chinese number words. For example, Chinese refer to the number 12 as "ten-two" rather than use a single word, such as "twelve".

The unique feature of the Chinese language: A variety of classifiers (number unit)

Another important feature of Chinese is that uses classifiers much more than Western languages. This is related the concept of ratio lv (率) , which is key to Liuhui's commentary on Jiuzhang suanshu (九章算术刘徽注) (Guo, 2012). As in many East Asian languages, classifiers are required when using numerals with nouns. For example, the English expression "an apple" has to be replaced in Chinese by an "ones" (ge, 个) apple. There are many classifiers in Chinese that have no corresponding English words, and each type of object that is counted has a particular classifier associated with it. As a weakening of this rule, it is often acceptable to use the generic classifier "ones" in place of a more specific classifier. "Ones" was originally used to describe the magnitude of numbers related to the number unit. "Ones" has become the only general classifier in the Chinese language, the most important "number unit", and it has been elaborated into other larger number classifiers (i.e., different number units) to describe larger magnitudes, such as tens (shi 十), hundreds (百), thousands (千), and ten thousands (万) in analogy with the metric units of mm, cm, dm, and m in English.

These linguistic practices led in ancient Chinese numerals to the writing of numeral in two lines to indicate both the digit and order of magnitude or number unit or measurement unit (Fig. 1). In the example below of a rod numeral, from Yongle Encyclopedia[6], the top line reads 7,1824, in which the first line of characters "七一八四二" stands for number value "71, 842", and the line below it specifies that 万 means ten thousand, qiān (千) means thousand, bǎi (百) means hundred, and shí (十) means ten , and bu means a step , an ancient Chinese measure unit. The number units have the same position as the measure units.

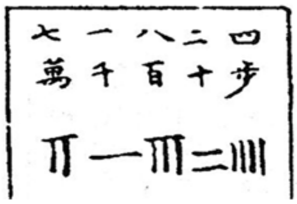

Fig. 1: Number/number unit written by Jiaxian mathematics of Song dynasty (960–1279)

Chinese spoken numerals: The most advanced number name

Due to the linguistic practice of specifying specifies, as explained above, it is not surprising that Chinese numerals evolved to combine the digit and the units together into one line. Both spoken numerals and written numerals are consistent, and this is a distinctive feature of Chinese.

> Chinese spoken numerals: 一个、二个、……;一十一个、一十二个、……; 二十个，二十一个….
>
> In English: One ones, two ones... one tens and one, one tens and two ones …two tens, two tens and one….

By contrast, the Arabian numeral 235 is written without any associated number unit of hundreds, tens, and ones, and the speaker is required to interpolate the units when reading. Also, number names in other languages may involve many irregularities. For example, For example, in French "eighty" or eight tens is referred to as "quatre-vingt" or "four twenties". In English "two-ten three-ones (二十三个)" in Chinese is different from the English number name "twenty three", wherein the number unit "tens" is combined with the number name "ty" in English. The uniqueness of Chinese spoken numerals, wherein the place value concept directly reflects Chinese oral/written numeration as an isolated number unit, has been listed as the simplest and most advanced number name in the world (Lam and Ang, 2004). This uniqueness reflects the following Chinese arithmetic tradition:

[6] http://en.wikipedia.org/wiki/Yongle_Encyclopedia

1. Numeration records both number name and number unit, which specifies the place value in a clearer way than others.

2. Chinese numeration fully follows the calculation framework in terms of place value.

The Chinese numeration is different from the western numeration where the number unit is rarely pointed out because of its strong calculation tradition.

Chinese ancient belief of WNA: World is calculation

Similar to the Greek concept of mathematics that number is world, the ancient Chinese believe that the only way of knowing the world is through calculation, which could be reflected in I ching (易經) in general. This is aslo expressed in the following quotation from the preface of Sunzi suanjing:

> Caculation is the whole of heaven and earth,the origins of all life, the beginning and end of all laws,father and mother of yinyang, the beginning of all stars, the inner and outer of three lights, the standards of five elements , the beginnging of four seasons, the origins of ten thousands matters ,and the general principles of six arts (Lam and Ang, 2004, p. 29).

Chinese ancient inventions of WNA: The most advanced calculation tools

With their advanced system of numeration, it is not surprising that the ancient Chinese invented the efficient systems of calculation embodied in the calculating rod and Suanpan (Chinese abacus) listed as UNESCO's heritage. The rod has a history of more than 1,500 years. Suanpan replaced rod due to quick speed, which has a history of more than 2,500 years, Suanpan is considered the fifth most important invention in Chinese history. It came into widespread use during 1368 to 1644. Its use has decreased since the 1980s, when schools abandoned Suanpan in favor of written calculation and then digital calculators.

Chinese ancient spirit of WNA: Place value is the most overarching principle

Compared with the number line and other representations of numbers, the Chinese rod and abacus have a special framework with unique decimal place value concept specified in terms of one-place, ten-place, hundred-place, and so on. This framework allows for visible and easier calculation. For example, ancient rod addition calculation of 362368 + 657783 from Sunzi Suanjing (Fig. 2) is similar to the current written calculation from high place to low place. With the unique framework, rod/Suanpan has been the most advanced calculation tool in the world to quickly deal with four operations for hundreds years. No tool from other civilisations has the capacity to equal the rod/Suanpan (Lam and Ang, 2004).

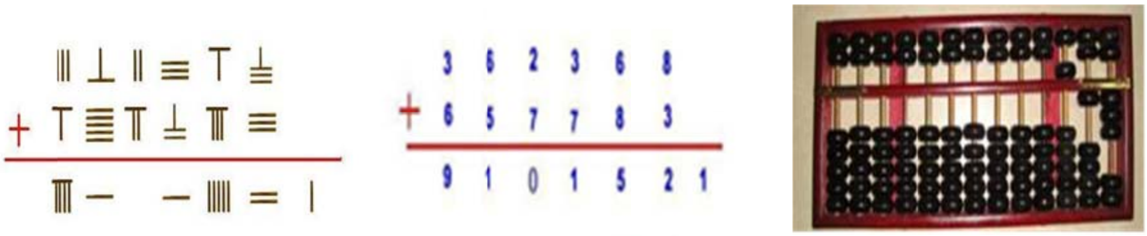

Fig. 2: A comparison of rod calculation with written calculation of 362368 + 657783

In the system of Chinese bead calculation (ancient computer), the numbers are represented by beads. Place is represented by its base-10 column as its hardware, wherein addition means composition of the beads and subtraction means decomposition of the beads. In this way, bead calculation is conducted by mental operation of one-digit under different places, indicating a unique feature elicited by the Chinese rod/bead:

1. Composition/decomposition of number is the most dominant concept.
2. Place value is explicitly made as the most overarching principle.

By comparison, if the number concept is represented by the number line, used with calculations by counting up or down, or skip counting, the associativity of addition is developed less naturally than with the composition/ decomposition model incorporated into the Suanpan. Also, place value is not the most dominate principle. The value of place value is evoked by this quotation from Carl Friedrich Gauss (Howe, 2010):

> The greatest calamity in the history of science was the failure of Archimedes to invent positional notation.

We leave it to the reader to imagine other ways in which the ancient tradition may influence current curriculum.

Materials and Methods

To identify the influence of Chinese tradition on the curriculum of WNA, we identify the key pieces of the knowledge package (see Ma, 1999) of the Chinese WNA curriculum. This includes the core concepts, core procedures, and core ideas of the whole number concept; the properties of the operations; and applications, as explicitly or implicitly presented in a Chinese textbook and the accompanying teacher's manual. For this purpose, we selected the text of (Mathematics Textbook Developer Group for Elementary School, 2005) used for over 30 years by the majority of students from many backgrounds. This is considered representative of the Chinese national curriculum by most comparative textbooks scholarship.

Results

Generally, the great importance traditionally attached to calculation is consistent with the current curriculum principle that whole number is "the first foundation of the whole subject" (Elementary Mathematics Department, 2005, p. 1). This heavy emphasis on calculation skills contrasts with American standards, and is reflected in higher requirements for speed in mental and paper and pencil calculations. Examples: addition and subtraction within 1-20 should be done at 8-10 operations per minute; from 20-100 at 3-4 per minute; multiplication within 1-10, 3-4 per minute; written two-digit multiplication, 1 – 2 per minute; all with 90% accuracy. Mental calculation is also emphasised. The four core features of this tradition of computation are presented below.

Three concepts of addition, subtraction, and number are connected

Chinese curriculum developers connected the three core concepts of addition, subtraction, and number in all chapters of addition and subtraction using the following principles:

1. Adding one into a number obtains its adjacent number.
2. Subtracting one from the adjacent number gives the original number again.

By this approach, not only are the three concepts of addition, subtraction and number tied closely together, but also connections are made between them and the concepts of inverse and of equation. This promotes not only doing and memorizing, but also reasoning. In contrast, in American curricula, the ideas of number, of addition and of subtraction are presented in 3 separate chapters, isolated from one another. Traces of this influence can be found in the Chinese unique numeration tradition:

1. Place value is more explicit than other numeration system.
2. Calculation procedure is embedded in numerations at same time.

For example, ten-two is embedded as an addition procedure of ten plus two. Two tens is embedded as a multiplication procedure of two times ten. The English name twelve and twenty could not convey the same calculation procedures or function in same way. This idea could be an implicit point, but a unique feature of the Chinese curriculum inherited from language tradition.

Place value is the overarching principle

Chinese curriculums do not have a chapter of place value similar to that in American curriculum, but it is permeated in all chapters with reading and writing number activities as overarching principle, in which the Hindu-Arabian number is implicitly translated into Chinese language by adding/deleting numeration units because of unique language habits. This point is important because place value is designed as implicit core knowledge of knowing the number unit in the Chinese curriculum, which is different from the calculation vocabulary in the chapter of calculation in American curriculum mandatory practice (Howe, 2011). For example, Hindu-Arabian number 24 is translated into Chinese language as two tens and four ones (二十四个), wherein numeration units of tens and ones are added during the process of translation. Through this approach, the place value concept is practiced and developed. The meaning and denotation of 24 is made connection by clarifying the numeration units during reading/writing the number using the unique Chinese approach (Fig. 3). This concept is rarely emphasised in most American curriculums. The place value of 10–20 is the key point for place value to differentiate ten-place and one-place for the first time in the chapter of knowing numbers 10–20, wherein the Chinese spoken number, 十一，十二，十三，十四， (ten one, ten two, ten three, ten four,…..) reflect “1” as ten-place and 1/2/3/4 as one-place. The number unit of ten as an isolated word is much clearer than the English name, such as eleven, twelve, thirteen, and fourteen. Through this concept, the place value meaning of tens and ones is developed timely.

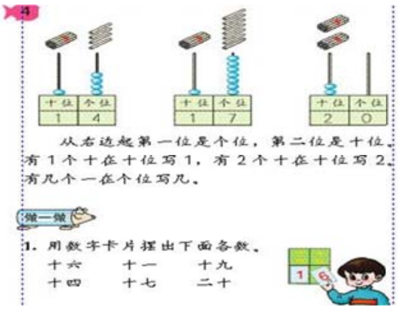

Fig. 3: Reading/writing number as a unique Chinese approach for place value

Chinese curriculum uses making-ten-solution in the first grade to develop the concept of place value, which generally assigns 30 hours (1/2 time of the first term) for conceptual foundation of addition/subtraction with re-grouping as core practice. Specifically, making-ten-approach has been designed critically for the composition of a tens unit, which is also an important aspect of the concept of place value (Ma, 1999) and in understanding the concept of addition with re-grouping inherited from the place value as the most overarching principle of bead calculation tradition. This concept is not found in the American curriculum. This difference could provide a possible explanation for the Chinese teachers in demonstrating much more conceptual understanding of subtraction with regrouping using “decomposing a higher value unit” compared with its American counterpart because of the curriculum learned.

Composition/ decomposition of a number as core approach

The approach of composition/decomposition is used six times with 1–10 in the Chinese curriculum, which implicitly forms a core practice for knowing number. The decomposition approach of 6, 7 is shown below. This approach aims to implicitly develop the association and commutation laws, number properties, and basic knowledge foundation of addition/subtraction operation flexibly. This approach is different from the flexibility of subtraction by addition in western literature. This approach is important for flexible addition, as well as for the explanations on why subtraction could inverse the operation of addition. In the American curriculum, composition and decomposition of number appear as one of the addition models, which is not the core approach but an optional one because it is possibly inherited from composition/decomposition of number as the most dominant concept of bead calculation tradition, inherited from Chinese bead calculation tradition mentioned above.

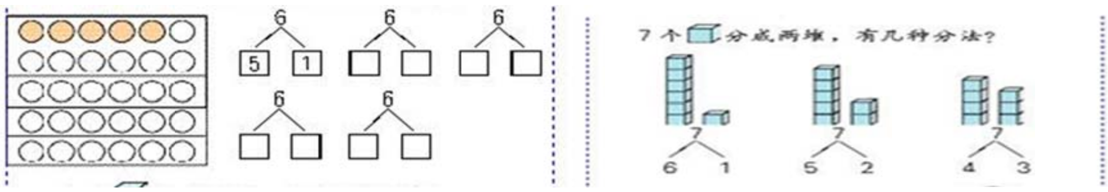

Fig. 4: The decomposition approach of 6, 7

Problem variation for the flexibly application

Systematic word problems include simple and complex problems called problem variation problems, which aims to deepen the concept connection and operations with flexibility (举一反三) (e.g., Bartolini Bussi, Sun and Ramploud, Sun, and Alessandro, 2013; Sun, 2011). The difference and sameness of the aforementioned models, which are rarely explicitly discussed in American curriculum, are key pieces of the knowledge package known as the flexible application of the four operations to the Chinese curriculum context inherited from mathematics application traditions. The American curriculum uses various models (e.g., model of taking away, model of comparing) to develop an understanding of the meanings of addition/subtraction, as well as strategies to solve word problems. However, none of these models appear in the Chinese curriculum.

Discussion and conclusion

This paper has discussed the influence on contemporary curriculum of the ancient Chinese tradition in whole number arithmetic. The ancient practices were reviewed from the linguistic and historic-epistemological perspectives. Influences on current instruction can be found in the following features: i) the concepts of number, addition, and subtraction are closely connected; ii) place value is the dominant principle; iii) composition and decomposition is the central approach to understanding number. iv) Problem variation is the central approach to flexibly application. These features are all consistent with the use of the Suanpan, and a focus on calculation for applications. The Suanpan makes place value explicit, and the calculation procedures of combining ones with ones, tens with tens, and so forth, are built into its structure. The model for numbers provided by the Suanpan may be contrasted with the number line, which is a continuous, non-digital model for numbers, and is not naturally connected with place value.

Acknowledgements

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LEVERAGING HISTORICAL NUMBER SYSTEMS TO BUILD AN UNDERSTANDING OF THE BASE 10 PLACE VALUE SYSTEM

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Abstract

Variation theory states that you cannot know something if all you know is that one thing. Thus to understand numbers and operations in the base 10 place value system one needs to experience number and operations within and beyond the base 10 place value system. In this study I leverage historical numbers systems, including the base 10 place value system, to design a series of tasks each explicating different aspects of understanding the base 10 place value system. The tasks allowed prospective teachers' (PTs') to (a) examine each system, (b) compare and contrast across the systems to identify similarities and differences, and (c) compare all systems to the base 10 place value system. Data examining 36 PTs' concepts of number before and after they worked through the series of tasks showed that almost all PTs developed a more sophisticated conception of number in the base 10 place value system.

Key words: elementary teacher training, number and operation, variation theory

Introduction

Preservice elementary teachers (PTs) often enter our university mathematics classrooms efficient at performing procedures but struggling when asked to explain those procedures conceptually (Ball, 1988; Ma, 1999; Thanheiser, 2009, 2010; Thanheiser et al., 2014). Procedural understanding without conceptual foundations often leads to overgeneralisations. For example in her study, Thanheiser (2009) showed that 60% of PTs did not recognise the overgeneralisation of the standard subtraction algorithm used in the United States (Fig. 1a) to a time context (Fig. 1b). These PTs incorrectly accepted regrouping 1 hour (cross out the 7 an make it a 6) into 100 minutes (putting a 1 in front of the 0 in the ten's place of the minutes) rather than into 60 minutes and thought the child correctly solved the problem (see Fig. 1).

$^{6}\not{7}\,^{1}08$	$^{6}\not{7}:^{1}08$
253	2:53
4:55	4:55
Fig. 1a: The standard subtraction algorithm used in the United States, regrouping 1 hundred into 10 tens.	Fig. 1b: A child's overgeneralisation of the standard subtraction algorithm to a time (not a base 10 place value system) context.

In addition, not making sense of procedures may then lead of an image of mathematics as a disjoint set of procedures one needs to memorise rather than an integrated set of concepts and procedures that make sense. Often PTs are not aware that a rationale exists for the procedures they memorised (Thanheiser et al., 2013). And finally, a combined procedural and conceptual understanding of numbers and operation in the base 10 place value system could lay the foundations for making sense of mathematics at all levels. However, developing this understanding is difficult since once the procedures are in place, it is hard make sense of the underlying mathematics (Pesek and Kirshner, 2000). One way to address this difficulty is to put PTs into a context in which they do not yet have the procedures available and thus have to make sense of the underlying mathematics and then compare and contrast between familiar and new contexts. This can be accomplished by allowing PTs to explore number and operation in different number systems and then make comparisons between the systems. In this study I examine how historical number systems can be leveraged to help PTs make sense of the base 10 place value system by stepping out of it and then comparing and contrasting between historic number systems and our base 10 place value system.

Theoretical Framework

Variation theory states that one cannot know something if all one knows is that one thing (Ling Lo, 2012). To really understand something one needs to know what that thing is and what it is not. Thus to understand something you need to know at least two things (what it is and what it is not) as well as the difference (variation) between those two things (Marton, 2009). "According to Variation Theory, meanings do not originate primarily from sameness, but from difference, with sameness playing a secondary role." (Marton in Lo, 2012 foreword). Thus, to fully understand numbers and operation in the base 10 place value system one needs to not only know the base 10 place value system but also other number systems and the differences between the base 10 place value system and those other systems. In the above example (time context) PTs did not understand the base 10 place value system and the way the subtraction algorithm builds on the base 10 place value system (i.e. regrouping 1 unit into 10 units of the next smaller size) well enough to discern that the subtraction algorithm cannot be applied (without modification) to a time (not a base 10 place value system) context, because the relationship between hours and minutes is not a 10 to one relationship.

In addition to variation it is essential that PTs perceive the tasks that they are given as authentic. One way of making a task more authentic is by connecting the university classroom to the real world (in the case of PTs the K-12 classroom) (Newman, King and Carmichael, 2007). Research has shown the importance of authentic tasks; "students who experienced higher levels of authentic instruction and assessment showed higher achievement than students who experienced lower levels of authentic instruction" (Newman et al., 2007, p. vii). PTs in particular are motivated by tasks for which they see a real connection to their future classroom (i.e. they can use that task with children).

Mathematically, to understand numbers in our base ten system one needs to understand that we have 10 symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and each of those symbols represents a different sized group of *ones* (for example, 3 represents 3 *ones*). Once we have ten *ones* we group those together into a group of *ten*. We repeat this process with the *tens,* gathering up to 9 groups of *tens* before grouping 10 *tens* into the next sized group of *hundreds.* Thus once we have 10 groups of any size we group those into the next sized group. Any number we have is then grouped into minimal groups of *ones, tens, hundreds, etc.* To represent these groups we reuse the same symbols we used to represent *ones*, and in our system the place of the symbol indicates its value. In a whole number the right most digit represents *ones*, the next digit to the left *tens*, the next digit to the left *hundreds*, and so forth. Thus in the number 2345, the 5 represents 5 *ones*, the 4 represents 4 *tens*, the 3 represents 3 *hundreds,* and the 2 represents 2 *thousands.* The successive grouping by tens also results in the ten to one relationship between adjacent digits (i.e. 1 thousand = 10 hundreds). This relationship is leveraged in many standard algorithms grouping and regrouping adjacent digits. To unpack these intricacies PTs need to understand that in our base 10 place value system (a) we have a limited number of symbols (0, …, 9), (b) each symbol represents the quantity of a particular sized group, depending on its location in the number, (c) the group are successively formed by grouping 10 of a unit type into the next unit type, (d) the size of the group is indicated by the digits' place in a number, and (e) a zero is used to indicate do not have any of a certain sized group. Three historic number systems (Unary, Egyptian, Mayan) are used to successively vary along these five components so PTs can build up their understanding of the Hindu-Arabic system we use today, a base 10 place value system (see Tab. 1). These historic number systems were chosen based on how they varied from the H.

Number System	Number of Symbols	Can a symbol represent more than one value	Grouping System	Place Value System	Need for zero?	Relationship of adjacent places
Unary	1	No	No	No	No	N/A
Egyptian	Infinite	No	Yes	No	No	N/A
Mayan	2	Yes	No	Yes	Yes	20 to 1
Hindu-Arabic	10	Yes	No	Yes	Yes	10 to 1

Tab. 1: Comparison between the characteristics of the number systems

Literature Review

Thanheiser (2009) identified 4 conceptions PTs hold when entering mathematics content courses for teachers (see Table 2). With only 30% of the PSTs holding a correct conception (reference units or groups of ones) and only 20% holding the most sophisticated conception (reference units). The reference units conception builds on the underlying base ten system and is required to explain all aspects of the algorithms. These results have been replicated across several studies at the beginning and the end of teacher education programs (Thanheiser, 2010, 2014; Thanheiser et al., 2013) showing that only 25% to 30% of PTs hold correct conceptions if those conceptions are not explicitly addressed in their teacher education programs. We also know that children "experience considerable difficulty constructing appropriate number concepts of multidigit numeration and appropriate procedures for multidigit arithmetic'' (Verschaffel, Greer, and De Corte, 2007, p. 565). Thus it is essential that we create activities especially designed to help PTs develop an understanding of base ten so they can help children develop a rich mathematical understanding.

Conception	PTs
Reference units. PSTs with this conception reliably conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on.	3 (20%)
Groups of ones. PSTs with this conception reliably conceive of all digits in terms of groups of ones (389 as 300 *ones,* 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*).	2 (13%)
Concatenated-digits plus. PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least sometimes. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones).	7 (47%)
Concatenated-digits only. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones,* 4 *ones*, and 8 *ones*).	3 (20%)

Tab. 2: Definition of conceptions in the context of the standard algorithm for the PTs in Thanheiser's (2009) study

Some research has explored the use of alternate bases with PTs to identify conceptions (Khoury and Zazkis, 1994; Zazkis and Khoury, 1993) and the development thereof (McClain, 2003; Yackel, Underwood and Elias, 2007) (Fasteen, Meluish and Thanheiser, 2015). Zazkis and Khoury used the context of base five decimal fractions to reveal PTs' conceptions of the underlying structure of the number system. McClain and Yackel et al. immersed PTs in a base 8 context in a semester long class and found that the development of PTs' concepts of numbers in base 8 mirrored the development of children's concepts in the base 10 place value system. PTs often struggle with alternate bases because they do not see their relevance to teaching K-12 (alternate bases are not typically taught in K-12). However, PTs often enjoy the use of history in mathematics classrooms as it "highlights the interaction between mathematics and society" (Wilson and Chauvot, 2000, p. 642) and is often a part of the K-8 curriculum. In this study I use historical number systems as a context to make sense of numbers and operation in the base 10 place value system by comparing and contrasting across the number systems and successively uncovering the complexities of the base 10 place value system. PTs typically enjoy learning about historical number systems and view them as authentic because they are relevant to their future teaching (ancient cultures including their number systems are part of the K-8 curriculum in the United States).

I build on some of my prior work (Thanheiser, 2014; Thanheiser and Rhoads, 2009), which examined the use of the base 20 Mayan numeral system as a context to explore shifts in the value of digits when comparing the values of a "one" with one, two, and six zeros attached at the end. This study showed that the Mayan system allowed the PTs to discuss the relationships of the values of adjacent digits in a way that is not possible in the base 10 place value system. In this paper I further examine how a collection of number systems (Unary, Egyptian, Mayan, and the base 10 place value system) can be used with PTs to successively vary one component of the system and thus slowly build up to the complex base 10 place value system. Thus, I am using variation in task design.

Methods: Data Collected

Data is drawn from two sections of a mathematics content course for preservice teachers with a total of 36 PTs (13 PTs in a summer course and 23 PTs in a regular school year course). Both courses were 4 credit courses in a quarter system. The summer course met 4 days a week for 4 weeks, each meeting lasting 2 hours and 20 min. The regular quarter course met 2 days a week for 1 hour and 50 min each over 10 weeks. All PTs were interviewed before and after the course to identify their conceptions of number using Thanheiser's (2009) framework (Tab. 2). The interviews were double coded with an agreement of 88% (the disagreements were resolved through discussion). Both groups experienced the same sequence of tasks exposing them to alternate number systems described below. All PT work was collected and scanned and read to make sense of how PTs approached each task.

Materials and Results of Task Design: Task Sequence and PT responses

The Unary Activity presented students with the idea that all number systems share one thing in common; they have a symbol for 1 (tally). Students were presented with a sheet of tallies (about five hundred of them) and asked to count those. The tallies were purposefully not lined up by rows to necessitate counting all tallies (rather than counting the tallies in one row and then counting rows). The context being that an ancient farm owner who lived in a time when only tallies existed recorded how many cows he had. The goal of the activity is to create an authentic need for grouping (needed to count without loosing track) and for symbols beyond the tally to record different sized groups. This activity is used to introduce number systems developed to record large numbers (grouping systems) in which groups of tallies are represented by new symbols (i.e. the Egyptian System). The *variation* emphasised between the Unary system and other systems is that other systems utilise various symbols to represent groups of various sizes while the Unary system only has groups of size 1 and thus only one symbol (the tally) (see Tab. 1). In the Unary activity PTs naturally grouped the tallies by 2s, 5s, 10s, or 20s. If they started with 2 or 5 they often grouped those groups again into larger groups. Thus, this activity is a motivation for grouping and explicates what grouping is used for, namely recording larger numbers. The activity also most often leads to groupings found in historical number systems.

The Egyptian Activity allowed PTs to explore the idea that while different symbols represent different sized groups; the location of the symbols does not matter. PTs were asked to convert numbers between the base 10 place value system and the Egyptian systems. Egyptian numerals were presented in mixed order (not ordered from largest to smallest) to highlight the fact that order does not matter in a grouping system. Numbers were also presented in non-minimal groupings (i.e. more than 10 tens listed in a number) to highlight that while easier to read, minimal grouping is not essential in a grouping system. Artifacts of children's mathematical thinking were used to discuss the fact that a symbol for zero is not needed in a grouping system. For example, PTs were first asked to convert 4508 into Egyptian symbols and then viewed a video of children doing the same and discussing how/whether to represent the 0 tens. Following the exploration of the Egyptian system PTs were then asked to perform operations (i.e. multiplication) in this system. The *variations* emphasised between the Egyptian system and the Unary system is that the Egyptian system has more than one group size and more than one symbol, each symbol representing a different sized group. The *variations* emphasised between the Egyptian system and place value systems are (a) that in a place value system the location of a symbol determines the size of the group it represents, whereas in a grouping system the value of a symbol is independent of its location, (b) in a place value system there is a need for 0 while there is not need for 0 in a grouping system, and (c) operations are quite cumbersome in a grouping system as compared to a place value system (see Table 1). In the Egyptian Activity PTs noticed that the place of the symbols does not matter, however, for ease of reading and writing numbers they (just like the Egyptians) would order the symbols from largest to smallest. This can then lead to a discussion of how our base ten system has the same underlying grouping structure (ones, tens, hundreds, etc.) as the Egyptian system. When asked to perform operations (such as multiplication) in the Egyptian system PTs realise how awkward such operations are in grouping systems. They were asked to double a fairly large Egyptian number and used repeated addition followed by regrouping. They were then asked to think about how to figure how many 30 times that number would and responses were along the lines of "this would result in a profane amount of symbol," and "it would be difficult," as well as "it would be really cumbersome," etc. Thus this activity highlighted the advantages of grouping systems (i.e. recording large numbers, values of symbols are fixed) and their limitations (cumbersome for calculations).

The Mayan Activity allowed PTs to explore a place value system in a base 20 system. First students familiarised themselves with the Mayan number system. They were presented with the first 30 Mayan numbers and then asked what a one with one zero (20), a one with two zeros (400) and a one with six zeros (64 x 10^6) represents. After this activity - which was designed to help PTs explicate the underlying base system resulting in a x20 relationship between adjacent digits – PTs were asked to invent addition and subtraction algorithms in the Mayan system. The *variation* emphasised between the Mayan system and the base 10 place value system is the explication of the underlying base (20 vs 10) and the relationship between adjacent unit types as x20 (Mayan) and x10 (the base 10 place value system). In the Mayan Activity PTs struggled identifying the value of a one with two zeros and a one with six zeros (see (Thanheiser, 2014) for a more detailed description of those struggles). The most common misconceptions were a one with two zeroes interpreted as 200 (since a one with one zero represented 20 and a zero was incorrectly appended to that 20) and a one with six zeroes as 20,000,000 (same line of reasoning). Two arguments were used by PTs to make sense of the one with two and six zeros. The first argument filled every place to capacity (19) before spilling into the next place, thus arguing that 19 ones and one more make a one with one zero (20), and 19 (20s) and 19 ones (399) and one more would make a one with two zeros (400). This argument is in line with a *groups of ones* conception (see Table 1). The second argument utilised the multiplicative relationship between adjacent places as x20, so the first place represents ones, the second 20, the third 20x20=40, etc. This argument is inline with a *reference units* conception (see Tab. 1). The power of this task derives from the fact that conceptions, which would not be easily observable in the base 10 place value system, become visible and can be examined by the PTs (i.e. appending zeros above) (Zazkis and Khoury, 1993). This can then prompt a discussion why procedures such as appending zeros work in the base 10 place value system. Along the same lines regrouping needs to be examined when working on adding and subtracting numbers, and the fact that we ungroup a group of larger size into the next smaller groups is explicated (as it is not hidden behind a procedure). PTs will also often quite naturally invent sense making algorithms in the context of the Mayan numbers and thus experience sense making connected to operations.

Compare and contrast the different systems. Once PTs make sense of each of these systems they are asked to describe a grouping system and a place value system and discuss the similarities and differences among them and identify the important aspects of a place value system.

PT's Conceptions and Discussion

Each of the activities helped PTs learn important information about numbers in the base 10 place value system by pulling the PTs out of their typical context and discussing the differences (variations) between systems. Almost all PTs developed more sophisticate concepts of number in the base ten system throughout the course. The distribution of the 36 PSTs' conceptions at the beginning of the course was 2 *reference units,* 7 *groups of one,* 18 *concatenated-digits plus,* and 9 *concatenated-digits only.* At the end of the course the distribution was 27 *reference units,* 4 *groups of one,* 4 *concatenated-digits plus,* and 1 *concatenated-digits only.*

Comparing and contrasting the different systems allows PTs to compare similarities and differences among the number systems, to explicated aspects of each, especially the base 10 place value system, and thus to build a better understanding of what the base 10 place value system is.

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WHOLE NUMBER IN ANCIENT CHINESE CIVILISATION: A SURVEY BASED ON THE SYSTEM OF COUNTING-UNITS AND THE EXPRESSIONS

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Abstract

Whole numbers are used in any civilization. In this paper, I will discuss the characteristics of whole numbers in ancient Chinese civilization. Chinese has been using the system of decimal numbers to count and calculate, but we can find that the system of numbers and its expressions have had some differences among different periods and different situations. This is a very complicated subject. In order to limit the length of the paper to 8 pages, I here only give a brief discussion on the basis of whole number expressions from the following aspects: representation by Chinese characters, manipulation with calculation tools, and representation by Chinese numerical symbols.

Key words: numerical notations, counting rods, whole numbers in ancient China

Whole numbers in ancient Chinese writing

In ancient Chinese writing, numbers are expressed by basic numerical characters, words or their combinations.

The basic numerical characters or words

We can classify the basic numerical characters or words into two types:

Nine characters for numbers from 1 to 9

Hindu-Arabic numerals	1	2	3	4	5	6	7	8	9
Chinese characters	一	二	三	四	五	六	七	八	九
Hanyu Pinyin (Chinese phonetic script)	yī	èr	sān	sì	wǔ	liù	qī	bā	jiǔ

Fig. 1: Chinese characters for numbers 1-9

Number units

Some characters or words are number units for the powers of 10 (Guo, 1993, p.1128). We list them from small to large order as follows:

Hindu-Arabic numerals	10	10^2	10^3	10^4								
Chinese characters	十	百	千	萬	億	兆	京,经	垓,姟,絯	秭	穰	溝	澗
Hanyu Pinyin	shí	bǎi	qiān	wàn	yì	zhào	jīng	gāi	zī	rǎng	gōu	jiàn

Hindu-Arabic numerals								
Chinese characters	正	載	極	恒河沙	阿僧祇	那由他	不可思议	無量數
Hanyu Pinyin	zhèng	zǎi	jí	hénghéshā	āsēnzhī	nàyóutā	bùkěsīyì	wūliàngshù

Fig. 2: Chinese characters for number units

Among the above number units, the names of the last five (from hénghéshā 恒河沙 to wūliàngshù 無量數) were from Buddhist Scriptures while others (from shí 十 to jí 極) were from Chinese own words (Li, 1934).

The first four number units shí 十, bǎi 百, qiān 千 and wàn 萬 always respectively represent 10, 10^2, 10^3 and 10^4. The other number units vary with different systems of notation of numbers. These words (from yì 億 to wūliàngshù 无量數) are called large numbers (dàshù 大數). Among different sources, large numbers may have very few differences in order.

There were differences among different systems of large number units (Li, 1934).

(1) 10 times of a number unit make the next.

This system was used probably during the period in pre-Qin period and possibly in Western Han dynasty as well.

(2) From wàn 萬 (10^4) on, wàn 萬 (10^4) times of a number unit make the next.

This system probably has been used since Western Han dynasty.

(3) Wàn 萬 (10^4) times of wàn 萬 (10^4) is yì 億 (10^8), from yì 億 on, wàn wàn 萬萬 (10^8) times of a number unit make the next.

We don't know whether this system is introduced from Buddhist Scriptures. This system was widely used from Han to Ming Dynasties.

(4) From wàn 萬 (10^4) on, the square of a number unit makes the next.

(5) From wàn 萬 (10^4) on, bǎi 百 (100) times of a number unit make the next.

Both systems were rarely used and were probably from Buddhist Scriptures.

Whole numbers expressed by the combinations of basic numerical characters or words.

The basic numerical characters or words can express any whole numbers by themselves or their combinations. The ways of the combination vary with different systems of large number units. But all systems have similarities in denoting numbers smaller than 10^5.

The expression of whole numbers smaller than 10^5

(1) The numbers in the pattern of *a* (*a* = 1, 2, …, 9) times of 10, *a* times of 10^2, *a* times of 10^3 and *a* times of 10^4 are respectively represented in the patterns *a* shí

十, *a* bǎi 百, *a* qiān 千 and *a* wàn 萬. This can be traced back to the oracle-bone inscriptions before 1000 BCE (Zou 2001, p.56-60; Guo, 2010, p.18-20).

(2) The whole numbers smaller than 10^5 are expressed by combining the combined numbers shown in (1) above and the nine basic numbers. A whole number represented as $a_5a_4a_3a_2a_1$ (a_1, a_2, a_3, a_4, a_5=0,1,…, 9) in the way of Hindu-Arabic numerals are usually expressed in ancient Chinese as in the pattern: a_5 wàn 萬 a_4 qiān 千 a_3 bǎi 百 a_2 shí 十 a_1.

If one of the a_1, a_2, a_3, a_4, a_5 equals to 0, then the corresponding part is omitted. But if the part is not the first and the last, the former part and latter part is connected by a word "ling 零" (or "ling ○") in later period. When one of a_2, a_3, a_4, and a_5 equals to 1, it is sometimes omitted. In few cases, there is a character yòu 有 or yòu 又 ("and" or "again") to be put after the units wàn 萬, qiān 千, bǎi 百 or shí 十. For example, bamboo strip no.0456 of the *Shu* 數 of Qin dynasty mentions a number "**six** wàn 萬 **three** qiān 千 **three** bǎi 百 **six** shí 十"(六萬三千三百六十) (Zhu & Chen, 2001, p.141), which is 63360.

There is another way named "*hewen* 合文" (combined writings) to express several wàn 萬, qiān 千, bǎi 百 or shí 十, in which a number unit and the basic numerical characters for denoting 1-9 are combined into a single character. For example, on oracle bone inscription, 30000 is represented as , which is a combination of ≡ (sān 三, 3) and (wàn 萬,10^4) (Qian, 1982, p.5-7).

The expression of whole numbers bigger than 10^5 or 10^5 itself.

With the first system of large number units, Chinese use the same way of expressing numbers smaller than 10^5 to express bigger numbers.

For the other systems of large number units, large numbers ($\geq 10^5$) are expressed in the combinations of several parts which are arranged in the order from big to small. The last part is smaller than 10^4 and is expressed as in 1.2.1, but "a_5 wàn 萬" does not appear. Each of the other parts is a multiple of a number unit from wàn 萬 on.

Whole numbers represented by tools for calculation

There are two main tools for calculations in ancient China: counting rods and the abacus. The former had been widely used until 14th century CE. The latter probably has been used since 11 century CE or even earlier. It had been very popular until 1990's and is still used today. Here I only discuss the first in order to save the length.

Counting rods

Counting rods are small rods which are arranged into special shapes to form numerical signs. Ancient Chinese usually used bamboo or wood to make counting rods, but sometimes they also used other materials. The segment of counting rods is usually round, but there are counting rods of which the segments are rectangle or triangle (Li, 1955, p.1-8).

The numerical signs put by counting rods have two forms: the vertical and the horizontal. We can list them as Fig. 3.

In the vertical form, a horizontal rod represents 5 while each vertical rod represents 1. Accordingly, in the horizontal form, a vertical rod represents 5 while each horizontal rod represents 1. In both forms, the rod which represents 5 of 6-9 is put on top while the rods each of which represents 1 are put below (Qian, 1982, p.8; Li & Du, 1987, p.6-11; Martzloff, 1997, p.185-187).

Vertical form	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
Horizontal form	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱
Hindu Arabic numerals	1	2	3	4	5	6	7	8	9

Fig. 3: figures of counting rods for expressing 1-9

Counting rods represent numbers by decimal place-value system. The vertical form is used to express the digits on the positions of units, hundreds, ten thousands and so on; while the horizontal form is used to express the digits on the positions of tens, thousands, hundred thousands and so on. A blank space is used for 0 (Qian, 1982, p.7-9; Du, 1991; Lam, 1986,1986,1988). For example, 68012 can be represented as 𝍥𝍰 𝍩𝍡.

In early period, ancient Chinese did not usually sit on stools. They first put a mat on the floor, and then went on the mat and sat on their own knees. Thus the front of the right knee was defined as the position of units (Guo, 1993, p.411). This rule can avoid the confusions like that both 600 and 6 are represented as 𝍥 which cannot show whether there are blank spaces or not.

Chinese introduced the notion of negative number into mathematics very early. According to Liu Hui, positive numbers and negative numbers are respectively represented with red rods and black rods; otherwise they are differentiated by slanted or upright rods (Li, 1955, p.1-8; Qian, 1982, p.55).

Chinese numerical symbols

Ancient Chinese also used numerical symbols to represent numbers. There are two sets of symbols (Yan, 1982).

Chinese characters for numerical symbols

This set of numerical symbols only consists of nine Chinese characters for 1-9 and one symbol for 0:

Hindu-Arabic numerals	1	2	3	4	5	6	7	8	9	0
Chinese characters	一	二	三	四	五	六	七	八	九	〇
Hanyu Pinyin	yī	èr	sān	sì	wǔ	liù	qī	bā	jiǔ	líng

Fig. 4: Chinese characters as numerical symbols

The representation of whole numbers by the above ten characters is based on decimal place-value system, and is similar to counting rods and Hindu-Arabic numerals. This notation is more concise than counting rods with two sets of numerical signs (Yan, 1982).

Anma 暗馬 (secret marks)

This set of numerical symbols originated from the figures formed by counting rods, and it was used in the way similar to counting rods as well. But a few figures were changed and a symbol was introduced to denote the empty space corresponding to zero (Yan, 1982).

In Southern Song dynasty, Yang Hui used the following symbols:

Vertical form	𝍠	𝍡	𝍢	×	ō	𝍥	𝍦	𝍧	⊠	○
Horizontal form	𝍩	𝍪	𝍫	×	〥	𝍮	𝍯	𝍰	〩	○
Hindu Arabic numerals	1	2	3	4	5	6	7	8	9	0

Fig. 5: Yang Hui's numerical symbols

These symbols are used to denote numbers by hand writing, and their usage is similar to counting rods. In Ming dynasty the way to use vertical and horizontal forms was not obeyed rigorously. The symbols from 4-9 were more frequently used in the horizontal form, and the 〥 for 5 became 8 or 攵, the symbol for 9 also has a variant 文. The set of symbols was called *anma* 暗碼 (secret marks), *mazi anshu* 馬子暗數 (marks for secret numbers) or *anzi mashu* 暗子馬數 (secret symbolic numbers). Because the *anma* was widely used by merchants of Suzhou 蘇州, it has also got the name *suzhou mzi* 蘇州碼子 or *suzhou ma* 蘇州碼.

Vertical form	𝍠	𝍡	𝍢	×/又	8/攵	𝍥	𝍦	𝍧	⊠	○
Horizontal form	𝍩	𝍪	𝍫			𝍮	𝍯	𝍰	〩/文	○
Hindu Arabic numerals	1	2	3	4	5	6	7	8	9	0

Fig. 6: *anma* (secret marks)

Many documents usually use different ways of numeral notations together.

Characteristics of whole numbers in ancient China

Adoption of decimal system, or even place-value decimal systems

As shown above, different systems of notation of numbers in ancient China are all of base ten, and very simple. The system in Chinese characters only uses a minority of basic number words and number units (usually five or six units are enough), and the multiple-rule to represent whole numbers. This can give a direct impression how much a number is. The other systems of notation are based on decimal place-value systems, and use even less basic symbols. When operating calculations by counting rods or Chinese abacus (*suanpan*), the intermediate steps are usually erased. This can make the operation only need a small space, and can avoid the trouble of preparing ink, brush and paper or bamboo slips. The other advantage of these two tools is that the arithmetic operations can be started either from the lower digit or from the higher digit.

Method of detached coefficients for modern algebraic expressions

The counting rods can express very complicated mathematical knowledge, including a set of higher degree equations with up to 4 unknowns. When ancient Chinese expressed algebraic expressions or equations, they used the method of detached coefficients, and took the positions of coefficients to indicate the unknowns and their powers. For example, we can express the mathematical relations given in the problem 3 of chapter 8 of the *Nine Chapters on Mathematical Procedures* (*Jiuzhang Suanshu* 九章算術, ca. middle of 1st century BCE)) as the simultaneous equations $\begin{cases} 2x+y=1 \\ 3y+z=1 \\ 4z+x=1 \end{cases}$. The *Nine Chapters* solves the problem by the method named *fangcheng* 方程 that looks like a matrix, as shown in Fig. 7-1. In every column of Fig. 7-1, the unknowns *x*, *y*, *z* and the constant are indicated by the positions of coefficients from top to bottom.

<table>
<tr><td>left</td><td>middle</td><td>right</td><td></td><td>left</td><td>middle</td><td>right</td><td></td><td>left</td><td>middle</td><td>right</td><td></td><td>left</td><td>middle</td><td>right</td></tr>
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<tr><td colspan="3">Fig. 7-1</td><td></td><td colspan="3">Fig. 7-2</td><td></td><td colspan="3">Fig. 7-3</td><td></td><td colspan="3">Fig. 7-4</td></tr>
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Fig. 7: Expression and operation of *fangcheng*

Ancient Chinese solved the *fangcheng* with a method similar to Gauss elimination. The structure of *fangcheng* and the corresponding method of elimination make the necessity of the introduction of negative numbers (Zou, 2010), which can be shown clearly in Fig.7-3 and Fig. 7-4.

Distinction between odd numbers and even numbers

The difference between odd numbers and even numbers seemed to be noticed in very early period. There were 36 stone knives unearthed from the Xuejiagang 薛家崗 Ruins (ca.5190 BCE±125) at Qianshan 潜山 in Anhui 安徽 province. Among 36 knives, only 1 knife with 4 holes, each of the others has an odd number holes (Shuo & Yang, 2003). These knives with odd number holes indicate that the ancient Chinese attached great importance to odd numbers.

Ancient Chinese connected odd and even numbers to yang 陽 and yin 陰, or something related to them.

Chinese mathematical documents usually pay much attention to 2. When simplifying fractions, ancient Chinese would usually first examine whether the numerator and denominator can be both divided by 2, and then would find other common divisors.

The "one" as the most important number

The Chinese character "yī 一" (one) has many meanings: one; whole, all; same, equal; the origin or beginning of everything; Tao; to unify, etc. The *Lao Zi* says: "From Tao there comes 'yī 一' (one). From 'yī 一' (one) there comes 'èr 二' (two). From 'èr 二' (two) there comes 'sān 三'(three). From 'sān 三' (three) there produces "all things" (道生一，一生二，二生三，三生萬物). This expression connects the numbers to Taoist's cosmology, which is still on the basis of the notion that one is the base of other numbers. In the third century, Liu Hui 劉徽 said, "Small things are the beginnings of large things, and one is the mother (origin) of numbers, therefore with respect to using rates they must be regarded as equivalent to one." (少者多之始，一者数之母，故为率者必等之于一) (Guo, Dauben & Xu, 2013, p.177-178). This expression admits that two magnitudes of the same kind should have a common unit to be measured. The cognition is related to the calculation of decimal approximate value of the square roots or cube roots which irrational numbers are known to us. It probably had influence to the fact that ancient Chinese did not find irrational numbers.

Common divisor and the thought on "prime to each other"

Ancient Chinese also paid attention to the relations between whole numbers. Both the unearthed *Suanshu Shu* 筭數書 (before 186 BCE) and the *Nine Chapters* record the method for finding common divisor of numerator and denominator of a fraction. The method can find the biggest common divisor, and is equivalent to the corresponding method in Euclid's *Elements*.

Ancient Chinese did not pay much attention to prime numbers. But they pay attention to the relation of two whole numbers which are mutually prime. This notion is very important in the problems of *dayan* 大衍, concerning linear congruences. The solution of these problems needs to obtain a set of whole numbers among which any two numbers are mutually prime (Qian, 1966). The method for finding common divisor can help ancient mathematicians to simplify a pair of whole numbers to make the numbers be mutually prime.

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THEME 2: WHOLE NUMBER THINKING, LEARNING AND DEVELOPMENT

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Introduction

Theme 2 addresses essentially cognitive aspects of learning and development of WNA, building further on the state-of-the-art as summarized in reviews of the research literature on WNA of, for instance, Fuson (1992) and Verschaffel, Greer and De Corte (2007). However, some papers view teaching as inextricably linked within any discussion about learning and development.

One of the key questions that this theme raises is whether there are synergies between different perspectives. And if so how can we integrate these different perspectives into a more coherent view about the foundations and development of WNA concepts and skills?

Several papers direct attention to new developments in the research field of early number (as described in a recent overview of English and Mulligan, 2014). Some re-conceptualisations of traditional aspects of early number and WNA are introduced. These focus on mathematics learning and development of other aspects such as visuo-spatial processes, pattern, kinaesthetic and embodied action and their relationship to development of number sense. The role of the number line as a mental model (Dehaene, 1997; Siegler and Opfer, 2003) is raised in several studies.

The inclusion of several cross-cultural studies on WNA in the middle elementary years provides a rich comparison with emerging research from Chinese researchers.

Studies of Number Development

Two large cross-sectional studies focus on developmental aspects of young children's number learning which provide a lens for re-examining 'traditional' features of number acquisition. A cross-cultural study (Cyprus and the Netherlands) of Kindergartners' number competence by Elia and van den Heuvel-Panhuizen focuses on counting, additive and multiplicative thinking.

Milikovic examines the development of young Serbian children's initial understanding of representations of whole numbers and counting strategies in a large study of 3- to 7-year olds. Culturally invented (formal) representations such as set representation and number line were found to be limited in their recordings.

Gould draws upon a large Australian large study of children in the first years of schools aimed at improving the numeracy and literacy in disadvantaged communities. A case study exemplifies how numerals are identified by relying

on a mental number line by using location to retrieve number names. This raises the question addressed in other papers focused on how the individual's brain processes numbers differently.
Obersteiner and colleagues propose a coherent five-level competence model for WNA in the lower grades of elementary school taking account of psychological perspectives.

Embodiment and Visuo-spatial Approaches to Number Sense

Mulligan and Woolcott provide a discussion paper on the underlying nature of number. They present a broader view of mathematics (including WNA) as linked to spatial interaction with the environment; the concept of connectivity across concepts and the development of underlying pattern and structural relationships is central to their view.

The Italian Percontare project (Baccaglini-Frank) built upon a collaboration between cognitive psychologists and mathematics educators, aimed at developing teaching strategies for preventing and addressing early low achievement in arithmetic. It takes an innovative approach to the development of number sense, which is grounded upon a kinaesthetic and visual-spatial approach to part-whole relationships.

Towards a Neuro-scientific Approach

Some papers reflect the increasing role of neuro-scientific concepts and methods in research of WNA learning and development.

Sinclair and Coles draw upon neuro-scientific research to highlight the significant role of symbol to symbol connections and the use of fingers and touch counting exemplified in their 'Touch Counts' iPad app.

Nesher and Shaul focus on the semantics and syntax of use of symbols '+' and '=' in an innovative experimental study utilising event related potential (ERP) to measure brain activity while performing different WNA tasks. They focus on the differential processing of sums less than 10 with young students.

Studies on Mental and Written Arithmetic in the Elementary Grades

Three Chinese studies provide new insights into mental and written whole number arithmetic by students in the middle elementary grades. He focuses on cognitive strategies for solving addition and subtraction problems. Yang highlights the conceptual difficulties of students' judging the reasonableness of results in whole number calculations. Ma et al. analyse and categorise students' systematic errors for three-digit multiplication and links these errors to teaching strategies.

In another, methodologically oriented, study Verschaffel et al. compare two kinds of empirical evidence (one verbal and one non-verbal) for children's use of a special type of strategy for doing mental subtraction, namely subtraction-by-addition.

In a South African study focused on early addition, Roberts turns our attention to the role of teachers by providing a framework to support teachers' interpretation of learners' representations when engaging with whole number additive relation tasks.

Further Questions for Group Discussion

(1) To what extent is basic number sense innate? To what extent is it affected by socio-cultural and environmental influences?
(2) What is the relationship of basic number sense to WNA. Is it the main precursor (predictor) of WNA competence?
(3) How can interdisciplinary and cross cultural studies contribute to shifting views of WNA learning, and what implications does this research have for practice?

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PREVENTING LOW ACHIEVEMENT IN ARITHEMTIC THROUGH THE DIDACTICAL MATERIALS OF THE PERCONTARE PROJECT

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Abstract

PerContare is an innovative Italian project, built upon collaboration between cognitive psychologists and mathematics educators, aimed at developing teaching strategies for preventing and addressing early low achievement in arithmetic. The paper describes two emblematic examples of activities proposed within the project to foster the development of number sense that are grounded upon a kinaesthetic and visual-spatial approach. A study within the project was conducted to investigate the effectiveness of the materials used in the experimental classes. Results revealed a higher performance of the experimental group on a number of items of the assessment batteries. Moreover, this group contained half as many subjects with performance below the cut off score on the AC-MT battery compared with the control group. This suggests that the didactical materials developed in PerContare do contribute significantly to diminishing the number of potential false positives in the diagnoses of dyscalculia.

Key words: calculation, dyscalculia, fingers, inclusive classroom, part-whole relation

Introduction

The PerContare project is an Italian inter-regional 3-year project (2011-2014) aimed at developing effective inclusive teaching strategies and materials to help primary school teachers (in Grades 1, 2, and 3) address low achievement, especially of students who are potentially at risk of being diagnosed with developmental dyscalculia (Butterworth, 2005). The teaching strategies and materials developed involve the use of digital and physical artefacts to help students construct mathematical meanings in a solid way, within the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008).

This paper focuses on two emblematic examples of practices proposed within the PerContare project (also see Baccaglini-Frank and Bartolini Bussi, 2012; Baccaglini-Frank and Scorza, 2013), aimed at fostering interiorization of part-whole relations and awareness of 'structural' aspects of natural numbers (1) through strategies that include particular uses of fingers, and (2) through manipulation of straws in bundles of ten. In the following section I will describe the theoretical grounding of the proposed practices, and then discuss the video on which this paper is based.

Theoretical Grounding

Studies in mathematics education have highlighted how sensori-motor, perceptive, and kinaesthetic experiences are fundamental for the formation of mathematical concepts – even highly abstract ones (Gallese and Lakoff, 2005; Radford, 2014). Various educators and researchers have designed didactical activities significantly based on bodily experience and on the manipulation of concrete objects. For example, Bartolini Bussi and Mariotti (2008), adopt a semiotic perspective, whereby student's use of specific artefacts in solving mathematical problems contributes to his/her development of mathematical meanings, in a potentially "coherent" way with respect to the mathematical meanings aimed at in the teaching activity. Also research in cognitive psychology – though from a different perspective – has identified specific and preferential channels of access and elaboration of information. For students with learning difficulties these include the non-verbal visual-spatial and the kinesthetic channels (Stella and Grandi, 2011).

Let us think about how these elements can apply to the domain of *number sense*. There is no monolithic interpretation of this notion across the communities of cognitive scientists and of mathematics educators, and not even within the community of mathematics educators alone (e.g. Berch, 2005). However, there seems to be a certain consensus about some features of the notion, which have important implications for mathematics education. The development of number sense is seen as a necessary condition for learning formal arithmetic at the early elementary level (e.g., Griffin, Case and Siegler, 1994; Verschaffel and De Corte, 1996) and it is critical to early algebraic reasoning, particularly in relation to perceiving the "structure" of number (Mulligan and Mitchelmore, 2013).

Moreover, literature from the fields of neuroscience, developmental psychology, and mathematics education indicate that using fingers for counting and representing numbers (Brissiaud, 1992), but also in more basic ways (Butterworth, 2005; Gracia-Bafalluy and Noel, 2008), can have a positive effect on the development of numerical abilities and of number sense. Across fields it is agreed upon that both formal and informal instruction can enhance number sense development prior to entering school. The importance of the role attributed to the use of fingers in the development of number sense by the research literature is highly resonant with the frame of embodied cognition.

Part-whole relations and numerical structure

Perceiving pattern and structure is a fundamental way of thinking that should be fostered in young children (e.g. Mulligan and Mitchelmore, 2013). Moreover, lack of the use of this way of thinking seems to characterise children with low mathematical performance. Indeed, Mulligan and her colleagues, over several studies, found that "low achievers" (as defined by their teachers) are more likely to produce poorly organised representations, they tend to use unitary counting exclusively, and appear unable to visualise part-whole relations. This led the researchers to an hypothesis that was confirmed in later studies: "*the more a student's internal representational system has developed structurally, the more coherent, well organised, and stable in its structural aspects will be their external representations and the more mathematically competent the student will be*" (ibid, p. 34).

Part-whole relations arise from what Resnick et al. (1991) have described as protoquantitative part-whole schemas that "organise children's knowledge about the ways in which material around them comes apart and goes together" (ibid.,

p. 32). The interiorisation of the part-whole relation between quantities entails understanding of addition and subtraction as dialectically interrelated actions that arise from such relation (Schmittau, 2011), and recognising that numbers are abstract units that can be partitioned and then recombined in different ways to facilitate numerical (also mental) calculation. Hands and fingers can be used to foster development of the part-whole relation, in particular with respect to 5 and 10, in a naturally embodied way.

Emblematic examples from PerContare

All the didactical materials are collected in an online teachers' guide, accessible for free (at percontare.asphi.it). Each activity is presented as follows: an estimate is given on the time necessary for the activity; then the teacher is guided through the preparation and given a suggestion for the task to propose; the next section briefly describes what the teacher can expect, based on the field-testing of the activity (this section may contain videos and commentaries of actual classroom outcomes); the next section describes the mathematical meanings that the activity intends to promote; then proposals on how to construct these mathematical meanings are given; and finally various student-sheets and possible homework is provided.

The various sections proposed for each activity in the teacher's guide are designed to help the teacher proceed according to the framework of Semiotic Mediation, keeping in mind what the objective-mathematical meanings for each activity are, and giving suggestions about how to help students develop them.

The "fingers game"

The first example comes from a video recorded in a first grade, in November, when the author (A) was proposing the "fingers game". She describes a configuration of fingers saying how many are up or down on each hand, while keeping them behind her back, and asks what number she is representing with the fingers that are up. After about 5 minutes of playing the game, A proposes to ask a 'harder' question.

> A: So now shall we do a harder one?
>
> Class: Yes!
>
> A: So, on one hand… I have three fingers lowered... three fingers lowered… and on the other I have two raised.
>
> Some kids: two.
>
> A: No, how many are raised?... Do it with your hands. [A looks at all students' fingers raised and lowered on each hand.]
>
> A: So, one hand has three lowered, and the other has two raised… How many fingers are raised?
>
> Class: Four, two...four...
>
> A: Let's see how different people did it. [A looks at all students' fingers raised and lowered on each hand.]
>
> A: Do it with your fingers.

Class: [unclear, children say various numbers between one and five. Some say and show 4.]

A: Very good. Three lowered...and two raised.

Child: four.

A: Four. Very good!

In this game the part-whole relation becomes embodied: ten is decomposed into five and five, and five is decomposed in all possible ways on the children's hands. Fig 1 shows representations of how part-whole relations can come into play in determining the total number of fingers raised.

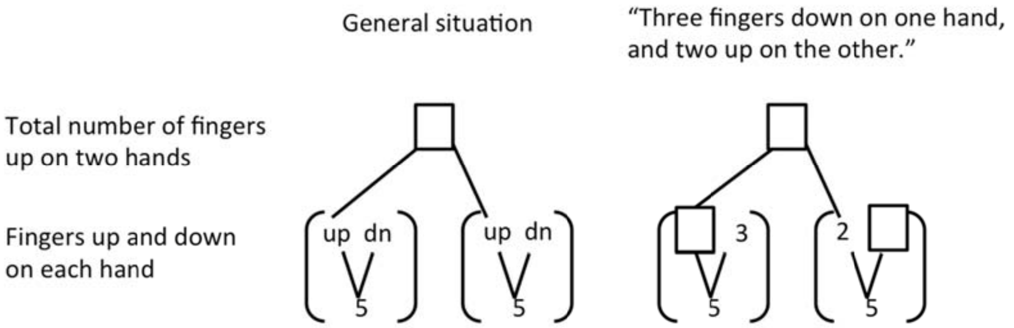

Fig. 1: Representation of the part-whole relation in the "fingers game"

The children actively engage in the game, attempting to reproduce instantiations of the described configuration. The game can easily be played with the entire class without leaving any student behind, because the teacher can look at the fingers raised and lowered on each student's hands and use this feedback to choose which students to explicitly engage in mathematical discourse. In classes in which the game was proposed for at least 5 minutes every other day for a month, children no longer needed to move their fingers and look at their hands to respond correctly, suggesting that they had acquired a stronger (dynamic) mental representation of their hands and fingers.

Mediation of decimal positional notation through bundles of straws

One of the artefacts introduced in first grade consists of straws that students learn to bundle up in groups of ten, through a process of discovery.

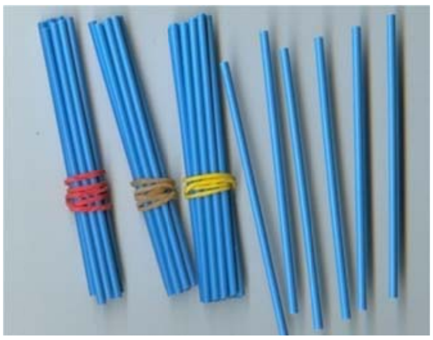

Fig. 2: Representation of the number 36 with straws in bundles of ten

The bundles of straws together with untied straws (Fig. 2) are then used to represent numbers given in different formats. Children discover that to count large numbers of straws it is easier to group the straws in bundles of ten, since this way they can use their ability to count by tens (even though initially there might not be deep meaning associated to the process). Moreover, children are used to making bundles of ten straws from other games proposed. However, children are not explicitly told 'how to' represent numbers with tied up and untied straws. The activity described below introduces this discovery. It is typically proposed around November-December of first grade.

The students are initially given 30 straws each and they are asked to represent the day of the month in which the activity is proposed, using their straws. They are initially invited to come up with ideas and share and discuss them. Once an agreement is reached, phase two proposes to ask students to

a) use the straws to represent a number (up to 30) given orally (verbal code);

b) use the straws to represent a number (up to 30) written in digits (symbolic code) on the blackboard;

c) use the straws to represent a number (up to 30) written in letters (visual-verbal code) on the blackboard;

d) write on their notebooks using digits the numbers represented with straws drawn on the blackboard.

The tasks proposed in this activity involve various transcoding processes (Dehaene, 1992): the verbal code, the symbolic code, and the visual-verbal code are used and put in relation with the structural "straw representation". Such a representation can support students with difficulties because it maintains an analogical format (there is exactly the number of straws that the given number represents) that also recalls symbolic aspects (the tens are grouped) of the numbers involved. Numbers in the "straw representation" maintain a physical connotation, activating the visual and kinaesthetic-tactile channels, and can act as a trampoline for students to pass from one code to the other.

The teacher is also invited to make use of horizontal parentheses under sets of straws to indicate the part-whole relationship s/he is attending to. For example, if the teacher wants to guide the students' attention to the composition of 36 as 'three ten' and 'six' s/he can put a horizontal parenthesis under the three bundles of ten straws on the left and write '3 ten' or '30' and a second one under the six untied straws on the right (see Fig. 2) and write '6'. A final horizontal parenthesis under everything can be used to mark the whole quantity '36'.

Soon after this activity the teacher is invited to use transparent boxes to hold bundles of straws (placed on the left, where the tens digit sits) and free straws (placed on the right, where the unit digits sit). Ten straws can be taken from the container on the far right and bundled up at any time. It is not necessary – like in the case of the abacus – to make a bundle as soon as there are ten straws. Making a bundle and placing it in the tens box makes recognising the number easier, but there is always the same number of straws in total. We have found that for numbers below one hundred the system of straws in boxes works quite well as an alternative for the abacus, which notoriously creates many difficulties for the students. Many of such difficulties seem to arise from the abstraction necessary in seeing a same ball of the abacus as 'one' or 'ten' based on whether it is put on the stick to the far right, or on the next stick to the left. Though the conventionality of the decimal positional notation is present in the representation with boxes of straws (as with the abacus), this artefact maintains a strong connection to the actual numerosity being represented, as it only gives a perceptually different structure to the same number of items being considered.

A study on the effectiveness of the didactical materials

Within the greater project, a specific study was carried out with the aim of gaining insight into the effectiveness of the didactical materials developed. A sample of 208 children (10 classes) was selected at the beginning of their first grade and followed until their third grade. No child with IQ score below average was included in the sample. The sample consisted of two groups: an experimental group of 100 children (5 classes) whose teachers used all didactical materials proposed, and a control group of 108 children (5 classes) whose teachers were not aware of the didactical materials. To both groups was administered a set of assessment tests on arithmetical abilities related to numbers and calculation, as in the typical tests used for diagnosing children at risk (Biancardi et al., 2011). The tests were administered three times to the classes of both groups, in the form of a game: in May of the first grade, and in January-February and again in May of the second grade.

The assessment battery for first graders contained the following tasks: (1) writing numbers (numbers under 1000 dictated in random order), (2) subitizing (numerosities from 2 to 7), (3) estimation (two numerosities were compared), (4) enumeration (counting a set of dots and writing the numerosity in symbolic notation), (5) magnitude judgment (choosing the symbol for the greater number), (6) quantity judgment (deciding whether two representations, one analogical and one symbolic, of a number referred to the same number or not), (7) insertions on the number line (placing a number on a number line with tacks and numbers 0 and 20 marked), (8) reverse counting (writing numbers in reverse order on the number line, starting from a given number), (9) additions (written operations, of which three need composition of tens), (10) subtractions (written operations, in which the greater number is within 10). For each task of each test the number of correct answers was collected.

The assessment battery for the second graders in January-February consisted of seven of the same types of tasks (1, 2, 3, 5, 8, 9, 10), that were only made more complex, and of three different tasks (decomposition, ordering increasingly and decreasingly). In May the assessment was the same as in February, only a task on multiplication was added. For each task of each test the number of correct answers was collected.

In order to verify the validity of the results obtained with the newly developed assessment batteries, in November of the third grade, the AC-MT battery

(Cornoldi et al., 2012) was administered to the whole sample of subjects, together with a test for collective evaluation of reading ability DCL (Caldarola et al., 2012), and the dictation of a sequence from a battery for the evaluation of writing and orthographic competence (Tressoldi et al., 2013).

Results and Conclusion

The results of the assessments at the end of the first grade show a substantially better performance of the experimental group on the following tasks: magnitude judgment, addition and subtraction. Moreover, in the experimental group four subjects of the 100 show low proficiency on at least four tasks of the battery, while in the control group eight subjects of the 108 appeared to be in this condition. The results of the January-February administration in second grade confirmed a significantly higher performance of the experimental group on the addition and subtraction, and also on the tasks on ordering increasingly and decreasingly. The third administration of the assessment battery in May of the second grade again confirmed these results.

As for the results on the validity of the assessment battery developed within the project, data show a significant correlation ($p > 0.05$) between the newly designed tasks and the standardised battery. In particular, there appears to be greater reliability ($\alpha = 0.8$) for the tasks that evaluate numerical knowledge. The comparison between the means of the scores obtained by the two groups on the tasks of the standardised calculation test (AC-MT) show a significant difference (t student = $p > 0.05$) on speed, operations and number knowledge. The experimental group appears to have higher mean scores on every task of the standardised test. Moreover, the percentage of subjects in the experimental group with performances at or below the cut off score on the AC-MT battery was about half of that of these subjects in the control group (7% vs. 13%). These findings in particular suggest that the didactical materials developed in PerContare do contribute significantly to diminishing the number of potential false positives in the diagnoses of dyscalculia.

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MAPPING KINDERGARTNERS' NUMBER COMPETENCE

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Abstract

In this study the number competence of kindergartners was investigated. Based on a series of items involving counting, subitizing, additive reasoning and multiplicative reasoning we collected data from a sample of kindergartners in the Netherlands ($N = 334$) and Cyprus ($N = 304$). A confirmatory factor analysis showed that the four-factor structure fit to the empirical data from the Netherlands, and that the competence of the kindergartners in Cyprus reflected a two-factor structure including extended counting and additive reasoning. With respect to this latter, common number component, the Netherlands children outperformed those from Cyprus. In both countries the children who were in the second year of kindergarten did better than those who were in the first year. In the Netherlands, out of the four components, multiplicative reasoning was the most difficult, whereas in Cyprus additive reasoning was more difficult than extended counting.

Key words: differences between countries, kindergartners, number performance, structure of number competence

Introduction

Number is the most fundamental topic of mathematics in primary school (Sarama and Clements, 2008), but children start learning about numbers and develop basic skills and concepts in arithmetic already before they are taught mathematics formally from Grade 1 on (e.g., Gervasoni and Perry, 2015). In fact, the foundation for children's understanding of number is laid in their preschool and kindergarten years when they learn about quantities, numbers, operations, and relations between quantities as a way of modelling their world (Nunes, 2012).

The number concept comprises several components, which children need to develop and link to each other to build a deep understanding of the concept (Sarama and Clements, 2008). Most studies on early number development have examined children's competences by focusing on these number components separately from each other. Taking a more comprehensive approach, this study aims to investigate the structure of young children's number competence with a focus on four major sub-domains of number development: counting, subitizing, additive reasoning and multiplicative reasoning. The nature of this structure in children in two countries is also a concern of the study.

The domain of number

Counting

Counting is considered a key component in the development of the concept of number (e.g., Sarama and Clements, 2008). By using counting in everyday experiences children construct basic knowledge about numbers resulting in being able to find the numerosity of a collection of objects. In order to succeed in this, children have to acquire the ability of oral counting (knowing the sequence of number words), the one-to-one correspondence between the set of objects and the number words, the ability to keep track of the counted objects and the objects that have not been counted and the cardinality principle (that the numerosity of a set of objects is indicated by the last number word of the counting process) (Baroody and Wilkins, 1999; Kilpatrick, Swafford and Findel, 2001).

Subitizing

Another way of determining the numerosity of a collection of items is subitizing. This means that children can recognise instantly the number of small collections (Baroody and Wilkins, 1999). The development of this ability is considered to take place even before children have learned to count objects reliably (Baroody, 2004). Based on the different mechanisms underlying subitizing, a distinction is made between perceptual and conceptual subitizing (Clements, 1999). Perceptual subitizing refers to directly seeing how many objects there are. Conceptual subitizing is quickly figuring out the numerosity of a larger collection of objects by viewing it as being composed of smaller groups of objects.

Additive reasoning

Children's early experiences with counting (Eisenhardt et al., 2014) and subitizing (Clements, 1999) form the basis for additive reasoning (addition and subtraction). This starts already at a young age. Most preschoolers can understand and solve simple additions and subtractions at the age of three, often by using real objects to model the tasks (Kilpatrick et al., 2001) through perceptual counting (see Eisenhardt et al., 2014). Playing with collections of objects supports children's development of the intuitive ideas of adding to/having something more and taking away/having something less (Baroody and Wilkins, 1999). Later, at the age of five or six, children acquire a basic understanding of part-whole relationships (Sophian and McCorgay, 1994), which is a great achievement in the development of the understanding of additive relations and of number sense in general. It means that they understand that any number can be represented as the sum of other numbers (additive composition) (Nunes, 2012), which helps them, for example, to solve missing-addend problems (Sarama and Clements, 2008).

Multiplicative reasoning

The domain of multiplicative reasoning, which includes multiplication and division, is clearly distinct from the domain of additive reasoning (e.g., Clark and Kamii, 1996; Vergnaud, 1983). Previous studies have shown that in the first grades of primary school, before formal instruction on multiplicative reasoning, children can resolve a substantial amount of problems in this domain (Bakker et al., 2014; Mulligan and Mitchelmore, 1997). A study by Carpenter et al. (1993) revealed that even children at the kindergarten level were able to solve various multiplication and division word problems. Research suggests that children's multiplicative knowledge is strongly influenced by the characteristics of the problems offered to them. A study of Bakker et al. (2014) showed that ‘equal groups’ problems were the easiest problems and that problems with pictures representing the multiplicative situation are easier than problems which are presented without countable objects. The same study also showed that multiplication and division problems were at same difficulty level, which could be accounted to an intuitive understanding of the connection between the two operations (see also Mulligan and Mitchelmore, 1997).

Cultural aspects in the learning of number

Most of the comparative studies between different cultures have concentrated on comparing the mathematics competences between Asian, Oceanian and South American students and students from the western countries. A common finding of these studies, which starts to appear even in the earliest years of children’s development, is that, for example, Asian children outperform Western children in the domain of number concepts (Anderson, Anderson and Thauberge, 2008; Starkey and Klein, 2008). A number of factors that were found to account for this difference include “linguistic regularities, parental and teacher mediation styles, different cultural expectations, and how mathematics is practised within different cultural groups, both in and out of school” (Anderson et al., 2008, p. 119). Whether these differences have been found also within Western countries whose cultural traditions may be closer to each other is to our knowledge unknown.

The present study

The aim of the study was to map kindergartners’ number competence by identifying its key components and investigating their performance in these key components in two countries. Considering the major subdomains of early number knowledge, we investigated: (a) whether kindergartners’ number competence can be distinguished into four factors, namely, counting, subitizing, additive reasoning and multiplicative reasoning, (b) how able kindergartners are in the number competence component each factor stands for and (c) whether the previous issues differ for children in different countries.

Methods

Set up of the study

A survey was carried out in the Netherlands and in Cyprus. The kindergartners’ number competence was assessed by administering two booklets, each containing items about counting, subitizing, additive reasoning and multiplicative reasoning.

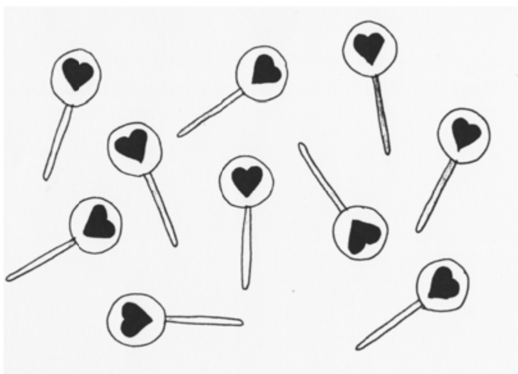
Participants

The participating children in the Netherlands were from kindergarten classes in 18 primary schools situated in the province of Utrecht. Each school participated only with one class containing first (K1) and second year (K2) kindergartners. The total Netherlands sample included in the analysis contained 334 children, 176 girls and 158 boys; 123 K1 children and 211 K2 children. The K1 children had an average age of 4.67 years and the K2 children were on average 5.69 years old. The participating kindergartners in Cyprus were from 10 primary schools situated in the province of Nicosia. Four schools had integrated kindergarten classes containing first-year kindergartners (K1) and second-year kindergartners (K2), while six schools had K1 and K2 children in separate classes. Also different from the Netherlands sample, the schools in Cyprus participated with more than one class. The analysis was based on 23 classes involving 304 children, 163 girls and 141 boys; 86 K1 children and 218 K2 children. The K1 children had an average age of 4.67 years and the K2 children were on average 5.61 years old.

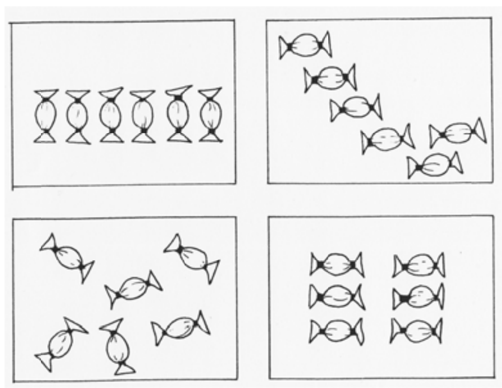
Items

To measure early number competence a series of pictorial paper-and-pencil items was developed (see Fig.1 for examples of items), including three items that refer to counting, three items that are about subitizing, three items that include additive reasoning and five items that refer to multiplicative reasoning.

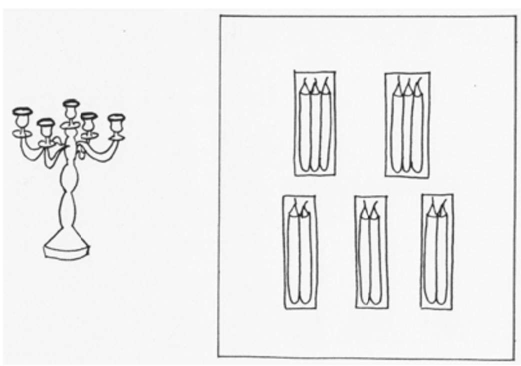
Lollipops: Mommy buys 5 lollipops. Put a circle around 5 lollipops.



Sweets: Underline the picture on which you can tell the fastest that there are 6 sweets.



Candle holder: Underline the boxes of candles you need for this candleholder.



Mittens: Underline the amount of mittens these children need in total.

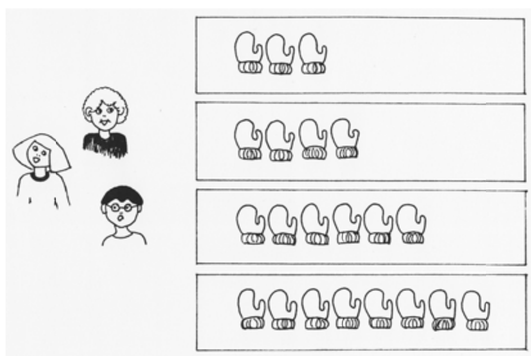


Fig. 1: Examples of items: *Lollipop* (counting), *sweets* (subitizing), *candle holder* (additive reasoning), *mittens* (multiplicative reasoning)

Every item covers one page and contains an illustration depicting a situation and/or a number of small illustrations that represent the possible answers. After a test item was read aloud to them, the children had to answer by underlining or putting a circle around the picture or pictures that represent the correct answer. Correct responses were coded as 1, and incorrect ones as 0.

Results

Components of early number competence

Confirmatory factor analysis (CFA) was applied, using MPLUS (Muthén and Muthén, 2010) to investigate whether different items in the topic of number can form different factors which reflect different types of competence (early number competence components) in the two samples. In order to evaluate model fit, three fit indices were computed (Marcoulides and Schumacker, 1996): the chi-square to its degree of freedom ratio (χ^2/df should be less than 2); the comparative fit index (CFI should be higher than .9); and the root mean-square error of approximation (RMSEA should be close to or lower than .08).

The results of the CFA are presented in Fig. 2. On the left the structural equation model is shown with the latent variables of the number competence components and their indicators for the Netherlands sample. We evaluated the construct validity of this model by examining whether the 14 items loaded adequately on each of the four number competence factors described above: counting, subitizing, additive reasoning and multiplicative reasoning. The CFA showed that this model reflected the empirical data quite well, as the descriptive-fit measures indicated support for the hypothesised model (χ^2/df = 1.10, CFI = .99 and RMSEA = .02). This means that students' early number competence in the Netherlands can be distinguished into four factors: counting, subitizing, additive reasoning and multiplicative reasoning. All factor loadings were statistically significant and most of them were rather large; the total range is from .36 to .86. The interrelations between the factors were significant and considerably strong, ranging from .72 to .95. In a general sense, this indicates that children in the Netherlands who were efficient in one number subdomain were quite competent in another subdomain and vice versa.

To evaluate the construct validity of this model in the Cyprus sample, CFA was used. The results of the analysis showed that the correlations between some latent variables (factors) were greater than 1 indicating that the four-factor structure did not make sense for the empirical data on number competence of young students in Cyprus. Another model with a smaller number of factors had to be explored. The model that best fitted the Cyprus data (CFI = .95, χ^2/df = 1.16, RMSEA = .02) was the one presented in the right part of Fig. 2, which includes two factors.

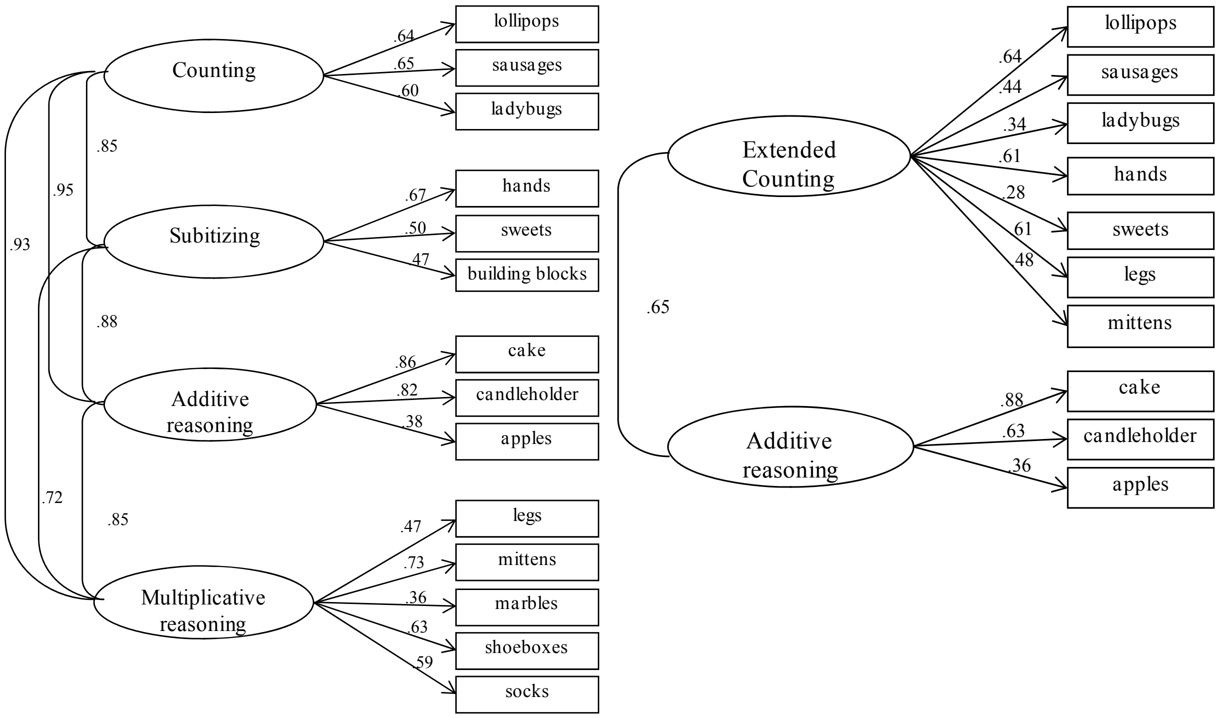


Fig. 2: Structural equation model for early number competence components in the Netherlands (left) and Cyprus (right)

In this model the items *cake, candleholder* and *apples*, were found to load adequately to one factor which stands for additive reasoning, as it is the case for the Netherlands sample. The items *lollipops, sausages, ladybugs, hands sweets, legs* and *mittens* were found to load adequately on the other factor. Although these items were initially used to measure three different number competences, namely, counting, subitizing and multiplicative reasoning, they can all be solved with the use of counting. Therefore, we considered the second factor in the Cyprus model to stand for counting. To distinguish this factor from the factor that stands for counting in the model of the Netherlands we named it "extended counting". This means that students' early number competence in Cyprus can be distinguished into two factors: extended counting and additive reasoning. As it is shown in the right part of Fig. 2, most factor loadings were rather high; the total range was from .28 to .88. It is to be noted that four items, that is, *marbles, shoe boxes, socks* and *building-blocks* are not included in the model because although we expected these items to be regressed on the factor "extended counting" their loadings on the particular factor were not statistically significant. This indicated that children's observed performance on these items was not related to the latent factor of "extended counting" and as a result they were eliminated from the model (Brown, 2006). The interrelation between the two factors, corresponding to students' competence in counting and additive reasoning, was significant and considerably strong (.65). This indicates that students in Cyprus who were competent in counting were efficient also in additive reasoning and vice versa.

Kindergartners' performance in each number competence component

Tab. 1 shows that for the total sample of kindergartners in the Netherlands, performance in counting ($M = .52$), additive reasoning ($M = .51$) and subitizing ($M = .55$) was significantly higher than performance in multiplicative reasoning ($M = .35$). These differences were found to be significant (due to limited space we left out detailed statistical information). Also, the children in the Netherlands appeared to perform better in subitizing than in additive reasoning. This difference was found to be marginally significant. For the common factor between the structural models of the two countries, that is, additive reasoning, kindergartners in the Netherlands performed significantly better than the kindergartners in Cyprus.

Similar to the results in the total sample of the Netherlands, we also found for the two kindergarten years that children performed significantly better in counting, subitizing and additive reasoning than in multiplicative reasoning. K1 children in the Netherlands demonstrated signifantly higher performance in subitizing than in counting and additive reasoning, which was not the case for the K2 children. For Cyprus, the results in the total sample was similar to those found for the two kindergarten years, namely that children performed significantly better in extended counting than in additive reasoning. Moreover, in both kindergarten years the Netherlands kindergartners outperformed the kindergartners in Cyprus in additive reasoning. When we compared the scores in the two kindergarten years for the various number competence components, we found that in both countries the K2 children significantly outperformed the K1 children in all components.

Component	NL				Component	Cyprus			
	M			M_{K2}-M_{K1}		*M*			M_{K2}-M_{K1}
	K1+K2	K1	K2			K1+K2	K1	K2	
Counting	.52	.34	.63	.29*	Ext counting	.49	.36	.54	.18*
Add reasoning	.51	.35	.59	.24*	Add reasoning	.38	.26	.43	.17*
Subitizing	.55	.45	.60	.15*					
Mult reasoning	.35	.23	.42	.19*					
N	334	123	211			304	86	218	

Tab. 1: Mean score for each early number competence component for the whole sample and each kindergarten year in the Netherlands and in Cyprus; differences in mean scores between kindergarten years in both countries (* $p < .01$)

Discussion and conclusion

This study provides evidence for the multidimensional structure of kindergartners' number competence. The investigated four-factor structure including counting, subitizing, additive reasoning and multiplicative reasoning indeed reflected the number competence, but only in the children from the Netherlands. For the children in Cyprus, a two-factor structure, including extended counting and additive reasoning was more adequate to capture their number competence. A possible reason for this finding could be that the children's profile might be influenced by the Cyprus kindergarten's mathematics curriculum and teaching practices, which emphasize counting and additive reasoning and give less attention to subitizing and multiplicative reasoning. To solve items referring to these latter competences, Cyprus children may have applied counting strategies instead, which were quite familiar to them and which could be applied because the items included countable objects. Another finding was that for additive reasoning, the component the two samples had in common, the Netherlands kindergartners outperformed the children from Cyprus. In sum, our study revealed differences in children from two countries in key components of early number competence. However, this conclusion should be taken with prudence, because our sample was small, and was not representative for the countries' population. A further limitation of our study was that our collection of items did not cover the full domain of number and operations. Further research is necessary to cancel out these limitations and also to identify more in-depth the sources of differences in number competence of kindergartners in different countries.

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RECALLING A NUMBER LINE TO IDENTIFY NUMERALS

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Abstract

In *Early Action for Success*, a strategy to lift the literacy and numeracy performance of students in the early years in primary schools serving disadvantaged communities in New South Wales (Australia), schools have engaged in the close monitoring of student progress in whole number knowledge. This has involved tracking progress in oral counting, strategies to determine the sum or difference of two quantities and identifying numerals. A videoed interview with one of the students revealed an unusual approach to numeral identification. This 7-year-old's process for identifying numerals appeared to rely on locating them on a mental number line and using their location to retrieve their names. This is at odds with the proposition that children learn to map number words and numerals onto a core representation of numerosity. It does provide support for the idea that different brains process numbers differently.

Key words: language of number, numeral identification, numeral recognition, representations of quantity, writing numerals.

Introduction

When children start the first year of formal schooling in public schools in New South Wales, their teachers conduct a one-to-one interview with each child to determine what whole number knowledge the children have (Gould, 2012). The teachers use a standardised interview that assesses knowledge of the forward sequence of number words, numeral identification, the use of counting to solve addition and subtraction problems, and instant recognition of small quantities (Fuson, 1988; Steffe and Cobb, 1988; Wright, 1994). This information is essential to planning teaching activities to meet the needs of all students and provides a snapshot of the school entry number knowledge of over 65 000 children with an average age of 5.3 years.

These various aspects of number knowledge have been described in a learning framework (Wright and Gould, 2002). However, to use whole number effectively, children must integrate many layers of verbal, procedural, symbolic and conceptual meaning. The knowledge of sequences of number words, forwards and backwards by ones and tens, and strategies used to answer addition and subtraction questions are interrelated and interdependent. For example, counting objects by allocating a number word to each object once, and recognising that the last number word stated corresponds to the total, clearly relies on correctly producing the sequence of counting words. Knowledge of the forward sequence of number words is effectively used in the service of quantifying.

Children in Australia, as in many other countries, must also learn that quantities are associated not only with number words but also Western Arabic numerals. Both "four" and "4" can be used as symbols to represent the quantity associated with a collection of any four things. Although words, objects and numerals can all be used to represent quantity, recent research suggests that the brain may process numeric symbols differently from number words (Shum et al., 2013).

Understanding how these connections across the representations of quantity operate is important. Neuropsychological models of number processing attempt to specify how the different representations of numbers are interconnected (Dehaene and Cohen, 1998). Children could conceivably learn to map spoken and written numerals onto each other. However, quantity and language do not appear to simply map to each other (Gelman and Butterworth, 2005) and different brains process numbers differently (Krause, et al., 2014).

The process in which a number is translated from one format into another one is referred to as number transcoding (Imbo et al., 2014). Considering numerals from the perspective of psycholinguistics, number symbols can be described in terms of their production and recognition (Mark-Zigdon and Tirosh, 2008). Selecting a specified numeral from a randomly arranged group of numerals in response to hearing it is described as *numeral recognition.* That is, numeral recognition is a receptive skill. Being able to name a specific numeral when it is shown to you is referred to as *numeral identification*. Numeral identification is a productive skill. Some children are able to recognise a numeral in response to an aural cue but not produce the number word in response to seeing the numeral. Numeral recognition and numeral identification operate in a mod akin to a dual carriageway. However, operating with number is not always simply based on verbal processes (Brysbaert, Fias and Noël, 1998). Noël and Seron (1997) have further argued that code-dependent intermediate representations may be used in mathematical operations.

As part of the New South Wales Literacy and Numeracy Action Plan, over 200 public primary schools have been closely monitoring the progress of students' whole number knowledge in the first three school years. This has required schools to regularly report students' progress in oral counting, developing strategies to determine the sum or difference of two quantities as well as identifying numerals. At the request of one of the schools the author conducted a clinical interview with a student who was experiencing difficulty linking the various representations of quantity and, in particular, numerals. The purpose of the clinical interview was to diagnose the student's use of counting to determine quantity and the association between quantity, number words and numerals. This paper reports on the results of the clinical interview and the implications of one use of a mental number line on the development of early number proficiency.

Materials and Methods

The male student who is the subject of the clinical interview was 7 years old and was the 26th student videoed on the day. Within this paper I use the pseudonym Jed to refer to him.

Jed was in Year 2, his third year in a primary school serving a low socio-economic community. The Assistant Principal at the school provided a detailed background of Jed's achievements at school and commented that the last time she worked with him; he counted his fingers starting from one to display seven fingers. That is, he used a perceptual counting strategy to make seven. It was also reported that Jed could not remember the numeral '7' for a whole week. This is extremely unusual for a student in the third year of school, as more than 50% of students starting Kindergarten (the first year of school in NSW) can identify the numerals 1 to 10 (Gould, 2012).

After the clinical interview, I asked the Assistant Principal to determine how Jed responded to writing numerals and to creating finger patterns for numbers. In particular, I wanted to know how far Jed could correctly write the sequence of numerals and if he could generate numerals in response to hearing them.

The clinical interview started with the task of determining how many small square foam tiles were on the table in front of the student. Twenty-four tiles of the one colour were provided to Jed, in no obvious order.

Results

Approximately halfway through counting the 24 tiles, Jed stopped and said it was "a tricky number". When he was asked to start again and count in a clear voice, he correctly counted 12 tiles and then stopped again; apparently uncertain of what came next. When asked, "What comes after twelve?" he again responded that it was getting 'trickier'. The interviewer then asked if he could continue counting if he knew that thirteen came next. Following this prompt, and with encouragement, he was able to state that fourteen came next. He then continued the oral count correctly to eighteen were he again stopped.

It is not unusual for students learning to count in English to learn the number words from eleven to twenty more slowly and with more errors than in many other languages. However, this challenge is usually successfully addressed in the first year of school.

Moving the tiles aside, the interviewer asked Jed to start counting from one and to go as far as he could. It is possible that keeping track of the count by matching the number words to the objects might have impeded Jed's production of the number words. Jed quickly gave a correct oral count to thirteen, omitted fourteen, said fifteen, and then recognised that this was not correct, and stopped. On his next attempt he again omitted fourteen and appeared to be uncertain of the counting words past thirteen.

When asked if he could count backwards from ten, it became evident that he could not. Jed was then asked a number of questions to determine the range of his memory for serial order. For example, he was asked to recite what he knew of the alphabet. He responded, "a, b, c, d, e, f, g, h" and could go no further.

Jed was then asked a series of questions to determine his awareness of numbers in his environment. His responses indicated that he has a home telephone, but did not know the telephone number. Jed knew the street name of his home but not the number.

Even within the range of numbers that he could successfully count, Jed appeared to be reliant on reproducing the whole string. For example, he could not start counting from 'ten'.

Linking numerals and number words

The interviewer sought to investigate any links Jed could make between the sequence of counting words and numerals.

Jed was asked how far he had previously counted and, as he could not remember, he was asked to count again. This time he stopped at "twelve". The interviewer reminded him that he had previously gone one number further. With encouragement he was able to state that the next number was "thirteen".

The interviewer selected the initial group of numeral cards because they were near the end of Jed's oral counting range. The numeral '8' was added to the group of numeral cards as the Assistant Principal had referred to Jed describing it as "two circles on top of each other". When asked which of the cards he recognised, Jed responded by selecting the '12'. He then tentatively identified it as 'eleven' and self-corrected to indicate 'twelve'.

Although Jed could not identify '15' he could, with effort, identify '5'. This suggested that he was not using the '1' as a positional tag for ten in reading numerals. That is, he may not be decoding the '15' as a '1' and a '5'.

When he identified the '5', he thought about it for some time before answering. When asked how he worked it out, Jed responded that he recognised the 3 then he went to the next number. He couldn't identify the '8', '15', '13' or '14'. Moreover, Jed couldn't really identify the '12' but rather appeared to be using some process such as counting to determine its name. When the interviewer returned to ask Jed to identify '5' he could be heard to say 'three' before *subvocalising* the count to '5'. The interviewer then asked if Jed remembered what the '3' is, and pictured where the '5' is, while gesturing a 'point count' process.

When presented with '4', Jed again appeared to be using '3' as a reference point to identify '4'. He confirmed that he imagined where the numbers are on a number line. Jed could readily identify the numerals '1', '2' and '3' but appeared to need to 'calculate' the names of other numerals. Although he worked out '7', he said he did this by recognising a number and counting on two. However, he did not rapidly identify '5' but rather determined the corresponding number word by counting from three.

Jed appeared to use a form of a mental number line from 1 to 10, with the location of the numerals often unclear above 5. When asked to identify '6' Jed repeatedly asked if it was upside down. He then answered that '6' was 'five'.

Jed's process of numeral identification does not align with his knowledge of counting word sequences. He could consistently count just past ten but he could not identify all of the numerals from '1' to '10'. He appeared to only be able to effortlessly identify the first three numerals. For other numerals, Jed gave every indication of accessing a mental number line arrangement of numerals and counting from the location of a known value.

Writing numerals and creating finger patterns

In the week following the interview, Jed was asked to write the numerals in order as far as he could. He wrote the numerals 1 to 15 correctly, then wrote the '1' for 16 and stopped to think. He looked back over the numbers he had previously written before completing '16'. Jed then wrote the numbers to 20 correctly but wrote '12' for '21' and stopped (Fig. 1).

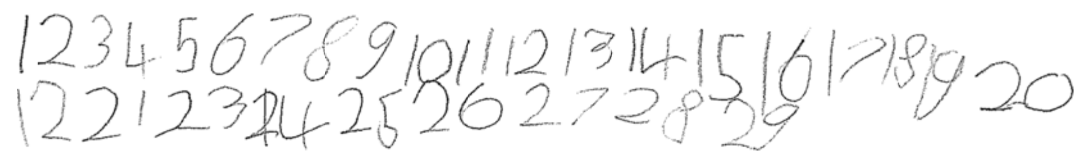

Fig. 1: Jed's recorded numeral sequence

When asked, "What comes next?" he said, "I don't know". He was then asked if there was a way he could work it out. He looked at the number '12' then said, "I know", and wrote '21'. Jed didn't correct his initial attempt at '21', which may have influenced his omission of '22'. The extent of his written sequence of numerals exceeded his correct oral counting sequence and his identification of individual numerals.

To determine if Jed could write individual numerals out of sequence he was asked in turn to write 3, 7, 1, 9, 4, 2, 5, 8, and 10. He wrote the numerals 1, 2 and 3 automatically but used his fingers to count from one for all other numerals. For '9' he counted 9 fingers correctly but wrote '6'. He then said, "I wrote that wrong", recounted on his fingers and corrected it (Fig. 2).

Fig. 2: Recording numerals out of sequence

Jed did not write all of the numerals from left to right in the order that he was asked to write them. He wrote the '3' and the '7' then went back to the left edge of the page to write the '1', as if he was visualising a number line. He did the same thing to record the '2'.

Jed was asked to create finger patterns to represent numbers between 1 and 10 (specifically 1, 5, 3, 7, 4, 10, 8, 2 and 6) and the only finger patterns that he automatically represented were 1, 2, 3, and 10. He counted all other numbers from one. Jed's process of creating finger patterns for numbers was closely aligned to his numeral identification.

Discussion and conclusion

Neuropsychological models of number processing have attempted to describe the way in which mental representations of numbers are interconnected. McCloskey (1992) proposed a model with a single abstract quantity representation surrounded by comprehension and production modules for Arabic numerals and number words, as well as calculation mechanisms. This model successfully explained the performance of a number of patients with neuropsychological impairments of number processing.

One of the limitations of the McCloskey model is the absence of direct pathways between various modules, such as seeing a number word and saying the number word. Campbell (1994) proposed an encoding-complex perspective where separate modality specific number codes exist. According to this model, number skills would be based on multiple forms of internal representation and could be realised in many ways. Number processing could vary as a function of cultural or idiosyncratic experience.

The idea that number processing could vary as a function of idiosyncratic experience aligns with Jed's method of recalling an ordered line of numerals to identify numerals. Rather than an instant process of identifying numerals, Jed's transcoding is performed as a semantic process; he gives meaning to the symbols by using a counting sequence. Jed counted to match a number word to a recalled sequence of numerals. His method is atypical in that people normally transcode between verbal and Arabic numerals by means of an asemantic system; the semantic route is usually only activated when the number has to be used in some other task (Power and Dal Martello, 1997).

Jed's method of identifying and recalling numerals is at odds with any proposition that children learn to map both number words and numerals onto a core representation of numerosity. Does the commonly used triple-code model adequately describe Jed's method of identifying numerals?

The triple-code model (Cohen, Dehaene and Verstichel, 1994) assumes that there are three different codes associated with number: Arabic (numeral), verbal and analogue magnitude. The Arabic code is responsible for multi-digit calculations while simple calculations and verbal counting are executed by verbal code. The first two codes are clearly notation dependent. The analogue magnitude code, used for comparing the size of numbers and number approximation, is considered to be notation independent. The triple-code model of neuropsychology associates activity in different parts of the brain with each node of the triple-code model.

The depiction of the triple-code model (Fig. 3) is based on Cohen, Dehaene and Verstichel (1994). It shows a semantic pathway, activating the quantity associated with a numeral, as well as an asemantic Arabic to verbal translation route.

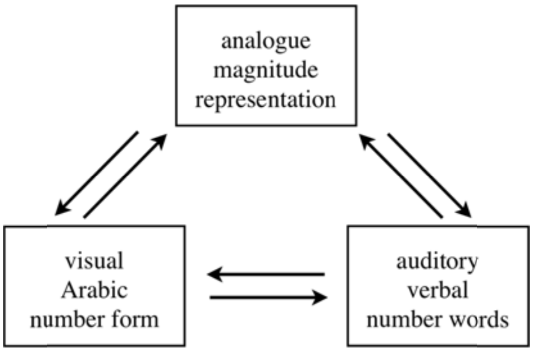

Fig. 3: Triple-code model

Numerals presented to Jed elicited attempts to visualise the location of the numeral, followed by seeking the corresponding number word by counting. Jed's retrieval strategy for numeral identification is not an asemantic reading pathway. In reading numerals his method relies on knowing "1, 2, 3" and using a count to identify the number word associated with the numeral. To write requested numerals beyond "3" he counted from one each time.

The mental coding pathways used when working with number may be more complex than many current models allow. Neuropsychological models of number processing, such as the triple-code model, do not adequately describe Jed's way of identifying Arabic numerals. Although Jed's method of identifying numerals is idiosyncratic and highly inefficient, it demonstrates that changing code from numerals to words need not follow a quantity translation or an asemantic reading pathway. Jed's interpretation of numerals is more the result of a time-intensive procedure than accessing an abstract representation of quantity.

Many neurological models associated with coding quantity do not include the use of finger patterns. Recent theories of neuronal recycling (e.g. Penner-Wilger and Anderson, 2013) suggest that redeploying cognitive resources might contribute to understanding how representing numbers using fingers could evolve as a component of cognitive number architecture. Children's finger strategies and their methods of recognising, identifying and recording numerals as different ways of representing quantity in the early years is clearly worthy of further research. Cases of atypical development in children's number knowledge can be of particular value in testing out predictions made from competing neurological models. The pathways we use for decoding and encoding numerals may be quite different at various times in our lives.

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HOW DO CHINESE STUDENTS SOLVE ADDITION / SUBTRACTION PROBLEMS: A REVIEW OF COGNITIVE STRATEGY

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Abstract

Chinese learners' excellent achievement in mathematics, especially in calculation, has caused a spread of curiosity from the international countries. Chinese learner's characteristics in mathematics may have some enlightenment for the international mathematics education. This article reviews the researches which involve 2 basic problems (simple addition / subtraction and equivalence problems) and analysed Chinese children's cognitive strategy in WNA.

Key words: Chinese student, WNA, cognitive strategy

Introduction

In recent decades, Chinese students' excellent performance in mathematics, especially in whole number calculation, has caused a spread of curiosity from the international countries, which drives the international scholars to study Chinese mathematical learner's characteristics (e.g. Biggs and Watkins, 1996; Marton, Dall'Alba and Lai, 1993, etc.). A comparative study (Liu, Xu and Geary, 1993) indicated that the addition calculating scores of Chinese students is 3 times as that of American ones. Specifically, Chinese students use more advanced strategies and faster retrieval speed (Tab. 1).

Strategy	%		Reaction time(ms)	
	Chinese	American	Chinese	American
counting fingers	—	36	—	4,300
verbal counting	4	28	2,300	3,900
decomposition and combination	10	7	1,800	4,000
retrieving	86	29	1,100	3,000

Tab. 1: The strategies in addition tasks

To make a close analysis of the table above, we may easily summarise that the usage and reaction time of retrieving[1] of Chinese students are both 3 times short as that of American ones. American students use counting strategy (counting fingers or verbal counting) more frequently than Chinese counterparts. In other words, Chinese students are advanced than the American peers both in term of strategy and efficiency. Comparatively, the usage of retrieving for American children is just significantly increased after admission and become the dominant

[1] Retrieving refers to the strategy that student calculate by recall the facts in the long term memory.

strategy only at grade 3 (Wo, Li and Chen, 2002). Actually, even to a larger extent, Chinese youths' performance in arithmetic tests is better than American counterparts in that they tend to use more cost-effective strategies (high accuracy and time-saving), according to some cross-cultural researches (e.g. Liu et al., 1996). To some extent, the frequent usage of retrieving may be closely related with Chinese curriculum philosophy which emphasises the "double-base"[2] (Zhang, 2006), "practice makes perfect" (Li, 1996) and "ingenious practice" (Stevenson and Lee, 1990). In this paper, the author will review the typical studies conducted by the cognitive psychologists and mathematics education scholars in China and try to explore the basic features of the WNA strategy for Chinese students.

Methods

We chose some high-cited Chinese papers published within latest decades from CNKI to investigate the typical strategies of Chinese students from age 5-11. Research studies[3] that appeared during the latest decades were included if they fell into the following criteria:

Published studies that focus on Chinese children's (or, students') strategy (or, methods) solving the whole number addition and / or subtraction problems, in addition, these studies also analysed the strategy development features. For the above criteria, studies include quantitative, longitudinal research published in academic journals, in conference proceedings, in Doctor's or Master's dissertations and any other available resources in CNKI, no matter it is conducted by scholars of psychology or mathematics education.

In this paper, the author would like to elaborate several typical research studies in order to underscore and highlight some particularly interesting, valuable or even little-known conclusions that are worth discussing. Other studies serve as support materials.

Results

Addition and subtraction strategy for students age 5-9

Wo and his colleagues (2002) conducted an experiment of 72 5-7 year-old children on the strategy characteristics of addition problems with different difficulty (Tab. 2).

[2] Double-base refers to basic knowledge and basic skills in fundamental mathematics.

[3] We define research studies as those that featured original analysis of the authors' or publicly available data.

Simple addition	Medium difficulty	High difficulty
(01) 2+9	(11) 10+8	(15) 51+66
(07) 9+2	(17) 8+10	(14) 246+7
(02) 4+9	(12) 4+15	(20) 7+246
(08) 9+4	(18) 15+4	(16) 231+16
(03) 3+6	(13) 9+15	
(09) 6+3	(19) 15+9	
(04) 7+9	(06) 21+15	
(10) 9+7		
(05) 7+8		

Tab. 2: The addition tasks

The study documented several strategies that Chinese students usually use: retrieving, counting from 1, counting from the smaller, counting from the bigger, make 10 and mental counting[4]. Mental counting is a particularly cost-effective strategy for Chinese children when compared with American counterparts (Wo, et al, 2002), because the Chinese pronunciation bytes of number are shorter than English ones, which can enlarge the digital memory span and facilitate the mental calculation procedure (Miller & Sigler, 1987; Sigler, Lee, & Stevenson, 1986). In addition, this is highly associated with the word formation in different cultures: in Chinese spelling, numbers beyond 10 are spelled as “ten one” “ten two”, etc., which underscores the concept of decimal (wei zhi, in Chinese) and facilitates the mental counting (Geary, Thomas, Fan, et al, 1993; Gwary, Frensch, & Wiley, 1993). A striking contrast is that Chinese pupils can even calculate the basic addition / subtraction questions within 100 mentally at their preschool stage (Liu, et al, 1993) while only a half American college students (Liu, Chen, Geary, & Salthouse, 1996) can do this.

The study also analyzed the developmental features of children aged 5-7 (Table 3). The chi-square shows that the strategy usage is significantly different among the ages ($\chi2$ (2, 422) = 30.905, $p < 0.001$). So, it can be easily concluded that with the age and experience accumulated, children tend to use the faster or more convenient strategies such as retrieving, make 10 and mental counting. Accordingly, the usage of counting (counting from 1, counting from the smaller and counting from the bigger) decreases.

The data in Tab. 3 indicate indicate that retrieving becomes the dominant strategy even at the age of 5 (50.0%), which is earlier than American peers for about 2 years (Wo et al, 2002). We can also easily find an obvious optimization in “counting strategy”: the usage of counting from 1 decreased from 17.4% to 0%

[4] Basically, mental counting refers to an “inner strategy” that student get the answer by heart without using written calculation.

and counting from the smaller decreased from 9.9% to 0.5% and counting from the bigger decreased from 10.9% to 1.0% (and be replaced by retrieving at the age of 7). In addition, the usage of mental counting increased from 1.8% to 3.1%, through it was not frequently used. It is also worth noting that mental counting is different from retrieving, for the former is “calculating” while the latter is “mechanical”. More specifically, children may resort to retrieving to solve the questions like “9+8”, while they may fail to use it to solve the ones like “97+29” because they haven’t input the answer into long-term memory. So, faced with such sophisticated questions, mental counting may be a suitable strategy in the case that no draft is accessible.

Strategy usage	5 year-old	6 year-old	7 year-old
Retrieving	50.0	61.8	82.7
Counting from 1	17.4	4.9	0.0
Counting from the smaller	9.9	6.5	0.5
Counting from the bigger	10.9	26.8	1.0
Make 10	9.0	0.0	11.5
Mental counting	1.8	0.0	3.1
Decomposition	0.0	0.0	1.0

Tab. 3: The strategies of children age 5-7

In addition to strategies, the usage, correct rate and reaction time of the basic strategies in each age group is reported in the following (Table 4). It can be clearly found that the correct rate of 5 year-old children from the highest to the lowest are making 10, counting from the bigger, retrieving and counting from 1. Furthermore, the reaction time of each strategy is significantly different. The correct rate of 6 year-old children from the highest to the lowest are counting from the bigger, retrieving and counting from 1. Furthermore, the reaction times of retrieving between counting from 1 and counting from the bigger are significantly different ($F(1, 63) = 37.018$, $p < 0.001$; $F(1, 83) = 24.225$, $p < 0.001$), while the reaction times of counting from 1 and counting from the bigger are not significantly different ($F(1, 35) = 0.193$, $p > 0.05$), which indicates that, for 6 year-old children, the reaction time of retrieving is obviously higher than for other strategies’, but the correct rate of counting is higher than that of retrieving. For 7 year-old children, the reaction time of retrieving and make 10 is not significantly different ($F(1, 169) = 0.001$, $p > 0.05$).

Age	Strategy	Usage	Correct rate	Reaction time	SD	Frequency
5	Retrieving	51.3	54.2	12.60	13.04	32
	Counting from 1	18.3	42.9	26.86	12.86	9
	Counting from the bigger	19.2	72.7	15.93	9.52	16
	Make10	7.8	77.8	6.60	7.59	7
6	Retrieving	61.3	73.7	6.27	4.84	56

	Counting from 1	8.9	37.5	17.48	5.13	8
	Counting from the bigger	27.4	82.4	15.51	12.32	28
7	Retrieving	82.1	95.5	2.82	1.46	156
	Make 10	11.6	95.45	2.71	1.16	22

Tab. 4： The usage, correct rate and reaction time of the basic strategies in each age group

In short, for 5-6 year-old Chinese children, retrieving and counting from the bigger are the dominant strategies. Especially for some children who have a lower correct rate of retrieving, counting from the bigger may be helpful to accuracy. At the age of 7, retrieving becomes the dominant strategy with the highest correct rate.

Chen and Geng (2005) conducted an experiment of 90 7-9 year-old children on the strategy characteristics in addition and subtraction problems with different difficulty (simple addition, simple subtraction, addition with regrouping, subtraction with regrouping, continuous addition[5], continuous subtraction[6], mixed operation of addition and subtraction).

The study documented that the strategy that student frequently use is array, oral calculation, formulation, retrieving, make 10 and gesture, while counting, decomposition are comparatively not used (Fig. 1).

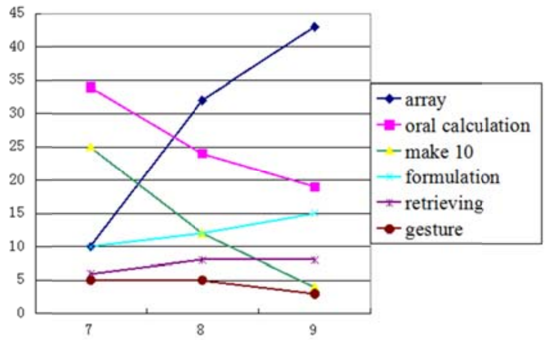

Fig. 1: The usage of the main strategies

The study also analyzed the developmental features of children aged 7-9: the chi-square shows that the strategy usage is significantly different among the ages ($\chi 2$ (2, 581) = 103.83, $p < 0.001$), which indicates that children tend to use array with the experience accumulated. And accordingly, the usage of mental calculation, make 10, counting and gesture are significantly different across ages.

Strategies of addition equivalence problems

Addition equivalence problem (e.g. Wang, 2000; Chen, Wo and Luo, 2005) is a special addition task like “a + b + c = d +__” and the basic structure of it is that,

[5] Continuous addition refers to the style of “31 + 21 + 52 = ?”.

[6] Continuous subtraction refers to the style of “123 - 31 - 41 = ?”.

there are 3 numbers added on the left and 2 added on the right and a "=" as the connection in the middle. Some scholars (e.g. Alibali, 1999; Perry, Church and Goldin-Meadow, 1988) pointed out that these mathematical tasks may be an ideal measuring tool for children's WNA strategy, and Chinese scholars have conducted some experiments to explore children's strategy characteristics based on this tool.

Take Chen's study (2005) as an example. Chen took 144 children aged 8-11 as participants and designed a group of questions which involve 12 addition equivalence problems. Specifically, these problems are organized in different types and difficulty according to the following dimensions: (1) The location of the blank. According to this dimension, the problems are differentiated into 2 categories: blank after the "=" (e.g. 9 + 7 + 5 =__+9) and blank located in the final (e.g. 3 + 6 + 8 = 6 +__). (2) The same number on both sides of the "=". According to this dimension, the problems are divided into 2 categories: with / without the same number (e.g. 9 + 7 + 5 =_+ 9, 7 + 5 + 8 =_+ 4). (3) The digit of the addend. According to this dimension, the problems are divided into 3 categories: all the addends are 1 digit, a two-digit addend in each side of the "=", all the addends are two-digits. According to the above 3 dimensions, the questionnaire contains 12 addition equivalence problems (Tab. 5).

NO.	problem	blank location	the same addend ?	the digit of the addend
1	9+7+5=_+9	after the " = "	yes	all are 1 digit
2	7+5+8=_+4	after the " = "	no	all are 1 digit
3	3+6+8=6+	in the final	yes	all are 1 digit
4	4+7+8=5+	in the final	no	all are 1 digit
5	7+9+18=_+18	after the " = "	yes	a two-digits in each side of the " = "
6	9+21+8=_+25	after the " = "	no	a two-digits in each side of the " = "
7	7+16+6=9+16+	in the final	yes	a two-digits in each side of the " = "
8	8+7+34=28+	in the final	no	a two-digits in each side of the " = "
9	15+22+26=_+22	after the " = "	yes	all are two-digits
10	33+15+24=_+30	after the " = "	no	all are two-digits
11	28+23+18=28+	in the final	yes	all are two-digits
12	17+32+19=36+	in the final	no	all are two-digits

Tab. 5: The questionnaire structure and tasks

The study summarised some typical strategies that Chinese student use to deal with the addition equivalence problems (Tab. 6).

Strategy categories	Answer examples
correct strategies	
combination	"there is a '3' in both side of the '=', so just put 4 plus 5 "
plus-minus	3+4=7,7=5=12,12-3=9
make the equivalence	3+4+5=12 , so the whole number of the right is 12 …
minus -plus	e.g. 6+25+8=_+21. 25-21=4, so 6+8+4=18
decomposition-combination	e.g. 3+6+8=4+__. borrow 1 from 6 to 3, then 4+5+8=4+_.
wrong strategies	
all the numbers added	e.g. 3+6+8=4+_. plus 3, 6, 8 and 4, then the answer is 21
add up to the "="	e.g. 3+6+8=4+_. plus 3, 6 and 8, then the answer is 17
add 2 numbers	e.g. 3+6+8=4+_. plus 3 and 6, then the answer is 9
guess	judge by interests, hobbies, intuition, etc.

Tab. 6: Typical strategies of equivalence problems

The data above yielded a strategy library for the addition equivalence problems which involves the strategies like combination, plus-minus, make the equivalence, minus-plus, etc. In addition to minus-plus and decomposition-combination (Chen et al., 2005), other strategies have been reported in the previous studies (e.g. Alibali & Goldin-Meadow, 1993; Perry et al, 1988; Rittle & Alibali, 1999). Take the question "8+7+34=28+_" as an example for minus-plus strategy, some children use the bigger number 34 on the left to minus 28 on the left which is close to 34 in number, and then get the answer 6 and plus it with the left 8 and 7. Take the question "4+7+8=5+__" as an example for decomposition-combination strategy, some children decompose 8 into 3 and 5, and then plus 4, 7 and 3. Additionally, children can use more than two strategies to solve addition equivalence problems even at the age of 8, and the usage of right strategies (Table 6) is gradually increased with age, and within the right strategies, the usage of efficient strategies (e.g. combination, decomposition-combination) are gradually increased while that of inefficient strategies are gradually reduced (e.g. plus-minus). Therefore, this study confirmed to Siegler's (1999) Strategy Development Model, that is to say, children's strategy on addition questions is time-adaptable, and due to the competition mechanism among the strategies, they resort to the high-automatic, accurate and "effortless" strategies.

In addition, the above strategies can be classified into several levels in terms of their understanding of "=". There are 2 basic meanings of "=" ---- one is operation (refers to operation process, "amount to", "generate", "get") and the other is relationship (refers to "balance", "equal to"). Students' understanding of the "=" directly affects their calculating strategy: if students can only understand that " '=' shows the operational orientation" – the ending of the operation – following with operation results, they can simply tend to use plus-minus strategy to solve the problem. If they can understand " '=' refers to balance", they will probably resort to combination, minus-plus, decomposition-combination, etc. to solve the problem. Furthermore, decomposition-combination is a more advanced strategy than combination, for it involves not only the apparent equal numbers but also the underlying equivalence.

Discussion

Children's cognitive strategy development is time-adaptable (Shrager, 1998; Siegler et al., 1999), and they tend to resort to the high-automatic, accurate and "effortless" strategies (e.g. retrieving) with the competition mechanism among the strategies. For Chinese children, retrieving is frequently used (around 50%) even at the age of 5, which is earlier than American peers for about 2 years (Wo et al., 2002). Additionally, mental counting is a particularly cost-effective strategy for Chinese children when solving these questions (Wo et al., 2002).

The cognition of addition equivalence is closely related to the understanding of "=" (operation or relationship). For those who can only understand that " '=' is a sign of a computing ending, they can simply tend to use plus-minus strategy to solve the addition equivalence problems. For those who can understand " '=' refers to balance", they would probably resort to combination, minus-plus, decomposition-combination etc. to solve the addition equivalence problems.

From the perspective of development, for the simple addition or subtraction calculations, the strategy usage is significantly different among the ages, which indicates that the strategy tend to be mature and automatic with age. For the equivalence problems, children can use more than 2 strategies to solve them even at the age of 8, while children age 9-10 tend to use just 1 or 2 strategy, which shows that children begin to form and develop the more dominant strategy (Wang, 1996). In other words, age 8 may be the transition time of addition strategy.

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ANALYSIS OF STUDENTS' SYSTEMATIC ERRORS AND TEACHING STRATEGIES FOR 3-DIGIT MULTIPLICATION

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Abstract

With the advances made in curricula reform, knowledge of learners and learning plays a significant role in pedagogical research. Based on students' error analysis, it could reflect teachers' understanding and integrating knowledge of subject matter, students and instructional strategy. Through the analysis of students' error patterns and teachers' instructional strategies for systematic errors in 3-digit multiplication, the research has observed students making three typical errors. Though teachers have some estimation and awareness of student error patterns, further exploration and application of the rules are still needed. Teachers' understanding of the nature of subject matter knowledge and students' error affect their instructional implementation. In addition, textbooks greatly affect teaching and learning. This research suggests student errors could be the new perspective to explore pedagogical content knowledge and serve as curriculum resources for research and practice in teaching. Moreover, errors could also help students think and communicate mathematically.

Key words: primary mathematics, pedagogical content knowledge, student errors

Introduction

Teaching and learning play a decisive role in curriculum reform. At the centre of pedagogy, learning is where curriculum reform should put greatest efforts. The current trend of international mathematical education reform puts emphasis on teaching for understanding (Hiebert and Carpenter, 1992). Understanding mathematics and instruction should centre on students' understanding.

Pedagogical Content Knowledge (PCK) affects teaching effectiveness. Researchers (e.g., Gess-newsome, 1999; Lederman and Latz, 1994; Li, 2006; Ma, et al., 2008; Veal and Kubasko, 2003) have shown that teaching performance and its effectiveness are related to PCK. PCK integrates teachers' understanding of specific content with their knowledge of students' misconceptions and errors as well as of relevant representation strategies for instruction.

Student errors are vital instructional and learning resources. Many studies have shown error analysis is a vital professional competence, which plays a key role in successful instruction (Burton, 1978; Huang, Huo and Xu, 2014; Weinstein, Husman and Dierking, 2000; Zhen and Liang, 1998).

Few researches on teachers' PCK and teaching strategies in primary schools are based on students' learning errors. Although some are built on specific content knowledge, they have not fully exposed the nature of content knowledge, resulting in general conclusions. As for student error research, pencil-paper tests help to gather large quantities of error types, but give limited insight into students' thinking process. When researching errors, we need not only know their patterns, but also explore why such patterns occur. So based on student errors, through qualitative research methods, we can, from the perspective of researchers, teachers and students, have deeper understanding of teachers' PCK by analysing students' erroneous thinking process and their causes as well as instructional understanding and representation on the teachers' part.

This research aims to answer the following questions:

- What are students' systematic error patterns in 3-digit multiplication, and their underlying reasons?
- What are teachers' knowledge recognition and analysis of students' learning errors in 3-digit multiplication?
- How is teachers' instructional decision-making based on students' learning errors in 3-digit multiplication?

Research Methodology

Theoretical framework

Many scholars such as Shulman (1987), Grossman (1990), Magnusson (1994), Marks (1990), Park (2010), Tsamir (1988), and Ball (2008), have tried to define PCK without reaching a consensus. Being the result of multi-domain knowledge interaction, PCK has been universally considered as the integration of knowledge of students' typical errors, teaching strategies and content knowledge in instructional practice. Knowledge of students is placed at the centre of PCK with knowledge of representation and content knowledge following. Therefore, the research of teachers' PCK can be carried out in those three aspects, exploring its function in thinking and integrating the aforementioned knowledge elements during instructional decision-making.

Among the diversified categories of student learning errors, systematic errors has been either widely cited or researched, through which the essence of mathematic learning, understanding and processing can be fully understood. This article departs from the error classification put up by Cox（1975), emphasising systematic error patterns and their causes. The repeated occurrences of systematic errors are a result of wrong algorithm or operation and are assumed to be caused by misconceptions or learning difficulties. The other type called random errors is mainly due to impatience, lack of thinking, obstructed memory retrieval, etc. (Errors that fall short of offering reasons or proofs are excluded from this research.)

Research context and participants

3-digit multiplication in primary mathematics was taken as a case context for analyse. Three teachers (T-g & T-s from Primary school F, City C and T-m from Primary school E, City A) are chosen from Grade 4, as well as all the students in their classes. All of them got consent from school boards, principals, the teachers themselves and students' parents.

Methods of data collection

Interview. Pre-and post-class interview of teachers were conducted to know teachers' understanding of the content, as well as its important and difficult parts when teaching, knowledge of students' learning, estimation of common errors, analysis of their causes, and error-related teaching strategies. Post-class interview aimed to realise the disparity between pre-consumption and real teaching, as well as shed light on questions raised during teaching. Student interviews took place during group activities and post-class. The interviewees were error-making students who were required to think aloud and describe their thinking process, displaying indirectly students' thoughts, error representation and process. Interviews from the perspective of experts were intended to find out teaching standards and orientation under this context. In addition, comments and analysis on teachers' knowledge status of the content, knowledge of students learning and appropriateness of teaching strategies when facing a certain situation were collected, as well as students' errors. Furthermore, interviews were designed to reveal what knowledge a teacher should know in an ideal state.

Observation. Classroom teaching was recorded during the whole process. Observation in class mainly focused on the teachers' activities as they taught certain contents mentioned in pre-class interviews, especially, teachers' awareness of student errors, attitudes, and teaching strategies. Samples of student errors were collected during the observation.

Material collection. Homework, exercises and test papers were all collected. Errors were picked out for coding.

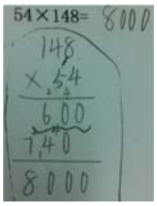
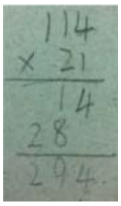
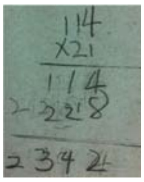
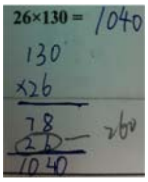
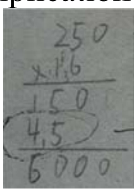
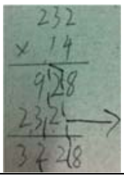
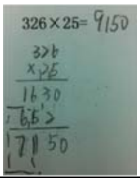
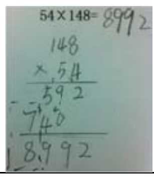
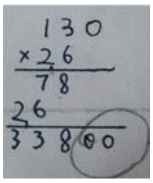
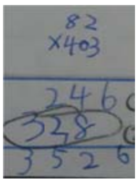
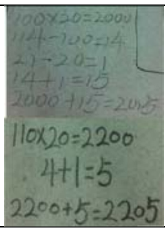
Methods of data analysis

In order to map student errors patterns, homework, exercises and test papers were analysed. Attribution analysis and categorisation were employed when coding. At the same time, the proportional distributions of students' errors were counted. Two external evaluators (subject expert E-m, primary mathematics researcher E-w) were introduced to review teachers' pre- and post-class interviews, as well as classroom teaching recording. They scored the teachers' PCK according to the PCK rubric developed by Xie (2013, revised from Park and Oliver, 2008). Meanwhile they analysed and commented on the data. The evaluators' scoring coefficients of the three teachers were $r_s=0.661$，$p<0.01$； $r_s=0.732$，$p<0.01$；$r_s=1$, which resulted in relatively high consistencies implying that the results are reliable.

Results

Students' error types and analysis

According to statistics of errors in students' homework and exercises, a comprehensive analysis of students' computational process and results were

Types	Frequency/ Percentage	Examples	Reasons
Multiplication misconception	13/15.29%	 	Students have failed to grasp the multiplication procedure of "Cross-multiplying the multiplicand by the ones digit and the tens digit of the multiplier, then adding the two partial products together" （E-m）
	7/8.24%	 	Some of the students understand the procedure, but not make clear the meaning of multiplying the multiplicand by tens digit of the multiplier, thus resulting in misconception of place value.
Computational error	Total Frequency/Percentage: 62/72.94%		
	33/38.82%	Multiplication error 	Multiplication errors happened in all three classes, reflecting students' loose grasp of multiplication tables.
	15/17.65%	Addition error  	
	10/11.76%	Carried number error  	Students lacked proficiency in carrying principles. They forgot the carried number, carried more than one digit forward, marked wrongly, etc.
	4/4.71%	"0" error  	It's worth noting that students with a "0"error are all in T-m's class, which means the teacher failed to help students understand the meaning of "0" in tens digit and in ones digit, resulting in students' computation errors.
Procedural error	5/5.88%		Students used multiplication decomposition to get the results, but failed to understand the place value of decomposed numbers, which led to faulty results.

Tab. 1: Students' typical errors

assisted by teachers' and experts' interviews as well as students' thinking aloud. The calculation of error frequency and proportion as well as the analysis of relevant causes among 1595 computation items of 3- and 2-digit multiplication show the following results: The total error frequency is 85, accounting for 5.33% of the whole sample. Random error frequency is 5, including copying wrong topic and wrong numbers, accounting for 5.88% of all error types; the other errors belong to systematic errors. Since this research puts emphasis on the analysis of systematic errors related to mathematics itself, the aforementioned careless and random errors are excluded from further analysis. Three typical error types and their related subtypes and details are given in Tab. 1.

According to statistics, error distribution is different in all classes. Procedural misconception and faulty decomposition are major mistakes made by students in T-g's class. Mistakes during computational process can be found in T-s' class. T-m's class showed a considerably higher proportion of procedural misconceptions and calculating mistakes. Based on observations conducted during classes and interviews, teacher T-g did not offer good scaffolding experiences during procedural instruction, to help bridge decomposition and vertical multiplication. Teacher T-s put more emphasis on conceptual than on procedural knowledge, while teacher T-m focused more on difficult and challenging exercises without students' thorough understanding of and familiarisation with the procedure, leading to a higher error proportion.

Teachers' understanding of students' errors and awareness activities

Teachers predicted and understood most of students' errors. The previous observations have indicated that the three teachers have a certain knowledge base in terms of students' understanding. Based on interviews, it can be indicated that the teachers can evaluate the learning situation accurately. They are on "proficient" level in understanding students' prior knowledge (such as the multiplication table, multiplication of two-digit with two-digit, multiplication of three-digit with one-digit numbers), and basic mathematic competence (various computational methods such as vertical algorithm and factor split, etc.). Based on experience from years of teaching, the teachers had views and presuppositions as to the cause of students' errors in this learning content. T-m realised that many of the students' errors were due to erroneous procedures, especially during the error-prone process in columns alignment. In an interview, T-m could not offer enough examples for students' recurring errors, and he incorrectly attributed most of the errors to failure in grasping the right procedure. In contrast, T-g and T-s can give more comprehensive examples and analysed the mistakes specifically. T-s could give a more coherent, comprehensive classification and attribution analysis (E-m, E-w). It was also observed that all teachers could not predict some of the students' errors, but during the interviews they able to reflect and analyse.

Teachers were not aware of students' errors patterns and misjudged some occasionally. In the study, it was observed that teachers could pre-set and realized most errors made by students, and, in turn, they could be more aware during the teaching process. But they cannot take care of all of students' errors due to time constraints. So teachers would simply attribute the errors to sloppy calculating without providing detailed strategies which can help students to analyse which part led to their calculation errors; rather, they would give students simple suggestions as "wrong, try again" or "be more careful". In the statistical analysis of students' errors, the continuous characteristics of errors are found in T-m's Class. The same types of errors made by the same students in the first class also appear in the second class, which shows the teacher's unawareness of the continuous characteristics of students' errors. In the instructional strategies based on a specific issue, it was observed T-s either puts too much emphasis on the unity and clarity of computation or demonstrates inaccurate professional judgment.

Teachers' instructional strategies related to students' errors

In instructional design, based on the presupposition of students' errors, teachers were able to relate their teaching plan with its implementation. Teachers employed appropriate instructional strategies based on analysis of students' errors. In the classroom, teachers were able to carry out teaching activities based on the knowledge of students' understanding. The main teaching procedure was to lead in through cases on textbook, use a variety of methods to calculate the "114 × 21", and then share their methods within groups and report one of their methods. During the process of independent calculating and group discussions, the teacher observed the students' calculations and helped them to correct errors by posing questions. As for some common and typical algorithms (including erroneous algorithm), teachers displayed them in the class and let the students verbalise the process of calculating and explain their reasons. Especially for those typical errors, the teacher asked the students to challenge each other by asking questions until the misconception was clarified.

Based on the evaluation of experts, as for their proficiency in teaching strategies, T-s, T-m, T-g were ranked "Excellent", "Good" and "Pass" respectively. T-s and T-m made proficient performance in teaching content, using strategies for instructional key points which challenged students' misconceptions, learning mistakes and learning difficulties. Based on students' feedback, teaching strategies and their adaption made by T-m were on the "Pass" level; T-g's teaching proficiency in the above-mentioned three aspects fell somewhere between "Good" and "Pass".

It was noted however that teachers should strive to improve their reflections. For instance, T-g was not specific enough in the analysis of students and teaching strategies for improvement. Instead, he thought that many students would be able to improve solely by practising more.

Conclusion and Implications

Conclusion

Three types of systematic errors displayed by students in multiplication of 3-digit with 2-digit numbers. Based on the problem solving process and errors analyses, students displayed three typical types of systematic errors: computational error, misconception of multiplication, and erroneous procedure, respectively accounting for 72.94%, 15.29% and 5.88%. Computational errors accounted for a large proportion of errors. Thus, teachers need to pay close attention to this kind of error while teaching through practicing well the multiplication tables and carrying with standard mark, to reduce or avoid the occurrence of errors. Besides, teachers should focus on helping students in learning the multiplication procedure, so that students can truly understand how to cross-multiply the ones and tens columns and add partial products together, especially the meaning of tens column multiplication and columns alignment.

Teachers ability to predict students' errors, rules exploration and application should be further improved. The research shows that teachers had some knowledge on students' errors types and could analyse the causes of errors, and presuppose teaching strategies. Teachers need to consciously discover and identify students' errors in teaching and should be able to further analyse their reasons. However, when facing students' errors, teachers neither were able to realise the instructional value of errors nor were they able to analyse the types of errors, their proportion, and related instructional adjustments consciously.

Teachers' understanding of the nature of subject matter knowledge and students' errors has impact on their instructional implementation. According to results from the experts' evaluation, the three teachers' subject knowledge was all at a skilled level. But during classroom teaching, the teachers' specific teaching strategies varied. The results of the experts' evaluation further showed that T-s's proficiency in grasping instructional difficult and key points was on the "Excellent" level, and T-g on "Good" level, and T-m upon "Pass" level. It was also observed that teachers tended to design and implement teaching plans based on personal understanding, ignoring previous strictly-followed instructional goals. Therefore, teachers with higher abilities can find core content in lesson quickly but teachers' behaviour could make students not pay more attention to important learning contents.

Textbook arrangement has great impact on teaching and learning. Teaching material is the medium between teaching and studying, which plays a very important guiding function, however, it is not hard to find cases where the specific design or content could mislead teachers and students. For example, the design of estimation impacts teachers' understanding of instruction and allocation of teaching time. Using graph paper affects students' understanding of place value, which leads to students' rigid application of graphing method and misconceptions of place value, resulting in erroneous products.

Implications

At present, most international researches focus on the current situation and the characteristics of PCK (Lederman and Latz, 1994; Hill, Ball and Schilling, 2004, 2008; Li, 2006; Lee, 2007; Ma, 2010; Park, 2011). However, there are few studies that explore teachers' PCK from students' errors perspective. Knowledge of students is at the core of PCK-integrated knowledge, which affects teaching strategies. Students' errors are relatively important in the knowledge of students, which can be affected by prior knowledge, misconceptions, learning difficulties, etc. Teachers' understanding of students' errors can reflect their understanding of subject matter knowledge and knowledge of students, as well as appropriateness of instructional strategies.

Students' errors could be resources to promote teaching research and practice. Some errors are recurring, regular, with certain continuity (unless teachers offer proper guide), with certain technical requirement, which can be attributed to specific difficult experiences in learning or some external interference (Radatz, 1980). Therefore, analysing and accumulating the types of errors and reasons could help evidence-based teaching plan and differentiated teaching.

Using error samples is a way to help students' mathematical communication and thinking. As a process of educational reform, requirements for students shift from mastery of knowledge and skills to students' learning itself and cultivation of thinking ability (as mentioned in US NCTM, the British curriculum standards, etc.). Mathematical communication among teachers and students by using errors is encouraged, which could enhance students' abilities to express their thinking in mathematics. Based on in-depth analysis of students' errors, teachers could help them form a stable and sound knowledge system by correcting students' understanding in an appropriate way.

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COUNTING STRATEGIES AND SYSTEM OF NATURAL NUMBER REPRESENTATIONS IN YOUNG CHILDREN

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Abstract

This study examines development of young children's initial understanding of representations of whole numbers and counting strategies. It was conducted with 661 children aged 3 to 7 years. The children were individually interviewed on various tasks involving different representations of numbers 1 to 10. The study provides evidence of the pathway to development in counting from pointing fingers to silent immediate recognition of numerical quantities. Key developments in understanding numbers at ages 3 to 7 appear to be (1) counting up, (2) constructing one-to-one correspondence, and (3) drawing a specified number of objects. The results indicate gradual acquisition of the system of natural number representations.

Key words: constructing knowledge, counting, natural numbers, system of representations

Introduction

How do children come to understand numbers? "How humans come to learn about the counting system of their culture is closely related to the nature of our initial representation of number, because in order to understand counting we must somehow relate it to our prior number concepts" (Wynn, 1992, p. 220). It seems that the study of children's initial learning about natural numbers has already provided a coherent body of research. Piaget's theory of cognitive development included explanation of children's development of understanding of numbers with a focus on conservation and one-to-one correspondence (reversibility). Later, researchers did not oppose Piaget's major findings about acquisition and utilization of conservation of concept of numbers (Sigel and Hooper, 1968) but did take a different stand on the issue. His point that children's understanding of numbers depends on their cognitive level was challenged by others (Bruner, 1966; Gelman and Gallistel, 1978; Le Corre and Carey, 2008). Bruner, Le Correy and Carey argued that knowledge of the counting principles is not innate but constructed out of representations.

Goldin and Shteingold assert that successful mathematical thinking involves understanding of different representations of the same concept and ability to rationalize similarities and differences among representations (Goldin and Shgteingold, 2001, p. 9). Kamii et al. (2001) reflected on Piaget's theory while discussing representations and abstraction in children's numerical reasoning. They assert that there are three kinds of knowledge (physical, social (knowledge of conventions) and logico-mathematical (mental relationships between concepts) and two kinds of abstractions, empirical and constructive.

Today, researchers are primarily focused on intervention studies in the primary grades (White, 2010). Earlier, it was also recognised that base-ten number concepts provide the basis of invented strategies of arithmetic operations (Carpenter et al., 1998). We believe that it is worth trying to bring new interpretation of the pathway to development of initial understanding of whole numbers. Our work is grounded in Bruner's theory of three modes of representations (Bruner, 1966). At age 3 to 7 years children are expected to rely on iconic representations. At age 7, they are expected to be in symbolic stage, thus they should develop correspondence between signs (symbols of numbers) and what they represent. (Note that the proposed stages of development do not imply different modes of thought as Piaget believed).

First, we attended to counting strategies. Verbal numeral sequences are considered to be the first explicit representation of natural numbers. Counting is a method to label quantities. Le Corre and Carey note that it is true that "the verbal numeral list deployed in a count routine is the first explicit representation of the natural numbers mastered by children growing up in numerate societies" (Le Corre and Carey, 2008, p. 651). Fuson (1992) remarks that counting is culturally determined. She explains that in developing counting strategies children need to learn a) the number sequence (language), b) the physical way of pointing, c) making correspondence between entities and numbers, d) methods to remember, and e) cardinal significance of counting (Fuson, 1992, p. 248). She reported that substantial competence of American preschool children in counting (Fuson, 1988). Fuson provided comprehensive analysis of counting strategies based on increasing integration of sequencing, counting and cardinality. Here, we make simple distinction between counting strategies based on three features: use of physical activities, loud or voiceless counting and speed of counting. We also examine when young children develop correspondence between certain representations and numerosity.

Our aim is to increase our understanding of the development of representations of numbers in young children. Through structured interviews we aim to gain further insight into children's understanding of numbers.

Materials and Methods

A sample of 661 children aged 3 to 7 years was individually interviewed. The interviewer read the questions from the protocol and recorded the answers in writing immediately. The obtained data was entered into a spreadsheet and these data was grouped. Bar graphs and tables were used to make clear findings. We also used χ^2 test to test independence of variables. (We calculated contingences tables using statistical program at http://www.physics.csbsju.edu/cgi-bin/stats/contingency.)

Sample

The sample was drawn from preschools in a large city of Serbia. There were 68 children aged 3 to 4 years, 157 children aged 4 to 5 years, 200 children aged 5 to 6 years, and 236 children 6 to 7 years old. Children were randomly drawn from the population of children in preschools in the city. Note that the attendance of preschools is obligatory in Serbia at age 6 to 7.

Interview items

Children were successively given ten tasks (Q1 to Q10) involving: a) counting objects, b) drawing certain number of objects and sets with certain number of elements, c) determining one-to-one correspondence between elements in a set and in string of circles, d) finishing up drawings of box diagram and of number line, and e) writing numbers. Since some tasks had two or three parts, children actually responded on 24 items. In this paper we will focus attention on a limited number of questions.

There were three counting tasks. In the first of those questions, Q1, a child was asked to count an array of stars. The interviewer recorded which one of the strategies the child used: (A) pointing by finger, (B) counting aloud, (V) counting voicelessly, (G) immediately responding, or (X) no answer. There were drawings of one, three and six stars in a row. In question Q3, children were asked to draw a set with appropriate number of elements (A) three elements, (B) five elements, and (V) nine elements. In question Q6 the interviewer asked a child to loudly count ten pencils. They simply recorded whether the child successfully counted pencils.

Then, there were tasks examining one-to-one correspondence with set representation. Children were asked to colour as many circles as there were cats in a set. There were two cases: (A) cats were drawn in a row in a rectangle field and (B) cats were spread unevenly within an ellipse.

Next, there were two tasks to “finish drawing“. In Q7, there was a drawing of box diagram with two equal boxes. Children were asked to finish the drawing “to have four“. Second, in Q8, they were asked to finish drawing on a number line “up to number five“.

Children were also questioned on their ability to write number symbols in Q9 and Q10(V). They had to write numbers given in words. Later, they were asked to write the biggest number they knew. In the last question, Q10 they were asked what is the smallest and the biggest number they knew.

Results

First we will discuss results related to counting strategies, than we will turn attention to representations.

Types of counting strategies used in Q2 may be seen in the Fig 1. Note that, from the youngest to the oldest, children by large did not make mistakes in counting one, three and six stars in a row which is in accordance with Fuson’s findings (Fuson, 1988).

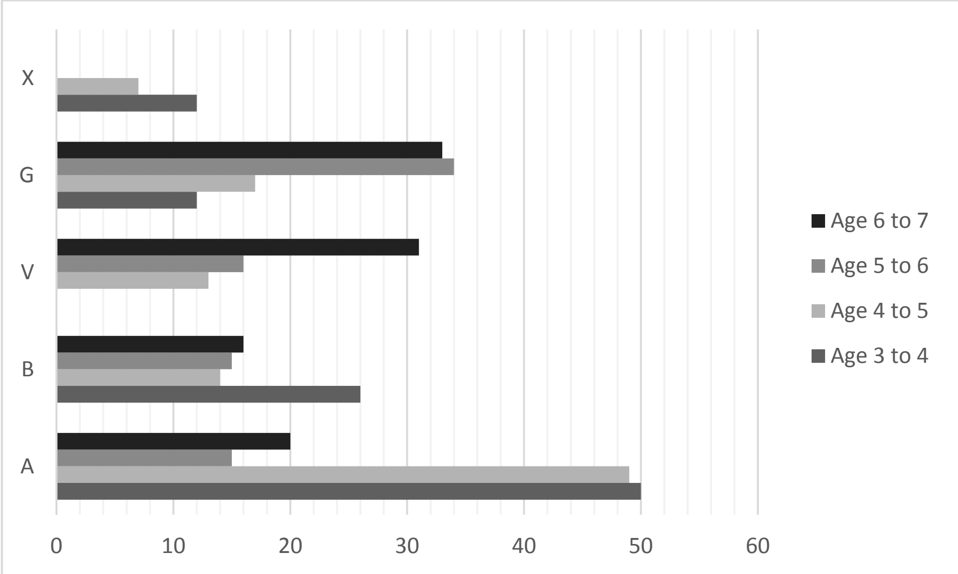

Fig. 1: Counting strategies, (A) finger pointing, (B) count aloud, (V) count voicelessly, (G) immediate response, (X) no response

While the youngest rely on fingers pointing strategy and count aloud strategy, older children tended to count voicelessly or to give immediate response. The difference in counting skill appeared to be significantly bigger when they were asked to count ten pencils. Only 26 percent of children aged 3 to 4 years successfully counted up to 10. The percentage of correct answers doubled within next year generation. Forty-six percent of 4 to 5 year old children counted up to 10. Almost all children aged 5 and older counted up to 10 without skipping any number. Precisely, success was 96 percent of children aged 5 to 6 and 97 percent of children aged 6 to 7 years.

In the following item, Q3 (A to V) children draw sets with three, five and nine elements. Tab. 1 displays percent of correct answers on each item, at different age level.

Item	Age 3 to 4	Age 4 to 5	Age 5 to 6	Age 6 to 7
Q3 (A)	16%	43%	65%	82%
Q3 (B)	7%	39%	60%	81%
Q3 (V)	6%	24%	54%	73%

Tab. 1: Set representation

The percentage of correct answers in Tab. 1 shows how successful children were in counting stars (Q1). Although all three items had basically the same request, it is evident that it was more difficult to count up to 9 than up to 3 when drawing objects. (We should mention that we judged answer incorrect when there was no line drawn to represent set.)

Next, we discuss Q4. This question we consider as a variant of conservation task whereas counting cats and then counting and colouring circles in a string shows understanding of one-to-one correspondence. Thus, mental activity of counting elicits children's deduction about equal cardinality of cats and coloured circles. In the Fig 2 we show an example of child's response. One can deduce from the drawing what was the path of that child's reasoning.

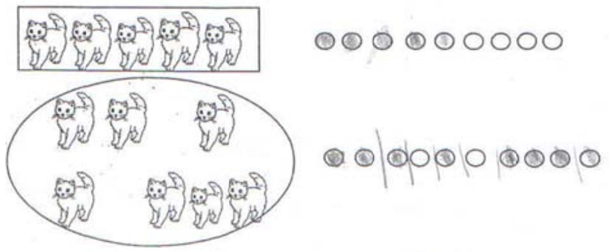

Fig. 2: Child's work, one-to-one correspondence

In the Tab. 2 we displayed number and percentage of correct answers at each age level for Q4A and Q4B.

Item	Age 3 to 4	Age 4 to 5	Age 5 to 6	Age 6 to 7
Q4(A)	23(34%)	85(54%)	179(86%)	226(96%)
Q4 (B)	12(18%)	82(52%)	156(79%)	216(92%)

Tab. 2: One-to-one correspondence

From Tab. 2 we could notice that slightly more children were successful in solving Q4A than in Q4B at each age level. In addition, we performed $\chi^{2 \text{ test}}$ of independence. We tested hypothesis H_0: Number of correct answers on Q4A and Age are independent variables. All assumptions for the test has been fulfilled. Since, X^2=186. $X^2 > \chi^2$ (3, N=661)=7.81, p=0.05 we rejected H_0 and, concluded that there is significant relationship between Age and Number of correct answers on Q4A.

We have observed that children had more problems when matching seven cats spread 'muddled' within the ellipse (Q4B) than when there were five cats in a row (Q4A). There could be two reasons for the result. One that it was easier to deal with a set of orderly placed objects. The other, that there were more cats in the later example.

Later we performed the χ^2 test of independence for variables Age and Number of correct answers on Q4B. Since X^2=172, $X^2 > \chi^2$ (3, N=661) =7.81, p=0.05. We rejected H_0: Number of correct answers on Q4B and Age are independent variables and conclude that there is a significant relationship between variables.

Next, we analysed children's understanding of different graphical (culturally invented) representations: box diagram and number line. We present examples of children's work (Fig. 3) but in this paper we will not discuss them further.

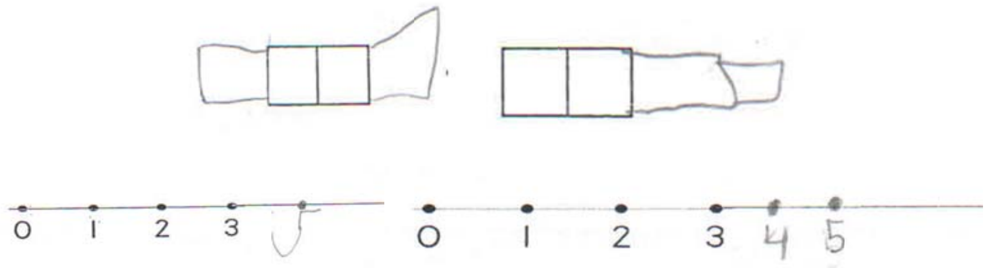

Fig. 3: Children's drawings, box diagram and number line representations

We displayed number and percent of correct answers at each age level for Q7 and Q8 (Tab. 3).

Item	Age 3 to 4	Age 4 to 5	Age 5 to 6	Age 6 to 7
Q 7	18(26%)	47(30%)	168(84%)	229(97%)
Q 8	1(1%)	46(29%)	104(52%)	176(75%)

Tab. 3: Understanding box (Q7) and number line (Q8) representations

It can be observed that Q7 was easier than Q8 for children at each age level. We tested hypothesis of independence of variables: Number of correct answers at Q7 (box diagram item) and Age. All assumptions for the χ^2 test for independence has been fulfilled. The χ^2 test was statistically significant, $X^2=282.$, $X^2> \chi^2(3, N=661)=7.81$, $p=0.05$. So, we rejected H_0: Variables Age and Number of correct items on box diagram representation (Q7) are independent and concluded that there is significant relationship between variables.

Later, we performed the χ^2 test for Q8. It was also, statistically significant, $X^2=148.$, $X^2=\chi^2 (3, N=661)=7.81$, $p=0.05$ so we rejected H_0: Variables Age and Number of correct items on number line representation question are independent and concluded that there is significant relationship between variables.

Analysis of responses on questions Q9 and Q10V showed that none of 3 to 4 year old children knew how to write symbols for numbers they were asked to do. Actually only 12 of them tried to do it. By the age 6 to 7, 111 out of 236 knew all number symbols while 55 more made only one mistake in writing numbers.

We may compare our results to Sinclair, Siegrist and Sinclair (1983). In their study they showed two, three and five balls respectively and asked children age 4 to 6 to somehow mark on a paper what they were shown. The researchers identified six types of representations of objects (1) global representation of quantity, (2) representation of the object kind, (3) one-to-one correspondence with symbols, (4) one-to-one correspondence with numerals, (5) cardinal value alone (sign or word, name of number), (6) cardinal value and object kind (e.g. 4 crayons). The order of the types of representations matches the age when it is predominantly used.

The easiest task in our interview was to draw as many candles as they have years on a cake. Except for the youngest children where 46 (68%) draw the right number of candles, all other groups of children were more than 90% successful.

Discussion and conclusion

In the reported study we were particularly concerned with the counting strategies. The analysis of the interviews with children unveiled a course of growth in initial understanding numbers.

Also, our findings show that as they grow older, children steadily exhibit development in the ability to use different types of representations. Certain culturally invented (formal) representations such as set representation and number line are limited in younger children which may explain children's relative failure in their use. It supports Kamii's (2001) findings about a close relationship between graphic representations and level of abstraction of number concepts as asserted by Piaget.

The findings indicate continuous development of whole number representations in children. Counting up, constructing one-to-one correspondence and drawing specified number of objects emerge to be key developments at age 3 to 7. Later comes understanding of set representation and lastly, number line representation.

Our study does not say anything about how children learned counting or about representations of numbers. However, we are not proposing that children individually came to understanding different number representations. We suggest that if we are going to make planned effort to introduce them, we should follow the route illuminated to some extent in this study.

Our aim was to contribute to better understanding of the beginning trajectory of learning about numbers. Here, we acknowledge the need for further study of initial development of children's understanding of the set of natural numbers. Understanding the concept of number is fundamental for learning mathematics. It is achieved to a certain level during early years. We should not underestimate this but also, we should not overestimate it.

Acknowledgements

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WHAT LIES BENEATH? CONCEPTUAL CONNECTIVITY UNDERLYING WHOLE NUMBER ARITHMETIC

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Abstract

Whole number arithmetic (WNA) is considered central to mathematics in the modern industrialised world. New developments in the cognitive and neurocognitive sciences, however, propose that WNA should be viewed in relation to the domain of mathematics more broadly and the real world interactions from which mathematics have developed. This may require mathematics to be conceptualised as a coherent domain that develops from human interaction, and that is reliant on spatial negotiation of one's environment. A lens on conceptual connectivity integral to an awareness of spatial pattern and structure, therefore, may offer a more complete picture of the connectivity that underlies WNA. A stronger focus on 'non-numerosity' attributes of mathematics learning and how they underpin WNA and mathematics more generally is proposed.

Key words: number, spatial reasoning, structural grouping, whole number arithmetic

Introduction

Whole number arithmetic (WNA) is integral to the modern mathematics curriculum, due in large part to requirements for a numerate workforce in modern industry. As a result, the teaching and learning of WNA within the subject of mathematics comprises an important component of curricula across industrialised societies (OECD, 2003). In these curricula, WNA is considered generally as a component of Number, one of several strands that have developed as the subject of mathematics (e.g., Kline, 1996). Mathematics in this context is assumed to be coherent subject whose teaching and learning is facilitated through use of such curricula. There has been traditionally an emphasis, however, on WNA and the strand of Number rather than on other strands, arguably through a focus on issues associated with numeracy and technology (e.g., OECD, 2003). Indeed, problems with learning of WNA, such as in dyscalculia, have been shown to have economic significance for GDP, due partly to effects on financial and statistical literacy (e.g., OECD, 2010).

The emphasis on teaching of WNA, however, has been subject to debate in recent curricula review, shifting direction to other strands such as algebra and statistics. There is increasing evidence of broad variability in the learning of WNA across industrialised societies as well as in the learning of mathematics concepts connected to WNA (e.g., Atweh et al., 2012). While there may be multiple reasons for this, the issue may centre on conceptual connectivity within mathematics (Woolcott, 2013), lack of coherency about mathematics as a complex whole (Mowat and Davis, 2010), and lack of a complete view of the relationship of mathematics to other domains (Lakoff and Núñez, 2000).

Modern cognitive and neurocognitive perspectives

Recent developments in the cognitive and neurocognitive[7] sciences indicate that such issues may be examined using a broad perspective that considers human interaction with the world as the primary basis for teaching and learning (Woolcott, 2011). Mathematics learning may be seen from this perspective as determined by a wide range of human interactions, sometimes seen in terms of socio-cultural and physical factors as an interaction of genetics and environment (e.g., Kovas and Plomin, 2007). These factors may be connected in a complex way, however, that may allow approaches based in systems views (e.g., Davis and Renert, 2014; Woolcott, 2011). Within these contexts, the human activities that underpin the learning of mathematics, or learning in general, are yet to be fully described (e.g., Lakoff and Núñez, 2000).

Is it just about whole number arithmetic (WNA)?

This discussion paper proposes that an emphasis on the teaching and learning of WNA in mathematics may be misdirected in that it is not focused on underlying common features and relationships. The need for mathematics expertise may be better served through a new approach to viewing how the strands in mathematics are connected as a coherent whole and how these develop from interaction with the three-dimensional world. An important consideration within this approach is how the ability to generalise from environmental interaction, inextricably linked with spatial ability[8] and spatial reasoning, may function in the development of mathematics generally (e.g., Devlin, 2012).

Any such approach requires an understanding of the particular features of accumulated culture that underlie mathematics, particularly in terms of mathematics considered as a social construct (Dehaene, 2009). Consideration, therefore, must be given to the body of knowledge, skills and experiences (culture[9] in the sense of Tomasello, 1999) that comprises mathematics in a given individual and consideration of how the culture that constitutes school mathematics accumulates across a given society (Woolcott, 2013). Emergence through spatial interactions is an important feature of such accumulation (e.g., see Davis and Renert, 2014).

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[7] The term 'cognitive' is used for studies generally found in cognitive or educational psychology and 'neurocognitive' for studies specifically derived from biology or, in particular, neuroscience

[8] Spatial ability is used here as an umbrella term and includes spatial sense or spatial differentiation.

[9] Not to be confused with the more diffuse concept of culture used in terms such as socio-cultural.

Conceptual connectivity and exceptionality

Conceptual connectivity can be seen as a lens for focusing, in both cognitive and neurocognitive studies, on the environmental interactions that underpin mathematics. The notion of connectivity has been examined in studies of exceptional students with both high and low performance in mathematics, including cases where learning of WNA is influenced by pathology, as well as in cases of 'twice exceptional' students, who have high ability in mathematics but low ability in other subject domains (Woolcott, 2013). This lens has been used also in examination of developmental dyscalculia in cases where there are difficulties in mathematics learning, including WNA and spatial reasoning, due to individual differences in processing of environmental information (e.g., Butterworth, Varma and Laurillard, 2011). Mulligan and colleagues (e.g., Mulligan et al., 2013) have applied a connectivity lens in studies of mathematical pattern and structure with young gifted children, indicating that WNA may also be enhanced through awareness of structural relationships.

Arithmetical or mathematical?

Mathematics education research, and studies in cognitive neuroscience, have indicated that the development of abstract arithmetical notions and procedures in school mathematics may depend on such attributes as number sense, subitizing (the rapid and accurate perception of small numerosities), comparison of numerical magnitudes, location on a number line, axis differentiation and symmetry (e.g., Dehaene, 2009; Mulligan et al., 2013). This view of development has support from neurocognitive studies of students (and non-school age adults) who were exceptional in that they did not have, or could not use, all of these attributes (e.g., Butterworth et al., 2011; Dehane, 2009). Modern teaching and learning of WNA has incorporated some of these features into interventions for low ability students and students who are performing below specified benchmarks in mathematics, but with a focus on numerosity rather than other mathematical attributes (e.g., Wright, Martland and Stafford, 2006).

Non-numerosity – links to spatial reasoning

Some studies have argued that there may be benefit in a stronger focus on 'non-numerosity' attributes of mathematics learning and that these may actually underpin WNA and mathematics more generally. In studies based in cognitive psychology, for example, Mulligan and colleagues (e.g., Mulligan et al., 2013) have shown that WNA, and mathematics in general, can be improved through a pedagogical approach that targets the development of abstract generalisations rather than focusing on WNA. Such studies, which examine the development of spatial aspects of patterns and spatial structures across mathematics concepts, indicate that such features as differentiation of foreground/background, alignment (collinear or axis), unitizing and equal grouping, transformation, recognition of shape and equal areas, are critical to mathematical development. These features can be improved through intervention (Mulligan et al., 2013).

In related classroom studies, this approach has been shown to improve the overall mathematics performance of students of low ability, including their performance in WNA (Mulligan et al., 2013). The features include elements of spatial differentiation as well as the categorisation and analysis of patterns that are critical to spatial reasoning. This implies that such features, including ability to develop co-linearity, spatial organisation and spatial memory, may underpin development in WNA as well as other mathematical concepts.

The Awareness of Mathematical Pattern and Structure (AMPS) construct

The ability to generalise spatial structures, based on similarities and differences observed by an individual, has not traditionally been considered relevant to the domain of Number learning, including WNA. An example would be the spatial structure of a 10x10 grid for representing 100. Mulligan and colleagues (Mulligan et al., 2013) have described and measured a general underlying construct, AMPS, that lies beneath and connects mathematical concepts and relationships. The AMPS construct involves, for example, the following structural groupings.

Sequences: recognising a (linear) series of objects or symbols arranged in a definite order or using repetitions, i.e., repeating and growing patterns and number sequences.

Structured Counting: counting in groups, such as counting by 2s or 5s or on a numeral track with the equal grouping structure recognised as multiplicative.

Shape and Alignment: recognising structural features of two- and three-dimensional (2D & 3D) shapes and graphical representations, constructing units of measure, such as co-linearity (horizontal and vertical coordination), similarity and congruence, and such properties as equal sides, opposite and adjacent sides, right angles, horizontal and vertical, parallel and perpendicular lines.

Equal Spacing: partitioning of lengths, other 2D or 3D spaces and objects into equal parts, such as constructing units of measure. It is fundamental to representing fractions, scales and intervals.

Partitioning: division of lengths, other 2D or 3D spaces, objects and quantities, into unequal or equal parts, including fractions and units of measure.

Mulligan and colleagues (e.g., see Mulligan, Mitchelmore and Stephanou, 2015) have applied the analysis of AMPS to the assessment of early mathematics based on pattern and structural relationships. The development of the AMPS construct is based on the complex connectivity between these structural groupings where some features are salient across structural groupings and some are more integral to a particular structural grouping (e.g., Woolcott, Chamberlain and Mulligan, 2015—under review). For example, Sequences and Structured Counting both involve the idea of equal groups or units represented in a linear way. These may be linked to Shape and Alignment where students may count using a 2D grid. Equal Spacing and Partitioning both involve division into equal parts. A student's AMPS level reflects how these interrelationships may occur for that individual.

If we take AMPS as a general indicator, then we cannot assume in teaching WNA that a child may already know, perhaps intrinsically, how spatial information can be represented as Number. Number may be an artificial classification or representation of information that the child has interpreted spatially. Consider, for example, that subitizing may require knowledge that spatially differentiated circles (let's call them 'dots') in a field can be classified by a child as number or, in a similar way, that a shape (square) or a pictorial graphic can also be classified in terms of number, but that this classification must, in fact, be learned or imposed rather than intrinsic to the learner. A spatial pattern (of dots, say) can also be classified in a number of different ways, for example, in terms of the spatial arrangement, such as an array, or in terms of partitions according to colour.

The current focus in early mathematics on spatial differentiation, primarily in relation to number, may limit opportunities for development of alternative spatial differentiations, even though such alternatives may have a positive effect on the learning of WNA. In a similar way, teaching multiplication only as repeated addition, for example, does not consider those students who use spatial structure to develop multiplicative thinking; they can generalise that repeating equal groups is different from repeating unequal groups. Students who use multiplicative thinking may be able to see a pattern in multiplication facts, such as the 3 x pattern, by visualising groupings based on spatial structuring. Such students may be able to subdivide a larger group using equal-grouping structure, rather than by adding and unitary counting.

Non-numerosity – examples based in neurocognitive studies

The view that WNA relies partly on the development of non-numerosity, such as in strands other than Number, is supported also from modern neurocognitive studies. Dehaene and others (e.g., Dehaene, 2009) for example, have shown that mathematics development in a school context depends, not just on such specific WNA attributes as number sense, but also on such cognitive features, arguably inbuilt, as axis differentiation and symmetry. Some neurocognitive studies, in fact, have argued further that the learning and remembering of mathematics depends on features that enable categorisation, abstractions or generalisations from learned information (e.g., Edelman, 2007). Based in studies of exceptional students, for example, Geake and colleagues (e.g., Geake, 2009) have defined gifted students as those who most effectively employ abstractions, as fluid analogies, for explanation and clarification. Analogising can be conceptualised as a process underpinning creative aspects of intelligence.

Mathematics as patterns and rules

It can be argued from Geake's (2009) studies and those of Mulligan and colleagues (e.g., Mulligan et al., 2013), that analogising and generalising may be what underpins knowledge connectivity within mathematics and between mathematics and other subjects. In line with such views, the prominent mathematician and somewhat controversial mathematics education spokesperson, Keith Devlin, has suggested that mathematics is 'more than arithmetic' and that generalisation of rules from identification and analysis of patterns is, in fact, the key to mathematical thinking (Devlin, 2012, p. 1). Although Devlin argues the separation of mathematics from other subject domains on the basis of the development of laws and axioms from such generalisations, these may be a feature of cognition in a general sense. What Devlin sees as important, in relation to WNA, is that the generalisations and abstractions required for mathematics do not necessarily involve patterns that are exclusively the domain of number, but rather these patterns may include patterns of shape, motion and behaviour as well as other patterns both real and imagined. This parallels the views of Mulligan and colleagues about AMPS.

Temple Grandin, a truly exceptional savant with autism, has argued further that extreme differences in ability to abstract or generalise are a feature of people on the autism spectrum (e.g., Grandin, 2009). She has argued that this ability may be sub-categorised for some high functioning individuals with autism as either thinking in pictures, thinking in patterns, or thinking based on verbal logic. This appears to imply that those who are gifted in mathematics require ability to generalise from numerical patterns, rather than from pictures (spatial representations). However, in line with Devlin's view, Grandin (2009) has suggested that this may be because modern mathematics does not consider adequately the spatial sense required for people who predominantly 'think in pictures'. These thinkers may benefit from mathematics teaching approaches that are not based solely on analysis of and generalisation from number patterns.

Mathematics and interaction with the world

A broad perspective that can be drawn from such studies is that all students categorise information from spatial interactions with their 3D world and the generalisations and abstractions from these categorisations are what constitutes their conceptual connectivity. What is generally described as a concept is arguably a description of the complex network of connections that make up such categorisations, abstractions and generalisations. Only some of these may pertain to the artificial construct of mathematics, a subject based generally in societal expectations (e.g., Dehaene, 2009). Mathematics may be underpinned, therefore, on conceptual connectivity based in spatial negotiation (e.g., Woolcott et al., 2014).

Such complex connectivity, however, relies on an already operational system that did not evolve with institutional education in mind (Sylwester, 1995), even if some of the operational features of this system are considered as related to mathematics. Lakoff and colleagues (e.g., Lakoff and Núñez, 2000) have argued, based on the environmental interactions of such an operational system, that mathematical coherence can be argued from the point of view of mathematics being developed as a connected set of embodied metaphors and related abstractions and inferences (although see Edelman, 2007 for an alternative argument). Mowat and Davis (2010) have shown how such mathematical coherence may be examined through use of complex networks based in the view of Lakoff and Núñez (2000) and this idea is being examined using empirical data (e.g., Woolcott et al., 2015; Woolcott et al., 2014).

Implications for further research

It may well be, based on viewing WNA and mathematics through the lens of connectivity, that the place of WNA may need to be reconsidered in relation to underlying patterns and structures inextricably linked to spatial ability and spatial reasoning. Spatial negotiation may arguably be the basis of the categorisations, generalisations and abstractions that form the basis of the growth of individual and human culture and cultural accumulation (in the sense of Tomasello, 1999). This is in concord with the view that spatial ability is a spontaneous part of Sylwester's (1995) already operational system, and that this ability is crucial in human negotiation of the 3D world. Such an argument supports the views, summarised above, that WNA may benefit from new perspectives on mathematical coherence, such as AMPS, through examination of how spatial ability and spatial reasoning underpin such coherence.

This paper suggests, therefore, that studies that examine the place of WNA in mathematics may need to draw upon newly-developing environmental connectivity theories, as well as the neurocognitive studies that emphasise human connection with environment (e.g., Edelman, 2007). Such theories may be useful in examining, from a scientific perspective, how learned information is accommodated in the artificially constructed subject domains, including mathematics, that have developed in the industrial education model and that are driven by historical practice (e.g., OECD, 2003, 2010).

Integration of such theories may be useful in examining how analysis of perceived environmental differences and similarities, including analysis of patterns, as well as resultant generalisations and abstractions, are separated as the domain of mathematics. Such examination may provide a different picture of how WNA and other components of mathematics are linked to other subjects in some contexts, but separable in others. Spatial ability and spatial reasoning may be the underlying connection here. Through the integration of modern science and complexity theory with educational theories and practices, it may be arguable that WNA is connected in a complex way to an underlying construct, AMPS—what lies beneath.

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ON THE SEMANTICS AND SYNTAX OF '+' AND '=' SIGNS

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Abstract

The research reported here is on the semantics and syntax of the signs '+' and '=' as exhibited in an event related potential (ERP) study. The experiment examined three variants of addition sentences consisting of sums smaller than 10, and investigates the differential processing of the numbers vs. the '+', and '=' signs.

Key words: addition, ERP, math signs, semantics

Introduction

The research reported here is on the semantics and syntax of the signs '+' and '=' as exhibited in an event related potential (ERP) study. ERPs are used to collect brain activity while performing different cognitive tasks. This method permits direct observation of information processing at different levels of analysis. ERPs consist of various discrete components, which are related to different stages of information processing. The components reflect the time course of sensory and cognitive processes with millisecond resolution that cannot be directly inferred from behavioural measures (i.e., reaction times [RTs]). The experiment was conducted with students who are skilled in math and supposed to solve addition facts with numbers smaller than 10 in an automatic fashion .We believe our results have implications for the teaching of arithmetic in early grade.

Additive triplets are 'privileged triplets'

Not every combination of three numbers makes up a true addition sentence. There are 'privileged triplets' of numbers that make true addition sentences (Baroody, 1999; Nesher and Katriel, 1977). The triplet (2, 3, 5) is an example of a 'privileged' triplet; These three numbers create a correct addition sentence (2 + 3 = 5). (2, 3, 7) is not a 'privileged triplet' because these three numbers are not part of any correct addition sum and they are not connected to each other. In fact, there are only a few 'privileged triples' whose sums are smaller than 10, and they occupy the attention of first-grade children for many months. In learning the links between numbers that are 'privileged triplets' they form automatic associations (LeFerve et al., 1988), so that in an addition exercise such as '2 + 3 = ?', the number '5' is retrieved automatically from long-term memory (Szucs and Csepe, 2005). It is not clear how these facts are stored - is it the whole equation, or are the numbers and other signs taken separately into account? LeFerve et al. (1988), in their Stroop experiment, have demonstrated the possibility that numbers are additively associated and their sums are obligatory activated even in stimuli that omit the '+' sign and the task does not demand finding the sum.

The semantics of the '+,' '-' and '=' signs

In the current study we differentiate between the semantics of the specific numbers and the more general signs '+,' '-' and '='. Usually, in early teaching of simple addition facts one tends to emphasise the correct numerical answer. However, except the specific numbers (i.e., 2, 3, 5), other signs appear in the addition sentence (the '+' or '-,' and the '=' sign) which act as general rules and operate on all numbers. There is a difference between the '+,' '-' and '=' signs. They carry differential syntax and semantics. The '+' is an operation sign while the '=' is a relation sign. Moreover, addition and subtraction both describe the same situation of parts and a whole (Sophian and McCorgray, 1994). The underlying structure of the numerical sentence a + b = c (or b + a = c) and c – b = a (or c – a = b) is the same. However, the surface structure differs. In addition the numbers on the two sides of the '+' sign refer to the parts (named 'addends' in numerals) and the number after the '=' sign refers to the equivalent whole amount (the 'sum' in numbers). In subtraction the role of the numbers differs. The number left of the '-' sign refers to the sum and the number to the right of the '-' sign is one of the addends. The number to right of the '=' sign in subtraction refers to the second addend. The above interpretation encompasses the semantics of the signs '+', '-' and '='. From this point of view, it is understood why a string of symbols such as '3 = 4 + =', is syntactically unacceptable.

Competing triplets

The phenomenon of 'competing triplet' was studied under several names: 'associative confusions' (Winkelman and Schmidt, 1974), 'associative lures' (Zbrodoff and Logan, 1986; Lemaire et al.,1994), 'relatedness' (Ashcraft, 1995, Desmet et al., 2012), and 'consistent–inconsistent lures' (Domahs et al., 2007). The above effect indicates the associative nature of arithmetic facts in long-term memory and that the presentation of the operands primes not just one answer, but rather a set of results (Niedeggen and Rosler, 1999; Szucs and Soltesz, 2010). We use the term 'competing triplet' rather than 'related facts' to stress the fact that we deal with a more constrained situation, since there is just one 'competing triplet' to each additive problem that can also be the correct solution for the subtraction of the two given numbers (e.g., 3 – 2 = 1). Being a possible correct solution for subtraction renders the competition of the 'competing triples' even more plausible.

How is the automated solution achieved?

The status of the arithmetic signs in the automatic process is not clear. Are the specific numbers with their associative connection to plausible privileged triplets (Campbell and Graham, 1985; McCloskey, 1992) attended to first, followed by or in parallel to the operation signs which dictate which triplet is the correct solution? Or, alternatively, are the addition facts stored as an entirety as a proper mathematical sentence? The cross interference between operations, such as supplying an incorrect multiplication answer that is correct for addition (e.g., 8 x 4 = 12), might support the hypothesis that there are stronger links among triplets of numbers and that the operations signs are activated separately.

Usually, an addition stimulus (exercise) consists of two among the three numbers of the expected triplet (i.e., 3 + 2 = ?). If the facts are stored in long-term memory as full sentences (including the '+' sign) then there is one and only one true reply (5 in the above example). However, if numbers are attended to first with a repertoire of competing associations, and the semantic considerations (of the +/- and = signs) are made separately (Domahs et al., 2007), then the required third number can belong to two additive triplets. For example, given 3 and 2 (as above: 3 + 2 = ?), either 5 or 1 could be triggered (by the triplets [3, 2, 5] and [3, 2, 1]). This might produce a conflict. The triplet [3, 2, 1] is a 'competing triplet' to the triplet [2, 3, 5] in the above example.

The structure of the experiment

In this study we employed the well-established 'distance effect' as a relative measure to the 'competing triplets' effect. The distance effect states that when comparing two numbers it will take longer to decide which one is bigger when the distance (difference) between the two numbers is small and less time when the distance is large. **The research question was**: How do different types of answers to canonical addition, sums smaller than, 10 affect the performance and brain activity of regular calculating students? There were three possible conditions: (a) a correct result (b) an incorrect result belonging to a 'competing triplet' with a large distance from the correct result, and (c) a result which deviates from the correct answer by +1or -1 (see Tab. 1). Thus, the (b) condition is the larger distance and the (c) condition is the smaller distance from the correct answer. One hundred and forty four sums were presented randomly to the centre of the computer screen for 2000 ms with an ISI of 700 ms. The task was a verification task. An addition problem (a + b =) was presented with an answer and the subject was requested to decide whether the answer was correct or incorrect by pressing a button on the keyboard. One third of the sums were correct and the rest incorrect with planned distracters. All the sums were created with the following 9 sets of numbers: (2, 3, 5), (2, 4, 6), (2, 5, 7), (2, 6, 8), (2, 7, 9), (3, 4, 7), (3, 5, 8), (3, 6, 9), and (4, 5, 9), omitting triplets that include 0, 1 and equal addends. These are all possible combinations for sums smaller than 10.

(a) correct	**(b) competing triplet**	**(c) +/- 1**
3 + 5 = 8	3 + 5 = 2	3 + 5 = 7
5 + 3 = 8	5 + 3 = 2	5 + 3 = 7

(a) Sample of a correct sum which represents a privilege triplet.
(b) Sample of a competing triplet, correct numbers with incorrect sign or incorrect order of the numbers.
(c) Numbers which do not compose a additive triplet and the result is close (+/- 1) to the correct answer.

Tab. 1: The three conditions of the different types of addition sums and examples

According to the distance theory condition (c) should take longer to process than condition (b). However, if condition (b) will take longer, it will support the hypothesis that there is an interference of the 'competing triplet'.

ERP considerations: By using ERP techniques (Bentin, 1989) it is possible to detect brain processing activity that occur before RTs measure the participant's overt reply. Areas of brain specialisation can be identified by observing variations of amplitude and latency in ERP components across different scalp locations (see Halegram, 1990). Several ERP components have been recognised while processing different mathematical and numerical stimuli.

Behavioural findings

The mean accuracy and reaction times for the different types of answers are presented in Tab. 2. No significant differences were found concerning accuracy between the three types of sums. All subjects performed above 98% correct. Significant differences were found in the reaction times ($F(2, 30)= 48.06$, $p < 0.01$). The subjects answered fastest to stimuli with the correct result for the triplet (898 ms), then to the near answer (949 ms; $p < 0.01$), and most slowly to the answer which represented a 'competing triplet' (1047 ms; $p < 0.01$).

	Correct	**Near (+/-1)**	**Competing**	**F**
Triplet				
Accuracy (% correct)	99.5	98.4	99.3	NS
Reaction time (ms)	898	949	1047	48.06
***p<0.001				

Tab. 2: Reaction-time and accuracy measures of the three types of sums

ERP findings

The following are **some** examples of our ERP findings capturing the differences among the three conditions of the experiment at the early component (N170) and the late component (P600) (see Fig. 1).

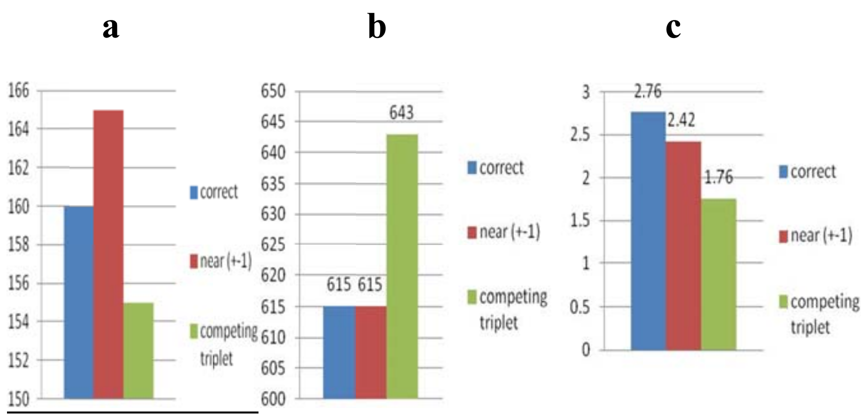

Fig. 1: Mean latency and amplitude of ERP components for canonic addition with three different types of answers. (a) N170 latency, (b) P600 latency, (c) P600 amplitude

N-170 is a negative component and is thought to reflect categorical coding and perceptual analysis of the visual stimuli. (Tanaka and Curran, 2001) proposed that the N170 reflects expert visual object recognition. At about N170 significant differences were observed in the latency and the amplitude of some electrodes. An effect of 'privileged triples' vs. 'non-privileged' is detected. Thus, at first there is demarcation between conditions (a) and (b) vs. (c) There was no difference between the correct answer and the 'competing triplet' which appeared about 13 ms later. See Fig. 1(a). **N400** has been found to be a domain general index of semantic congruence. Niedeggen and Rosler (1999) suggested, in their study of multiplication problems, N400 as an arithmetic component. Further studies of additive sums indicated also the N400 amplitude as a function of the incongruency (Szucs and Soltesz, 2010). We observed significant difference in the amplitude of N400: the highest amplitude was for the near answer (+/- 1) which was higher than the correct,; the lowest amplitude was for the 'competing triplet' answer. **P600** (LPC-late positive component) is a positive component which is syntax related (Martin-Loeches et al., 2006). It is elicited when a sequence has an incorrect ending. It has been suggested that the P600 occurs when a stimulus is difficult to integrate into a structure of the preceding context. The P600 has been observed in mathematical rules violations too as Niedeggen and Rosler (1999) found for unrelated answers (e.g. 4 * 8 = 26). In our experiment significant differences were found in P600 amplitude and latency [see Fig. 1 (b) and (c)]. Thus, the demarcation was found between conditions (a) and (c) vs. (b).

Final remarks

The behavioural results of our study are similar to previous studies which have shown that verification of correct addition sums are performed faster than incorrect sums; (Szucs and Csepe, 2005). The 'competing triplet' took the longest time to decide whether the sum is correct or not. The ERP results can help us understand that different types of symbols are processed at different time frames, and only after approximately 600 ms the integration between all the information is obtained.

Educational implications: In first grade we usually teach the number facts of 'privileged triplets' that become associated in the brain. These triplets should be fostered within their **underlying** additive structure, accentuate their linked **surface structure forms,** and emphasize more the semantics of the signs and not merely the numbers, and the task of finding a correct answer (number).

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WHOLE NUMBER ARITHMETIC—COMPETENCY MODELS AND INDIVIDUAL DEVELOPMENT

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Abstract

Competence models have been developed to describe levels of competence in mathematics and particularly in the domain of whole numbers. So far, only descriptions of what competence at different levels actually means are available, but current models do not describe how children can reach the next level. In this article, we propose a fine-grained description of the five levels of competence in the domain of numbers as proposed in a competence model for the primary school level that was based on theoretical and empirical research. Moreover, we discuss students' errors on three items to show how such a detailed analysis can provide additional information about how to support students in their development. We suggest that with such a combined empirical-psychological perspective, competence models can provide guidance for instruction.

Key words: arithmetic, mathematical competence, number and operations, patterns and structures, whole number reasoning

Introduction

Whole number arithmetic is an essential part of mathematical competence at the primary school level. However, what mathematical competence at this level actually means and how it develops remains a matter of current theoretical and empirical educational research. In the past decades, competency models and levels of mathematical competence have been described in the context of international large-scale assessments such as PISA (Programme for International Student Assessment). In these studies, levels of competence were identified empirically by students' performance on a set of items. These models are useful tools in order to better understand what constitutes mathematical competence. However, from a mathematics educational perspective, competency models based on empirical data only have some important constraints. For example, they do not provide information on how to support students in reaching higher levels of mathematical competence. Obviously, competence models should better take into account theories of mathematics education, but theory-based descriptions mostly lack the idea of levels. Theoretical models have been used to describe which mathematical competences children are expected to have acquired at the end of primary education. On the basis of such models, for example, standards for school mathematics have been introduced in many countries, which usually do not include a differentiation between high and low performance (e.g., CCSSI, 2010). For theoretically derived models it is therefore important to validate them empirically, because individual competence does not necessarily develop in accordance with the inherent structure of the learning content. Furthermore, in order to understand student's development of mathematical competence, cognitive aspects of mathematical learning need to be considered.

The aim of this article is to shed light on primary school children's mathematical competence in the domain of (whole) numbers. First, we describe number competence on the basis of a competence model for school mathematics suggested by Reiss, Roppelt, Haag, Pant, and Köller (2012). In particular, we describe in detail what competence means at the five levels of competence specified in the model. We then report on the results of a German national assessment of third graders, which was based on the theoretical model. Here, we focus on individual children's responses to specific items and discuss how such data can offer insight into children's cognition. For this purpose, we make use of psychological theories that may explain the developmental steps a child has to take in order to reach the next level of competence. An important example of such a theory is the concept of the zone of proximal development, which dates back to Vygotsky (1978). In essence, the zone of proximal development describes the discrepancy between what learners are able to do and what they are able to do only with the help of instruction. In our context, focussing on this zone can help describing in more detail how children at a certain competency level can be supported by their teachers in order to reach a higher level of competence.

The Competence Model

Levels of mathematical competence can be described for both the entire subject area and the different content domains. To understand competence in the domain of whole numbers, we focus on the content domains of *number and operations* and *patterns and structures*. The latter plays a particularly important role for a deep conceptual understanding of number and the structure of the number system. In the following, we describe in detail what whole number competence means on five competence levels (from Level I – lowest competence to level V – highest competence) in both content domains. This competence model was developed by Reiss, Heinze, and Pekrun (2007), based on detailed analyses of the mathematical learning content and theories of cognitive development. The model was validated by these authors as well as by Reiss and Winkelmann (2009). The following description is a further development of this model, which is supported by further empirical data (Reiss et al., 2012).

Number and Operations

Competence Level I includes technical background knowledge such as routine procedures on the basis of simple conceptual knowledge. On this level, students know the basic structure of the decimal system such as the classification of numbers into ones, tens, hundreds etc. They also know every basic single-digit multiplication and addition problem. Subtraction and addition of lower numbers can be completed partly written. While doing this, students are able to check for the accuracy of their solutions. Written addition can be used correctly if two summands are used. Written subtraction can be used if there are no carries over ten. In simple problems, students make use of the relationship between addition and subtraction. Strategies that students have learned during their first years at school – such as doubling a number – are applied to higher numbers. Simple numbers, especially those within the 100s and 1000s, can be shown on a number line with appropriate scale and the size of that number can be compared with other numbers along the sequence.

Competence Level II requires the simple use of basic knowledge for routine procedures within a clearly defined context. Students on this level use the structure of the decimal system when dealing with various representations of numbers. They recognise ordering principles and use these principles when continuing number patterns or during structural counting. Simple problems related to basic types of calculation are conducted mentally, partly written or fully written; occasionally, students find the solutions through systematic trial and error. During such trials, students conduct rough estimations and use those to determine the value range of their solutions. They correctly use both fundamental mathematical terms (such as "sum") and basic mathematical procedures to solve simple word problems.

Competence Level III includes the recognition and use of relationships within a familiar context. The numbers that were taught as part of the curriculum are securely read and written in various representations (such as in a place value panel). Also, the number zero can be assigned correctly. Students are proficient in every type of partly written or written calculation procedure that is part of the curriculum, but division is limited to single-digit divisors. They can use basic procedures of mental arithmetic even in unfamiliar contexts. They can transfer the multiplication table to a larger range of numbers, conduct rough estimations securely, and round the results meaningfully, even when the numbers are high. Students recognise the relationship between addition and subtraction, as well as between multiplication and division. They can recognise and communicate simple structural aspects (e.g. in relation to sequences of the multiplication table) if the contents were practiced before. In addition, they model simple object matters and find solutions, as long as the numbers used are within the number range covered by the curriculum.

Competence Level IV describes the secure and flexible use of conceptual knowledge and procedures within the curricular scope. Students solve problems securely using all types and variations of the calculations taught as part of the curriculum of fourth grade[10]. This includes particularly written division. During calculation, students systematically use the attributes of the decimal system and

[10] Note that in most German federal states, primary education includes grades 1–4.

the relationships between operations. They also apply this knowledge when investigating number sequences, for example when finding incorrect numbers in a sequence or when explaining the underlying procedures for the sequence. Different calculation procedures are combined flexibly and solutions are estimated or rounded appropriately. Students use solution strategies such as systematic trial and error even for more complex problems. Rules for calculation are well known and can be applied meaningfully. Complex situations can be modelled adequately and worked on correctly, and solutions can be presented appropriately. The students' conceptual knowledge also includes special technical terms, which the students can use and communicate appropriately.

Competence Level V is the highest level. It includes modelling complex problems and independent development of adequate strategies. Difficult mathematical problems can be solved correctly using various strategies. Relationships between numbers are recognised according to the situation. Mathematical rules, such as the factorability of natural numbers, are used in problem solving. Based on basic mathematical principles, even difficult solutions can be worked on and are solved through procedures such as systematic trial and error. Special aspects such ass calculations with fractions or decimals do not pose any problems. Also, students are able to comprehend and describe different approaches to a solution.

Patterns and Structures

On *Competence Level I*, students recognise simple principles that create patterns of familiar numerical relationships, such as easy sequences from the multiplication table, especially doubling of numbers.

Children on *Competence Level II* are able to apply a given rule for continuing simple number sequences, and recognise incorrect numbers within such sequences. They also recognise the basic structure in simple pictures or number sequences (such as a sequence of addition by a small number). They securely indicate numbers up to a thousand on place value panels; in addition, they can change those numbers along the place value panel following instructions. Students recognise and use proportional attributions (such as doubling).

Competence Level III means that students recognise the principles behind relatively complex patterns and can continue such patterns. During this process, students use their analytical abilities. When asked to determine a specific element at a given place, they do not require concrete materials or active manipulations to solve the problem. They can recognise and explain the principles in number patterns if numbers are used that are part of the curriculum. In addition, students selectively manipulate numbers within a number sequence and meaningfully interpret the results. They also recognise and interpret proportional relationships. Finally, students recognise and interpret functional relationships in simple real-life contexts. Specifically, students use proportional relationships for modelling and problem solving.

On *Competence Level IV*, students can analyse complex patterns and can continue these patterns graphically or numerically. Number patterns are recognised even if patterns are not based on addition of a given number or on multiplication by a given factor, and even if the number sequences are not given as numbers but as terms. In addition, students recognise the relationships among various presentations (for example visual and numerical representations) even if the sequences or patterns are difficult. Following instructions, they can independently and systematically change the presentations of numbers on a place value panel even if the numbers are very high. They also utilise proportional relationships when modelling and solving word problems.

Competence Level V includes the ability to securely deal with difficult number sequences (i.e., including square numbers or sequences that include several different calculation techniques). Children can recognise principles even if they need to combine different mathematical operations and are able to explain these principles. They create arithmetical patterns following given criteria and thereby develop their own strategies. Proportional relationships of simple fractional numbers and decimal fractions can be applied, and students can interpret tables that show such proportions. They can even model, analyse and use unfamiliar functional relationships in real-life contexts.

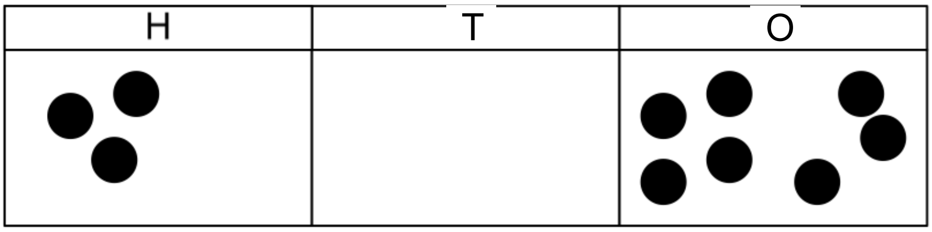
Materials and Methods

The model outlined above was used as a basis for annual national assessments in German third-grade classrooms. These assessments intend to make teachers understand better how their students perform compared to other students. Moreover, teachers may see not only whether solutions are wrong or right but also how tasks were solved. In the following, we report on students' performance on three items, which were presented in the pilot studies of the 2009 and 2011 assessments. Each item was presented to at least 550 students. Previous analyses of the whole data set showed a high fit to the dichotomous Rasch model, so that item difficulty and student competence could be located on the same metric scale. This allowed a detailed description of what students are able to achieve at a certain level, and therefore an empirical evaluation of the theoretical competence levels of the model. In the following analyses, we are particularly interested in students' incorrect answers to specific items in order to describe what prevented students who gave these answers from solving the item correctly. We will present the items together with the empirical results.

Results

Item 1: Place Values

In this item, the students were asked to explain the error that Paul has made when representing the number 370 in the place value table (see Fig. 1). According to the model, this item represents competence level II, because it requires basic understanding of the decimal number system and the ability to use a well-known representation.



Paul wants to show the number 370 in a place value table. However, he makes a mistake. Explain his mistake.

Fig. 1: Item 1 – Place Values

The empirical competence level of this item was the same as the theoretical one (Level II). 56 % of the children were able to provide a correct solution. Of the responses that were coded as incorrect, 23% were non-responses, suggesting that these children did not understand the place value system as such or the representation used in this item. When children provided incorrect answers, they most frequently showed a basic understanding of the place value system but gave incomplete responses to the question. Many children argued that Paul has only paid attention to the ones, or that he has ignored the tens. Such answers were coded as wrong because a correct response required reference to the transposition of the ones and the tens. This illustrates that the dichotomous coding is not sufficient to provide detailed information for instruction. The closer look on student responses shows that some students did understand the decimal number system and how numbers can be represented in the place value table, they were just unable to provide complete argumentations for the observed problem situation. The educational implication could be that those children need particular support of their argumentation and communication skills.

Item 2: Number Pairs

This item required understanding of numerical relationships. The task was to give reasons why one of the number pairs did not match the other pairs (see Fig. 2). To do so, it was necessary to refer to the sum of each two numbers of a pair. Accordingly, the theoretical competence level required to solve this item was Level III ("recognise and explain the principles in number patterns if numbers are used that are part of the curriculum").

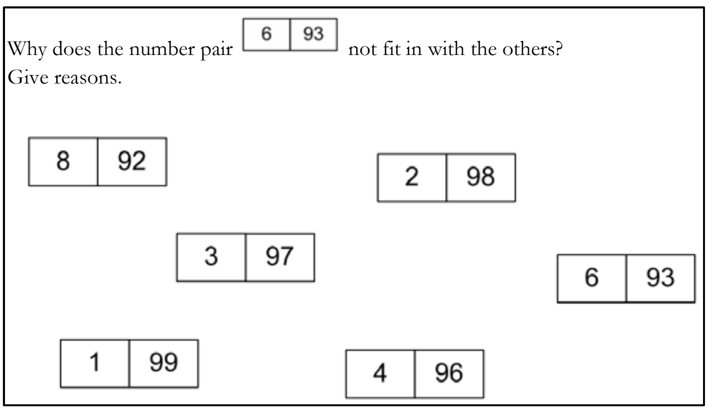


Fig. 2: Item 2 – Number Pairs

The data analysis showed that this item was on level IV, with 34% of the children solving the item correctly. The children who did not solve the item correctly referred to different irrelevant aspects of the item, for example, that the number 5 is missing in the pattern, or that the number pair given in the question already existed in the set of pairs. Some tried to apply operations other than addition to the number pairs (e.g., multiplication: "93 is not a multiple of six."). 32% of the students did not show any work and thus did not provide any information on their problems or misconceptions. From a psychological perspective, in order to solve this task correctly, one needs to recognise that all the number pairs consist of one number that is close to 100 and another number that is numerically small. This in turn requires the ability to quickly activate numerical magnitudes of number symbols, an ability that has been referred to as "number sense" (e.g., Dehaene, 1997). Although we can assume that most children in grade 3 have actually acquired a basic sense of numbers, it is probably not self-evident that they are able to make use of it in a problem situation. As an educational implication, we may conclude that support of applying basic understanding of numerical magnitudes in problem solving situations could help children reach the next level of competence.

Item 3: Number Sequence

In this item, the number sequence needed to be completed (see Fig. 3), which required understanding that each number can be generated from the previous one by subtracting a number that is increased by one each time, starting with 11. The second part of the item (part b) required the ability to communicate the strategy used for solving the first part (a). Accordingly, the theoretical competence levels of the two parts of this item were IV and V, respectively.

Look at the number sequence.

111 100 88 __________61

a) What is the missing number?

b) Write down the rule for calculation!

Fig. 3: Item 3 – Number Sequence

The empirical data showed that 27% of the students were able to fill in the missing number (as required on competence level IV) but just 14% could write down the rule for calculation, which would be required on competence level V. Most of the children who found the right missing number but failed on the second part did not give any rule or explanation. Most likely, these children did have in mind the rule for calculation but were unable to write it down, probably because the number that needed to be subtracted varied from one number of the sequence to the other. Obviously, these students had difficulties in formulating and communicating their (correctly applied) solution strategy. As a consequence for education, explicit training on mathematical argumentation and communication could be effective to increase children's competence level.

Discussion and Conclusion

We presented a detailed description of what competence means on the five levels of competence as proposed in the model by Reiss et al. (2012). In addition, we discussed student errors from a psychological point of view in order to explain why those students who gave incorrect responses failed to reach a certain level of competence. As the examples have shown, many students showed basic abilities in the domain of whole numbers, but had difficulties in applying their knowledge in problem solving situations or giving reasons for their solutions. Establishing mathematical argumentation and communication in the mathematics classroom could be beneficial in two ways, firstly, to enhance students' mathematical competence, and secondly, to provide the teacher with valuable information on students' difficulties and their stage of development.

To conclude, we consider a combined empirical-psychological perspective as pursued in this article as fruitful, because empirically based models of mathematical competence can provide reliable information about the mathematical abilities of a broad population of learners. The psychological perspective can add important information on students' cognition, and therefore inform teachers about ways to support their students.

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INTERPRETING CHILDREN'S REPRESENTATIONS OF WHOLE NUMBER ADDITIVE RELATIONS IN THE EARLY GRADES

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Abstract

This paper proposes a framework to support teachers' interpretation of learners' representations when engaging with whole number additive relation tasks. It builds on previous South African research on a conceptual framework for the specialisation of modes of representation in early grade mathematics. Using a combination of empirical data of learners' representations of additive relations and literature, the adapted framework is exemplified focusing on shifts within modes of representation which denote a move from counting to calculating.

Key words: additive relations, Foundation Phase, models, representations, South Africa, whole number

Introduction

Globally there is acknowledgment that children's early number development progresses from early counting in ones into mental calculation procedures which make use of group-wise actions (non-unit counting strategies) which require reified notions of number (Sfard, 2008). This learning trajectory from counting to calculating is succinctly described by Askew and Brown (2003) as a progression from 'count all, count on from the first number, count on from the larger number, use known facts and derive number facts'. These researchers also suggest that there is a related progression in children's representations of whole number arithmetic from limited number sense being related to representations that are closer to concrete actions, to good number sense being related to more compressed symbolic representations (Askew and Brown, 2003).

There is growing consensus in South Africa that one of the major factors inhibiting learners' mathematical progression is continued using of counting in ones strategies for mathematical calculations. This has been highlighted by Hoadley (2012) drawing on several earlier studies. In the Foundation Phase, Ensor et al. (2009) identified that learners remain reliant on counting-based strategies for calculations, and do not shift to more abstract calculation strategies. This was confirmed by Schollar (2008) who reported the prevalence of concrete counting strategies well into the Intermediate Phase.

In this paper I put forward a conceptual framework for interpreting children's external representations of whole number additive relations in the early grades, which takes into account the identified concern of the lack of shift from counting to calculating.

Methods

This paper forms part of a broader PhD classroom based study on children's use of narrative in mathematics learning where the content focus was whole number additive relations in the early grades. Narrative encompasses both oral storytelling and other external representations such as gestures, drawings or writing. The study formed part of a broader development project where the ongoing professional development of Foundation Phase teachers to improve teaching and learning of mathematics was in focus.

The paper draws on the mathematics education literature on modes of representation and an empirical base of children's diagrammatic representations from a ten-day Grade 2 teaching intervention in a poor township school in South Africa. It focuses on the representations developed by a particular child: Retabile (pseudonym), over the course of these ten consecutive school days. Where necessary a contrasting example from another child's work is presented. I keep the child invariant, and period of consideration short (10 days) to demonstrate that the early stages of whole number arithmetic is a messy process where a child moves between different calculations strategies and uses multiple modes of representation within a limited timeframe.

Results and Discussion

In this section I explain my use of modes of representation and its origins. I then propose a framework for interpreting representations of whole number additive relations and exemplify the use of this framework for the iconic, indexical and symbolic modes of representation with examples from Retabile's work.

Modes of representation

Several researchers have identified the importance of representations in supporting children's problem solving processes in mathematics, referring to these with various terms for example: picture and models (Deloache, 1991), representations (Eisner, 1993), models, images, and tools (Askew, 2012). I use the concept 'external representations' (hereafter 'representations') to denote children's markings, drawings and writings in mathematics. This is taken to be a form of communication which makes children's internal representations (thinking) visible to self and others.

Dowling (1998) drew on the work of Peirce to develop three modes of signification: iconic, indexical and symbolic in an analysis of representations in a mathematics textbook scheme in England. Ensor et al. (2009) used Dowling's work to categorise the representation of number evident in early grade South African classrooms describing each mode as follows:

- **Concrete apparatus** which entailed the manipulation of physical objects…This apparatus was used for counting and for calculation-by-counting strategies)
- **Iconic** (*images of everyday context realistic depictions*) apparatus including photographs, cartoons or drawings. This apparatus was used as concrete apparatus by could not be manipulated in the same way.
- **Indexical** (*images of everyday contexts – generic rather than realistic depiction of everyday contexts*) apparatus features drawing of sticks, tallies, dots, circles and

other shapes represent everyday objects. This apparatus was used for counting and calculating-by-counting tasks

- **Symbolic – number based** *(use of numerals to represent numbers)* apparatus including number lines (structured or semi structured) number charts, number cards. This mode of representation supported calculation without counting but could also be used for calculation-by-counting tasks
- **Symbolic-syntactical** (*use of mathematical notation to produce mathematical statements*). This mode of representation is abstract and entails the deciphering and production of mathematical statements. It relies on known number facts, and facts which can be derived without counting.
- **No representation used.** This refers to tasks which learners are asked to carry out which did not entail the use of modes of representation.

(Ensor et al., 2009, p.17)

Venkat and Askew (2012) note the shifts in specialisation from concrete to abstract representations in the Ensor et al. (2009) framework but suggest that there are gradations *within* these representational categories which relate to building calculating that is likely to involve some reified number facts and some counting.

I take up Venkat and Askew's modification and exemplify it through Retabile's work, arguing that working *within* a particular mode of representation, makes it possible to discern shifts from counting to calculating when attention is focused on the structure of, and actions on, representations.

Conceptual framework for interpreting representations

I constrain attention to representations of whole number additive relations in the early grades. I conceptualise these representations as defining an 'example space' where 'examples are usually not isolated; rather they are perceived as instances or classes of potential examples' (Watson and Mason, 2005, p. 51). Consistent with the notion of example spaces I consider 'dimensions of possible variation' and the 'range of permissible change' within the example space (Watson and Mason, 2005, p. 51).

There are several important features to which I have attended when interpreting learner's representations. Firstly, I consider the particular *circumstance* that determines why a learner is using a particular representation, be it their own invention, self-selected from a range of teacher presented options, or prompted by a teacher. Secondly mathematical representation occurs within a social context making the learners' representation dependent on the nature of the mathematical *task*. Thirdly central to mathematical thinking is the ability to *flexibly* move between multiple representations. I recognise that circumstance, task and flexibility are important dimensions of possible variation when interpreting learner representations as the backdrop for this paper.

The *modes of representation*, as defined by Ensor et al. (2009), are useful distinctions and I consider these to be the *first dimension* of possible variation when interpreting learners' representations. The other three dimensions of possible variation are dimensions for interpreting representations within a particular mode of representation. The *second dimension is the arrangement* of elements within a representation, referring to the spatial positioning of elements in relation to each other. The *third dimension is the group-wise* depictions evident in a child's representation. The way children group items and act on their groups, suggests shifts of attention between considering each object as a member of a group and acting on the group itself. In the context of early grade learners' work on additive relations, these groups are commonly single objects (one's), pair-wise, 5-wise or 10-wise groups. Together the arrangement of objects, and the group-wise depictions may be considered to define the *structure* of the representation. The learners' *actions (the fourth dimension)* made visible by their gestures or markings are significant because they depict change or movement in their representations. Changes children make to representations depict a process and offer insights into the chronology of the development of their representation.

Tab. 1 provides a summary of the conceptual framework for the analysis showing the four dimensions of possible variation and their related range of permissible variation.

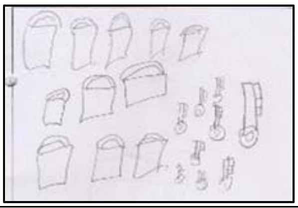
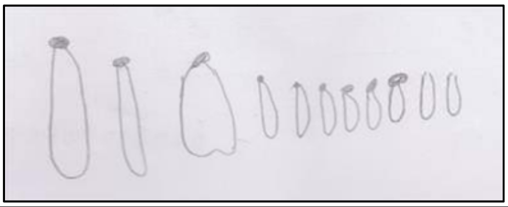
Dimension	**Range of permissible variation**				
Representation mode	Concrete	Iconic	Indexical	Symbolic	Syntactical
Structure: Arrangement	None	Horizontal linear	Left-right partition	Top-bottom partition	Array
Structure: Group-wise	1s	2s	5s	10s	Other
Action	Enclosing	Gesturing	Erasing or moving	Depiction of change	Other

Tab. 1: Conceptual framework

I turn now to three of the modes of representation – iconic, indexical and symbolic – to illustrate how I use each dimension of possible variation to interpret what Retabile was communicating. The sequencing of the modes of representation in this paper should not be used to interpret a progression by Retabile in relation to these modes. She used various modes of representations for different tasks and at different times over the10 day intervention period.

Iconic modes of representation

The following example pair contrasts different arrangements evident in Retabile's iconic representations.

Task	There are 11 locks but only 9 keys. How many keys are missing?	There are 11 bottles but only 9 lids. How many lids are missing?
Representation mode	Iconic	Iconic (some evidence of shift towards indexical)
Arrangement	Left right partition of locks from keys	Horizontal linear
Group-wise	1s	1s
Action	None evident	1:1 matching (compare/difference model). It is assumed that 11 bottles were drawn first, then the lids were matched to each bottle

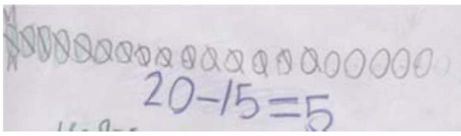
Fig. 1: Iconic representation of compare (matching) problems

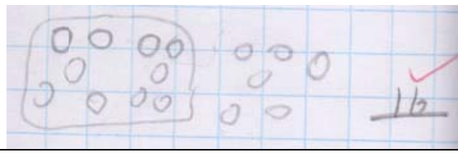
Notice Retabile's lack of a discernible arrangement depicting the relationship between each lock with a particular key. Contrast this to the bottles and lids problem where she *arranges* the two sets showing the 1:1 relationship between the items in each set as each lid touches each bottle. The two bottles that are missing their lids are discernible in her representation.

While both examples reflect Retabile using a counting in one's strategy, comparing the examples reveals a shift *within* the iconic representations, to noticing number relations in a structured way (making the relationship between elements in two sets discernible by their arrangement).

Indexical representations

Consider Retabile's use of indexical representations in response to different tasks.



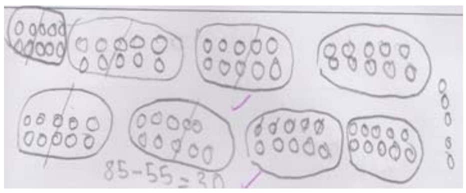


Task	Draw different ways to show 5 = …	Draw 16 (so we can quickly see how many there are)
Representation mode	Indexical and syntactical	Indexical and symbolic
Arrangement	Horizontal linear	Dice arrangements of 5s
Group-wise	1s	5s and 10s
Action	Marking take-away (in ones)	Enclosing 1 group of ten

Fig. 2: 'Counting in ones' indexical representations

In the first example Retabile's arrangement was linear with no apparent grouping. She then acted on this representation by crossing out each one of the 15 iconic sweets. She supported this with a syntactical representation of the related number sentence: 20 – 15 = 5. The second example shows a shift towards a group-wise depiction of 16 where the dice patterns of 5s are evident. The action in the representation is the enclosure of two groups of five to depict one group of ten. Retabile also expresses this with symbolic representation of the number symbol 16. Both representations make use of a 'counting in ones' strategy, however the second example shows Retabile imposing a group-wise structure through her *arrangement* of the ones and her *action* of encircling the group of ten.

That counting strategies can be shifted from counting in ones towards a group-wise concept of ten *within* the indexical mode of representation, is evident with this contrasting example from another child's work.



Task	85 – 55 = …
Representation mode	Indexical and syntactical
Arrangement	Pair-wise groups of ten
Group-wise	10s (although all ones still visible)
Action	Marking take-away (in groups of 10)

Fig. 2: Group-wise indexical representation

Like Retabile this child uses a take-away model of subtraction, showing this by crossing out elements in her representation. However this child's take-away action is group-wise as she crosses out each group of ten, and not each one within the group of ten. This learner continues to show tens as pair-wise groups of ones.

Symbolic syntactical modes of representation

The type of indexical representations shown above (for unit-counting and 10-wise depictions using ones) can be shifted by introducing number symbols as thinkable objects into these indexical representations.

Here Retabile uses the operation symbol + to signify the mathematical actions. In the first example she encloses the 2 + 8 to depict a group of 10. In the second example she breaks down the 30 and 40 into groups of 10 (which she depicts with enclosed number symbols). This work suggests use of known facts, as Retabile recognises the 8 and 2 as making 10, and breaks down 30 and 40 into three and four groups of 10 respectively. She provides no evidence of working in ones in her representation. The representations show a mixture of symbolic syntactical representations and indexical representations. Her indexical action of enclosing ten ones to show one group of ten is now used on the number symbols.

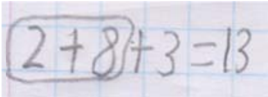
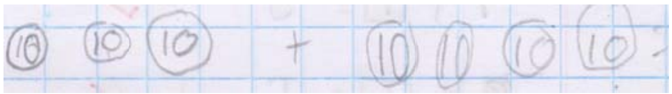
		
Task	2 + 8 + 3 =	30 + 40 =
Representation mode	Syntactical	Syntactical
Structure: Arrangement	Left to right linear	Left to right linear
Structure: group wise	10s	10s
Action	Enclosing group of ten + and =	Enclosing group of ten + and =

Fig. 3: Syntactical representation

Concluding remarks

The selection of examples from Retabile's work show that, when appropriately supported, she moves between the various calculation strategies: demonstrating the use of known facts for particular tasks, and a counting all strategy for tasks with less familiar numbers or problem situations. In a short period of time (10 days) she makes use of several modes of representations at different stages of a problem solving process and in response to different tasks. This 'messiness' reflects the complexity of the learning process that is often absent from neat theoretical frameworks of likely learning progression.

The conceptual framework I put forward and exemplify with Retabile's work acknowledges the need for children to work flexibly with multiple modes of representation. It rejects the notion that a particular representation type is automatically mapped to a particular calculation strategy, seeing the interplay between learner talk, gesture and representation as providing the only evidence of calculation strategy adopted. I note that the context of a representation including circumstance, task and flexibility are important features, although I do not elaborate on these here. I do elaborate on several dimensions of possible variation to which a teacher can attend when interpreting learners' representations: representation type, structure (arrangement and group-wise) and action.

The conceptual framework is intended to guide practitioners and researchers when interpreting and responding to children's mathematical representations. Representations are one of the means that children use to communicate mathematically. It is hoped that this conceptual framework helps others to be better able to listen to the children in their care.

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'A TRILLION IS AFTER ONE HUNDRED': EARLY NUMBER AND THE DEVELOPMENT OF SYMBOLIC AWARENESS

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Abstract

In this paper, we draw on neuro-scientific evidence that questions the typical developmental sequence posited by researchers, of a movement from considering actions on concrete objects, to the culmination in abstract mental structures. A clear hypothesis emerging is that what is significant in the learning of mathematics is not being able to link symbols to objects in a manner that is often considered accessible or natural, but being able to link symbols to other symbols. There are also neuro-scientific studies that implicate the significant role of fingers and touch in the learning of early number. We report on the use of the innovative 'TouchCounts' iPad app, which supports children in the development of number. What children become engaged and energised by is precisely the development of symbol-symbol awareness. There is a need for further research into how students gain symbolic fluency, particularly in relation to early number.

Key words: early number, neuro-science, ordinal, symbolic awareness, touchscreen

Introduction

This paper is focused on whole number thinking, learning and development. We address, in particular, the question:

> How can we integrate different perspectives about the foundations and development of whole number arithmetic concepts and skills? (ICMI, Study Call)

A first distinction is important, between ordinal and cardinal aspects of number. Typically, ordinality refers to the capacity to place numbers in sequence; for example, to know that 4 comes before 5 and after 3 in the sequence of natural numbers. Cardinality refers to the capacity to link numbers to collections, e.g., to know that "4" is the correct representation to denote a group of four objects. We believe the current emphasis on cardinal awareness in learning number may be misplaced (Coles, 2014b). In this paper we make a start on exploring: what is involved in developing ordinal awareness of number? What are the potential affordances? How might help children attend to ordinality? We draw on three sources of evidence: first neuro-scientific; second (only briefly) the pedagogy of Gattegno and Davydov; and third an empirical study.

Neuro-scientific evidence

Experiments in the 1970s appeared to suggest that ordinality occurred in young children at a much earlier age than cardinality (Brainerd, 1979). Recently, the kind of ingenious psychological experiment conducted in the twentieth century, has given way to brain research. One of the findings of broad agreement from neuro-science is that humans share an early (in evolutionary terms) Approximate

Number System (ANS), our 'number sense' which we use to judge the relative size of groups of objects (Neider and Dehaene, 2009), i.e., the ANS is a non-symbolic form of numerical awareness. Research is currently being undertaken to try and map out how the ANS links to our use of numerical symbols, since there is evidence that ANS acuity is correlated with later mathematical achievement (e.g., Gilmore et al., 2010).

Mathematical proficiency requires the co-ordinated action of many brain regions (Susac and Braeutigam, 2014). Research in recent years has shown that how children understand and manipulate number symbols (e.g., Hindu-Arabic numerals: 1, 2, 3, …) is an especially crucial building block (Lyons et al., 2014). How well a child is able to reason about direct relations between number symbols (e.g., whether three numbers are in order of size) is one of the strongest predictors of skills such as mental arithmetic (Lyons et al., 2014). Children with developmental dyscalculia, a persistent problem in mathematics education, also show consistent deficits in number symbol processing (Noël and Rousselle, 2011).

Neural evidence shows that accessing ordinal information is what distinguishes abstract number symbol use from a more perceptually grounded sense of magnitude. Accessing ordinal information from numerical symbols relies on a different network of brain regions and shows qualitatively different behavioural patterns when compared to ordinal processing of perceptual magnitudes (Lyons and Beilock, 2013). One does not need to access a perceptual feeling or 'sense' of 1,000,001 to know that it is one more than 1,000,000, because 1,000,001 immediately succeeds 1,000,000 in the whole number count list. A simple assessment of how quickly a person can judge the relative ordinality of three numbers predicts how well that person can do highly sophisticated mental arithmetic problems with a correlation of $r = 0.7$ (Lyons and Beilock, 2011). This relation holds when controlling for other numerical skills and ordering in non-numerical domains (e.g., letters).

Evidence from a large sample of Dutch children ($N = 1391$) in years 1-6 of school has corroborated the notion that symbolic number skills overshadow non-symbolic skills in terms of how well the former predict mathematical achievement (Lyons et al., 2014). Beginning in year 2, the ability to assess the relative order of number symbols is an increasingly strong predictor of mathematical achievement. How well children can directly access the order of highly familiar number sequences (e.g., 3 – 4 – 5) maps onto the individual variance in children's mental arithmetic achievement (Lyons a Beilock, 2011).

To summarise the findings detailed above, consider the following five aspects of number sense: the ANS; non-symbolic cardinal processing; non-symbolic ordinal processing; symbolic cardinal processing; symbolic ordinal processing. What Lyons and Beilock (e.g., 2011, 2013) have found is that symbolic ordinal processing is the 'odd man out' of the set, in neuro-imaging and behavioural tests. A clear hypothesis to emerge from this work is that students' awareness of ordinality may be distinct from awareness of cardinality and, in terms of developing skills needed for success in mathematics, that ordinality is the more significant. Neural and psychological sources of evidence converge to show the ability to understand and manipulate number symbols is crucial for further mathematical success. The capacity to articulate relations between numerals flexibly, such as their relative order, is particularly crucial. While these studies point to the importance of ordinality, they do not shed light on what is involved in ordinal awareness, nor how it may be developed.

Pedagogical insights

When students underachieve in mathematics, they are generally offered concrete resources and materials. The neuro-science suggests work on linking symbols to objects may reinforce the very way of thinking that underachieving students need to overcome to become successful. We hypothesise that what these students need is support to work with symbols in their relationship to other symbols. This hypothesis represents a radical challenge to current practice in the UK and Canada (and elsewhere) where, as stated above, the emphasis in the first years of schooling is firmly on linking number symbols to collections of objects. Current emphasis is also placed on cardinality as well, based in part on Butterworth's (2005) work, which emphasises children's arithmetic abilities – which involve working with numbers as objects, whether through subitising or adding on (see Clements, 1999). Even the influential work of Gelman and Gallistel (1978), which identifies five counting principles, there is a substantial and driving focus on cardinality.

There is an intriguing parallel, however, between our hypothesis and the (perhaps neglected) work of Gattegno (1974) and Davydov (1990) both of whose curriculum for early number were based on developing awareness of relations between lengths (Dougherty, 2008), where what are symbolised are relations between objects (greater than, less than, double, half), rather than, say, using numerals to label 'how many' objects are in a collection. Gattegno introduced work on place value, as a linguistic 'know-how' and not something that required 'understanding'. He developed a tens chart that can be used to provoke awareness of number relations (see Coles, 2014a for recent work on the chart). He also made extensive use of fingers (both the teacher's and the children's) as haptic symbolic devices for working on number relations, with a focus on correspondence and complementarity. We see awareness of number, in these curricula, arising out of linguistic skill and awareness of relations in a manner that does not emphasise a cardinal focus on counting collections. Linguistic skill and awareness of relations (which we see as aspects of ordinality) can be used to answer questions (as suggested above) such as what number is one bigger than a million.

In the remainder of this paper, we investigate how *TouchCounts* (Sinclair and Jackiw, 2011) might shed light on how children develop ordinal awareness. Elsewhere, we have described how it supports the development of finger gnosis

(Sinclair and Pimm, 2014) and offers opportunities for children to work with cardinal aspects of numbers (Sinclair and Heyd-Metzuyanim, 2014). In this paper, we focus on children's attention to the numerical symbols featured to study the possibly unique opportunities that *TouchCounts* offers for working with symbols in relation to other symbols. We analyse an episode in which a group of four-year-old children engage in a typical task that involves trying to "get to 100".

Methodology

The data for this paper comes from a larger research project aimed at studying the use of *TouchCounts* with young children. The setting is a day-care facility in which the first author set up a desk in one corner of the room and children (ages three to five years old) were allowed to come and go as they wish (as per the culture of this day-care). This means that there were sometimes eight to ten children crowded around, and sometimes only one child. It also means that not every one of the twenty-four children in the day-care participated in the study; conversely, some children participated in almost every session. Most of the time, there are three to four children involved at a given time. The research sessions took place once every one or two weeks for a total of 22 sessions.

The researcher allowed the children to explore, but also proposed certain tasks, depending on the child's experience using *TouchCounts*. In this case, and throughout, the first author was the adult working with the children, and so the pronoun "I" will be used to describe her actions and intentions. The sessions lasted about one hour, but the children tended to spend between five and fifteen minutes at a time. In addition to exploring the effectiveness of different tasks, the study aimed to examine how the children attended to the visual, tangible and aural dimensions of the environment, and how this related to their actions, gestures and speech around number. For this paper, we have chosen an episode that occurred in one of the later research sessions in which four four-year-old boys are involved. We chose this episode because it highlighted the way in which the children attended to number symbols. Our aim is to show what is possible when children are in a context that allows for attention to symbol-symbol relationships, in order to propose that research is needed to explore the potential, at primary school, of a more ordinal emphasis on number.

Materials and Methods

The TouchCounts Operating World

We focus our description of *TouchCounts* on the Operating World, which was the one used in the episode we will analyse. Tapping on the screen in this world creates autonomous sets of numbers. The user starts by placing one or several fingers on the screen, which immediately creates a large disc that encompasses all the fingers, which bears a numeral corresponding to the combined number of fingers touching the screen, together with each one of the fingers creating its own, smaller (and unnumbered) disc centred on each fingertip (see Fig. 1a).

When the fingers are lifted off the screen, the cardinal numeral is spoken aloud, and the smaller discs are then lassoed into a 'herd' and regularly arranged around the inner circumference of the big disc (see Fig. 1b). The small discs move slightly to emphasise that they are to be seen as one unit.

The herds can be interactively touched, either to move them around on the screen or to operate upon them. After two or more herds have been produced they can either be pinched together (addition) or torn apart (subtraction or partition). In the case of pinching together, a finger on each herd merge together to dynamically become one herd that contains the digital counters from each herd, thus adding them (Fig. 1c). The new herd is labelled with the associated sum (Fig. 1d), which is said aloud. The new herd keeps a trace of the previous herds, which can be seen by the colours of the small discs. Multiple herds can be pinched together. See (see www.youtube.com/watch?v=lTnxT8oNA_0).

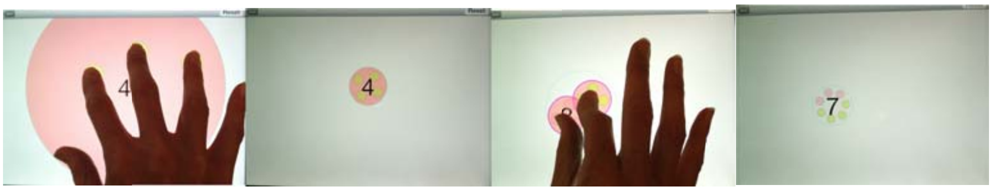

Fig. 1: (a) Creating a herd of digital counters; (b) The rearranged herd; (c) Pinching two herds together; (d) The sum of two herds.

Children can create and merge herds without planning to or even knowing that they are also adding or subtracting. Indeed, the pinching gesture draws on one of the four grounding metaphors for addition, that of *object collection* (see Lakoff and Núñez, 2000). Unlike with the calculator, *TouchCounts* first requires the production of herds that will be labelled by a numeral (indicating "how many" are in the herd) and then enacts the gathering/splitting mechanisms in which the two herds join or separate, both visually and temporally.

Children can pinch two herds relatively easily (though some children inadvertently create new herds instead of touching existing ones). They can do this, obviously, without knowing what the sum will be, and without knowing that the transformation that occurs reflects the operation of addition. When pinching herds, the children's attention will not likely be focused on counting, since *TouchCounts* manages the sum, but on the result of the pinching operation.

Attending to ordinality

The episode, which lasts about thirty minutes, begins with one boy sitting in front of the iPad, creating and merging herds on the screen. He has made 155. Another boy, Henri arrives, and begins making herd as well, but not as expertly. The first boy Ned, goes away, and Henri continues working on merging herds. By the time Jordan and Dipak arrive, Henri has made 39. George says "39 Henri". He continues making his herd bigger and Dipak says "Make one hundred. Henri, make one hundred." Ned returns and, having heard the comment, asserts "I can do a hundred and fifty one." He goes away while Jordan and Dipak watch Henri work. Dipak says "If you could make one hundred that would be awesome." Dipak laughs at a certain point when Henri has made 68 and says "six eight?" and Jordan says "sixty-eight". Henri makes 76 and he and George both say "seventy-six". Dipak says "Make a trillion, Jordan". Five minutes after they have begun, when Henri has made 80 (and there are other small herds on the screen) he says "okay, who wants to go next?". Jordan tries to put herds of 80 and 2 together but ends up making some new herds. He finally gets 82, and Dipak says "eighty-two". Henri says, "Let's make a trillion."

Jordan: Look how big this is (*see Fig. 2a*)
Dipak: Wooooow
Henri: What the heck? You need to use two fingers, not just one finger (*Jordan is trying to merge 88 and 1 but ends up creating new herds*)
I: What do you have there already?
Dipak: Two eights
I: Two eights?
Henri: Eighty-eight
I: Eighty-eight?
Henri (*looks up to the left*) And when there's two of the same number that's eighty-eight, forty-four, five, what is the number with two fives? (*Henri holds two fingers up, see Fig. 2b*)
Jordan: (*Jordan merges herds together to obtain 96*) Now look, sixty-f, sixty-six!
I: (*Responding to Henri*) Fifty-five
Dipak: Sixty-six?
Jordan: (*Makes a herd of 2 and merges it with the 96*)
iPad: Ninety-eight
Jordan: (*Makes another herd of 2 and merges it with 98)*
Henri: One hundred! (*clapping his hands while Nathalie takes Jordan's hands off the screen: see Fig. 2c*)
Ned: (*Coming back the group*) I thought you were supposed to make a trillion.
Henri: A trillion is after one hundred.

Fig. 2a

Fig. 2b

Fig. 2c

The boys continue making the herd bigger, now repeating the numbers after the iPad. They eventually make 204, and Dipal saying "we got to 100 and then we made two hundred and four"). They make 208 and decide to show the iPad to others in the room. When they return to the table, and since *TouchCounts* has been reset, they start over again. They eventually, after about two minutes, manage to make 100 again, with Henri clapping his hands in excitement.

In this excerpt, we are struck by Henri's interest in the numeral 88. When 88 appears on the iPad, Henri muses aloud about the general situation of 'when there's two numbers the same'; he appears to be attending to how to say the numbers. He knows how to say 44 but gets stuck on 55 (which interestingly is 'irregular' in the sense that it could, and perhaps should, be named 'five-ty-five'). A focus on number-naming is a pre-cursor to an ordinal awareness of relative size. In considering 44 and then 55, Henri perhaps demonstrates the beginnings of an awareness of this relative ordering. He is not attending to the cardinality of the numbers, in the sense of showing concern about 'how big' the numbers are, but seems to be taken by a symbolic pattern (two digits the same).

There is other evidence that the children's attention is on the ordering of symbols associated with the herds they have created. They not only repeat the named numbers that *TouchCounts* speaks aloud but make statements about the order of the herds, as in when Dipak asserts that they made 100 and then 204 and when Henri asserts that a trillion comes *after* one hundred. We see it as significant that he uses the phrase 'comes after' (an ordinal awareness) rather than, e.g., 'is bigger than'. Addition takes place in time on *TouchCounts*. Perhaps because they are working with large numbers, it is virtually impossible for them to actually attend to the quantity of coloured discs in the herd, which means that they are not attending to the relationship between the number symbol and the collection of disks. They are certainly attending to the growing size of the disc (at one point, George says that they should make the herd be bigger than the screen; and, at several other points, the boys talk about the size of the herd and their desire to make it bigger), which may support a more qualitative comparison between numbers—as in the comparison of 100 and 204 then a trillion and 100. The fact they are so motivated to get to 100 seems important in orienting attention towards number symbols, since these alone (or the *TouchCounts* voice) let them know if they reach their goal. We hypothesise that since the iPad takes care of labelling and saying the number names, the children's attention can shift to seeing how the digits relate to the spoken names and the order of numbers much larger than typically used in school at age 4.

Discussion and conclusion

This paper suggests that ordinality, arising out of awareness of symbol-symbol relationships (for example as encoded in the count) plays a vital, and often neglected, role in the early learning of number. Evidence for this proposal converges from three sources: neuro-scientific studies, the pedagogy of Gattegno (1974) and Davydov (1990) and an empirical study into the use of the innovative *TouchCounts*. There has only been space to offer one example of the data gathered on the use of *TouchCounts*, but this except showed children attending to and becoming engaged by symbol-symbol relations, sequencing and ordering, i.e., the more relational and ordinal aspects of number. Children of pre-school age (in the UK and Canada) are not exposed to numbers beyond 20 in the classroom, let alone the process of addition. Yet we see evidence of the mathematical thinking and playfulness that working more symbolically with number can occasion. We propose that research is needed to explore the potential of taking a more ordinal approach to the early learning of number. We are engaged in researching further the use of *TouchCounts* and other resources that can be used to work with children on awareness of ordinality and number structure. *TouchCounts* allows, or perhaps even forces, attention to be placed in the symbols that we care about and their connections to each other. In addition to their pedagogical insight, we venture that the kinds of results we have presented in this paper can provide neuroscientists with richer and perhaps more pedagogically-sound approaches to studying the development of number.

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ANALYSING SUBTRACTION-BY-ADDITION IN THE NUMBER DOMAIN 20-100 BY MEANS OF VERBAL PROTOCOL VS REACTION TIME DATA

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Abstract

This paper reviews and contrasts two lines of related studies done at our research centre on the use of the subtraction by addition strategy to solve symbolically presented two-digit subtractions: one wherein we used verbal protocol methods and one wherein the focus was on the analysis of children's reaction times. In all studies, we investigated if children made use of the subtraction by addition strategy and whether they switch between the subtraction by addition strategy and the direct subtraction strategy to solve these problems, and if so whether they base their strategy choice on the relative size of the subtrahend. The results of both lines of study were quite different. Whereas the first type of studies yielded little or no evidence for the use of the subtraction by addition strategy among children, in the latter ones children did show clear evidence of not only the use of this strategy but also its flexible application on the basis of the relative size of the subtrahend. Theoretical, methodological and educational implications are discussed.

Key words: flexible arithmetic, mental arithmetic, reaction times, strategy choice, subtraction, verbal protocols

Introduction

In the last four decades, a worldwide reform movement has changed some of the founding principles of elementary mathematics education. According to these reform-based ideas, instruction should no longer focus on solving school mathematics exercises quickly and accurately by means of the school-taught standard strategies (i.e., *routine expertise*), but children should solve mathematical tasks efficiently, creatively, and flexibly with a variety of meaningfully acquired strategies (i.e., *adaptive expertise*) (e.g., Hatano, 2003; Kilpatrick, Swafford and Findell, 2001; Verschaffel, Greer and De Corte, 2007). Although this idea of flexibility is also endorsed in the current attainment targets of mathematics education in Flanders (Vlaams Ministerie van Onderwijs en Vorming, 2010), Flemish publishers of mathematics textbooks and elementary school teachers still seem to value the fast and accurate execution of one strategy over the flexible use of different (self-invented) strategies (Torbeyns, De Smedt, Ghesquière and Verschaffel, 2009a). Moreover, there is still a lot of discussion whether strategy variety and flexibility should also be seen as important goals for mathematically weaker children (e.g., Kilpatrick et al., 2001; Verschaffel et al., 2007).

One of the mathematical subdomains in which strategy variety and flexibility can be especially aimed for and stimulated, is multi-digit mental subtraction. As for many other curricular domains, it has been shown that children develop various strategies to mentally solve subtractions (e.g., Beishuizen, 1993; Verschaffel et al., 2007). Most interesting within the context of the present paper is the observation that children sometimes solve subtractions by using an addition operation (Torbeyns et al., 2009a; Torbeyns, Ghesquière and Verschaffel, 2009; Verschaffel, Bryant and Torbeyns, 2012). They report, for instance, that they solve a problem such as 75 – 43 =. by asking how much needs to be added to the smallest number to get to the largest one, for example by doing 43 + 2 = 45 and 45 + 30 = 75, so the answer is 2 + 30 = 32. Consequently, by looking at the operation that underlies the solution process, two types of strategies can be distinguished: (1) *direct subtraction* strategies, in which the subtrahend is directly subtracted from the minuend (e.g., 75 – 43 =. by 75 – 40 = 35, 35 – 3 = 32), and (2) *subtraction by addition* strategies, in which one determines how much needs to be added to the subtrahend to get to the minuend (e.g., 75 – 43 =. by 43 + 30 = 73 and 73 + 2 = 75, so the answer is 30 + 2 = 32).

Over the years, several studies at our centre have investigated the use of the subtraction by addition strategy on symbolically presented subtraction problems, using Lemaire and Siegler's (1995) model of strategy change and strategy choice as a theoretical framework, and particularly focused on the parameter that refers to the adaptiveness or flexibility of individual strategy choices. In this model of strategy change and strategy choice, a strategy choice is called flexible if the individual chooses the strategy from his/her strategy repertoire that will lead fastest to an accurate answer. Based on a rational task analysis, direct subtraction is assumed to elicit few and/or small counting/calculation steps when the subtrahend is relatively small compared to the difference (e.g., 81 – 2 =.), but more and/or larger steps when the subtrahend is relatively large compared to the difference (e.g., 81 – 79 =.). Following the same logic, the opposite process is expected for the subtraction by addition strategy, i.e., few and/or small counting/calculation steps when the subtrahend is relatively large, but more and/or larger steps when the subtrahend is relatively small compared to the difference. This suggests that for problems such as 81 – 2 =., which are characterized by relatively small subtrahends, the calculation steps that have to be taken when performing a direct subtraction strategy are very small and easy, and such a quick and easy counting/subtraction process will lead very often to a correct answer. In contrast, for problems with a relatively large subtrahend compared to the difference such as 81 – 79 =., the calculation steps of the direct subtraction strategy are bigger and more error-prone compared to performing a subtraction by addition strategy.

Studies based on verbal protocol data

Remarkably, well-documented evidence of elementary school children's use of the subtraction by addition strategy is very scarce (e.g., De Smedt et al., 2010; Torbeyns et al., 2009a, 2009b). For example, Torbeyns et al. (2009a) asked Flemish second-, third-, and fourth-graders to mentally solve two-digit subtractions in two tasks. In the first task, the Spontaneous Strategy Use Task, children were asked to solve each problem as fast and as accurately as possible with their preferred strategy, and to verbally report both the answer and the strategy they used after solving each problem. Five out of the 15 presented two-digit subtractions had a relatively large subtrahend (as in 81 – 79 =.) and were assumed to trigger the subtraction by addition strategy. Still, children hardly applied this strategy spontaneously: Less than 10% of the second- and third-graders (resp. 4% and 8%) and only 15% of the fourth-graders spontaneously applied the subtraction by addition strategy on at least one subtraction with a relatively large subtrahend. In the second task, the Variability on Demand Task, children were asked to generate up to five different strategies for solving four two-digit subtractions. In this task, two subtractions were assumed to trigger the subtraction by addition strategy. Surprisingly, the frequency of subtraction by addition strategies hardly differed from the first task: Only 4% of the second-graders, 13% of the third-graders, and 21% of the fourth-graders reported subtraction by addition as a possible way to solve (at least one of) the two problems. On the basis of these results, the authors concluded that the subtraction by addition strategy was not included in most children's strategy repertoire.

Similar results were reported by Torbeyns et al. (2009b) when comparing two groups of second- to fourth-graders who were asked to write down their solution steps when solving symbolically presented two-digit subtractions. These two groups only differed in the instruction received about the subtraction by addition strategy. In the first group, the *no-instruction*-group, no instruction about this strategy was given, whereas the second group, the *instruction* group, had followed a mathematics textbook from first grade on that focused on the subtraction by addition strategy as a valuable alternative for the direct subtraction strategy for problems with a large subtrahend. Both groups of children were asked to solve the same 16 two-digit subtractions in whatever way they wanted, and to write down their solution steps. Half of the subtractions were designed with a large subtrahend and a difference smaller than 10, to elicit the use of subtraction by addition as much as possible. However, only 2% of the children from the no-instruction-group and not more than 15% of the children from the instruction group used subtraction by addition at least once. Again, these authors concluded that the subtraction by addition strategy was hardly used, even by children who had received instruction and practice in this strategy.

Taking into account the above mentioned results, De Smedt et al. (2010) tried to stimulate the use of subtraction by addition in third-grade children through an implicit and an explicit learning environment. Participants were divided over the two learning environments, both of which involved four training sessions. In the implicit learning environment, children were confronted with an unusually large number of two-digit subtractions with a relatively large subtrahend. This was done because Flemish textbooks hardly contain this type of subtractions, although they are most suitable for discovering the computational advantage of the subtraction by addition strategy. Children in the explicit learning environment were instructed to solve each problem once with direct subtraction and once with subtraction by addition, which was explained at the beginning of each training session. None of the children from the implicit learning environment reported using the subtraction by addition strategy in the test session halfway the training, at the end of the training, or in the retention session one month later. In the explicit learning environment, only 6% of the children reported using the subtraction by addition strategy in the test session after two training sessions, only 11% reported subtraction by addition by the end of the training sessions, and only 10% reported it one month later. From these low percentages, the authors inferred that – even in the explicit learning environment – third-grade children experienced great difficulties with picking up and integrating the subtraction by addition strategy into their strategy repertoire.

One important limitation of the studies reviewed above is that they relied exclusively on verbal or written data to detect the subtraction by addition strategy. These methods might, however, not be the best way to identify certain types of mental calculation strategies (e.g., Ericsson and Simon, 1993). Consequently, the results of previous research on children's use of subtraction by addition might represent an underestimation of their actual use of this strategy. In this respect, we point to the inconsistency between the verbal reports and children's reaction time data in De Smedt et al. (2010). The vast majority of the third-graders in this study only reported to use the direct subtraction strategy. However, if this had actually been the case, there should have been an increase in children's reaction times from problems with relatively small subtrahends (e.g., 81 – 2 =.) over problems with medium-sized subtrahends (e.g., 81 – 43 =.) to problems with relatively large subtrahends (e.g., 81 – 79 =.), because according to the above-mentioned rational task analysis, subtracting a larger subtrahend requires more and/or larger calculation steps. The observed reaction time patterns, however, argue against this interpretation: not only problems with relatively small but also problems with relatively large subtrahends were solved faster than problems with medium-sized subtrahends. These reaction time data thus suggest that the verbal report data were not always in line with the strategies actually applied by these children. More specifically, they indicate that the subtraction by addition strategy might have been used more frequently than suggested by the children's verbal reports.

We therefore aimed at studying the use of subtraction by addition with other, non-verbal methods for inferring strategy use. More particularly, we applied non-verbal methods for investigating the flexible use of subtraction by addition in both single- and multi-digit subtraction, first in adults and then in both typically developing children and children with mathematical learning disabilities. In the present paper, we focus on the reaction time studies with children in the number domain 20-100 (for other studies see Peters, 2013).

Studies based on reaction time data

In a first reaction-time study (Peters et al., 2013), we investigated fourth- to sixth-grade typically developing children's use of subtraction by addition by offering them four types of two-digit subtractions in both the standard subtraction format (81 – 37 =.) and in an addition format (37 + . = 81). We distinguished among four subtraction types on the basis of the combination of the magnitude of subtrahend (S) compared to difference (D), i.e., S < D or S > D, as well as the numerical distance between S and D, i.e., small or large. Small-distance problems were defined by S and D differing by less than 10, whereas in large-distance problems S and D differed by at least 10 and either S or D was a one-digit number. We analysed children's use of the direct subtraction and the subtraction by addition strategy in two steps. In the first step, based on Groen and Poll's (1973) analyses in the domain of single-digit arithmetic, we compared the fit of three linear regression models in which children's reaction times were predicted by respectively the subtrahend (S), the difference (D), and the minimum of the subtrahend and the difference (min[*D*, *S*]). The first model represents children only using the direct subtraction strategy: reaction times are best predicted by the size of the known subtrahend and thus indicative of the consistent use of the direct subtraction strategy, because it takes longer to subtract 79 from a given number than to subtract 2 from that number. The second model represents the consistent use of the subtraction by addition strategy: children's reaction times are best predicted by the size of the to-be-determined difference and thus indicative of the use of the subtraction by addition strategy, because it takes longer to determine how much needs to be added to get at a given number when the difference between both numbers is relatively large ("How much needs to be added to 2 to have 81?") than when it is relatively small ("How much needs to be added to 79 to have 81?"). Finally, the third model represents the flexible use of both direct subtraction and subtraction by addition, as reaction times are best predicted by the minimum of the subtrahend and the difference. In the second step, we compared reaction times on subtractions presented in the standard subtraction format and in the addition format. The results of both types of analyses converged to the conclusion that fourth- to sixth-children flexibly switched between direct subtraction and subtraction by addition based on the combination of two features of the subtrahend: If the subtrahend was smaller than the difference (e.g., 81 − 2=.), direct subtraction was the dominant strategy; if the subtrahend was larger than the difference (e.g., 83 – 79 =.), subtraction by addition was mainly used. However, this pattern was only observed when the numerical distance between subtrahend and difference was large.

Afterwards, we investigated the use of subtraction by addition in children with mathematical learning disabilities (MLD) (Peters et al., 2014). Especially for these children the idea of stimulating strategy variability and flexibility is still subject to discussion among scholars (Kilpatrick et al., 2001; Verschaffel et al., 2007). Some researchers and policy makers advise to teach MLD children only one solution strategy, others advocate stimulating the flexible use of various strategies, as for typically developing children. To contribute to this debate, we investigated the use of the subtraction by addition strategy to mentally solve two-digit subtractions in 44 children with MLD. We conducted a replication of the previous study, and thus again used two non-verbal research methods to infer strategy use patterns. First, we fitted three regression models to the reaction times of 32 two-digit subtractions. Additionally, we compared performance on problems presented in the standard subtraction and in the addition format. On the basis of these analyses, we found that MLD children – similar to their typically developing peers – flexibly switch between the traditionally taught direct subtraction strategy and subtraction by addition, based on the relative size of the subtrahend. These findings challenge typical special education classroom practices, which only focus on the routine mastery of the direct subtraction strategy.

Discussion and conclusion

We have reported on two lines of related studies on elementary school children's use of the subtraction by addition strategy to solve symbolically presented two-digit subtractions: one wherein we used verbal protocol methods and one wherein the focus was on the analysis of children's reaction times. In all studies, we investigated if children made use of the subtraction by addition strategy and whether they switch between the subtraction by addition strategy and the direct subtraction strategy to solve the problems. And, if so, whether they base their strategy choice on the relative size of the subtrahend.

The results of these two lines of study were quite different. Whereas the first type of studies yielded little or no evidence for the use of the subtraction by addition strategy among children, in the latter ones children did show clear evidence of not only the use of this strategy but also its flexible application on the basis of the relative size of the subtrahend. Based on the convincing reaction-time findings reported in the latter set of studies, we have to question why in previous research children did not report using the subtraction by addition strategy. Were they not aware of the calculation steps they had executed? Did they have difficulties in articulating precisely how they found the answer (and therefore reported the direct subtraction strategy, which they had learnt to verbalize during the numerous mathematics lessons wherein they had practiced that strategy)? Or, did they deliberately hide the use of the subtraction by addition strategy because they taught it was not valued, or even not allowed, to solve a subtraction problem in that way? All these explanations seem possible, and should be examined in more detail in further research. However, the two non-verbal research methods used in the presented reaction time studies have their limitations too (see Peters, 2013). Other research methods, such as using eye-movements and neuroscientific data, could therefore be used in future studies to further investigate and enhance the validity of both verbal and non-verbal research methods through triangulation.

From a practical point of view, teachers, teacher trainers, and material developers should be made aware of the possible problems linked to asking children how they solved a problem, and, more specifically, of the fact that children's verbal protocols may not always reflect their actual strategies.

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PERFORMANCE OF FOURTH GRADERS IN JUDGING REASONABLENESS OF COMPUTATIONAL RESULTS FOR WHOLE NUMBERS

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Abstract

To investigate the performance of fourth graders in and the use of methods for judging the reasonableness of computational results pertaining to whole numbers, 790 fourth graders from Southern Taiwan were selected to participate in this study. Results showed that the performance of the participants in solving problems related to judging the reasonableness of computational results was poor. Only one-fourth of the students could apply the number-sense-based method to answer the problems in the answer tier. Several misconceptions were found to exist.

Key words: fourth grade, number sense, reasonableness judgement

Introduction

Teaching and learning number sense are considered key components of mathematics education internationally (Berch, 2005; Dunphy, 2007; Menon, 2004; McIntosh et al., 1997; Yang and Wu, 2010; Yang and Li, 2013). In particular, studies have shown that a lack of number sense is likely to result in mathematics learning difficulties (Gersten, Jordan and Flojo, 2005; Jordan, Glutting and Ramineni, 2010; Mazzocco and Thompson, 2005). This shows the importance of number sense.

The capability to judge the reasonableness of computational results is considered a key characteristic of number sense (McIntosh et al., 1997; Yang and Wu, 2010; Yang and Li, 2013). In addition, *Principles and Standards of School Mathematics*, which is published by the National Council of Teachers of Mathematics [NCTM], states that all students should be able to "judge the reasonableness of computational results" (NCTM, 2000, p. 32). This shows the importance of judging the reasonableness of results. However, few studies have focused on this topic (Alajmi and Reys, 2007). Therefore, the objective of the current study was to examine the performance of fourth graders in judging the reasonableness of results pertaining to whole numbers.

"Judging the reasonableness of computational results" implies the application of flexible methods by a person to judge the reasonableness of an answer. For example, when estimating the height of 101 towers with 101 floors, children should use information they know, such as each floor being approximately 4 m high, and thereby determine the height of the 101 floors to be approximately 400 m. Consequently, 450 m is a reasonable answer, whereas 150, 250, and 350 m are not reasonable answers (Yang and Wu, 2010).

Methods

Sample

Seven hundred and ninety fourth graders from elementary schools in Southern Taiwan were selected to participate in this study. The schools considered in this study included those with large, middle-sized, and small student classes, and the students in these schools were from a wide range of socioeconomic backgrounds.

Instrument

A Web-based two-tier number sense diagnostic test designed by Yang, Li and Chiang (2007) to assess the number sense performance of elementary school children was used in this study. The first-tier test assessed the answers of the students to number-sense-related questions, and the second-tier test examined the reasons for the choices made in the first-tier test. The Cronbach α of the two-tier test was .828. Four categories were defined in the Web-based two-tier test, and each category contained eight questions. The test contained 32 questions in total. Because the focus of this study was on whole numbers, the questions of only one category (judging the reasonableness of computational results) are reported here.

Data Analysis

The test was administered online, and all answers were collected online. The answers were then analysed by using SPSS 17.0 for obtaining statistical results. The data were analysed on a computer. For the first-tier test, the computer classified the answers into correct and incorrect answers and calculated the percentage of both answers. For the second-tier test, the selections of the students were classified into a number-sense (NS)-based method, a rule-based method, misconceptions, and guessing by the computer, according to the choice they made.

Results

The eight questions related to judging the reasonableness of computational results, the percentage of correct answers of the sample students for each question in the first-tier test, and the percentage of responses for each question in the second-tier test are shown in Tab. 1. The results show that the average correct percentage for the eight items ranged from 31.8% to 60.8%. The average correct percentage for all the items was 51.7%. In addition, the results show that the percentage of students using the NS-based method to respond to the questions ranged from 7.9% to 38.2%. The mean percentage of students using the NS-based method to answer the questions was only 24.7%. However, the percentage of students who answered on the basis of misconceptions ranged from 24.2% to 51.3%. The average percentage for misconception for all the items was approximately 38.1%. Moreover, the range of percentages for guessing was 12.1% to 22.1%, and the average percentage for guessing for all the items was 15.7%.

Question	First-tier test	Second-tier test			
	Correct %	NS-based	Rule-Based	Misconception	Guessing
Item 1	51.6%	38.1%	7.6%	40.7%	12.6%
Item 2	31.8%	15.2%	21.1%	51.3%	12.1%
Item 3	52.0%	38.2%	7.9%	39.3%	14.4%
Item 4	45.6%	19.8%	20.0%	46.0%	13.7%
Item 5	52.7%	18.0%	31.0%	38.5%	12.6%
Item 6	59.0%	7.9%	43.0%	27.5%	21.3%
Item 7	59.9%	36.1%	16.7%	24.2%	22.1%
Item 8	60.8%	24.1%	21.5%	37.6%	16.6%
Total	51.7%	24.7%	21.1%	38.1%	15.7%

Tab. 1: Performance of students in judging reasonableness of computational results

In the first (answer)-tier test, the average percentages of the correct and incorrect answers for the eight items were 51.7% and 48.3%, respectively. In the second (reason)-tier test, the average percentages for the use of the NS-based method, rule-based method, misconceptions, and guessing for the eight items were 24.7%, 21.9%, 37.7%, and 15.7%, respectively.

The details of the responses of the students to each item are as follows:

Item 1. Whose statement is the most reasonable?

Joe: "I can fit 5000 textbooks into my school bag."
Lin: "I can lift a pig that weighs 5000 grams."
Kan: "I can fit 5000 M&M's into my mouth."

Joe 11.5%	4.1%	My school bag is very large, so 5000 textbooks can certainly fit into it.
	5.1%	Textbooks are very thin, so my school bag can hold 5000 textbooks.
	2.3%	I am guessing.
Lin * 51.2%	38.1%#	The amount of 5000 g is equal to 5 kg, which I can lift.
	7.5%	The amount of 5000 g is equal to 0.5 kg, which I can lift.
	5.6%	I am guessing.
Kan 21.2%	1.3%	I like eating M&M's, so I can fit 5000 M&M's in my mouth all at once.
	17.5%	Each M&M's is very small, so I can fit 5000 M&M's in my mouth all at once.
	2.4%	I am guessing.
Can't tell 14.9%	12.7%	I have not actually done it, so I can't judge.
	2.2%	I am guessing.

Notes. The difficulty index and discrimination power of this item were .526 and .266.
* indicates correct answer; # indicates the NS-based method.

Several misconceptions were found. For example, approximately 11.5% of the fourth graders believed that “My school bag is very large, so 5000 textbooks can certainly fit into it” or “Textbooks are very thin, so my school bag can hold 5000 textbooks”; 21.2% of the students believed that “I can fit 5000 M&M’s into my mouth.”

Item 2. How many digits are there in the sum of 2 three-digit numbers?

<table>
<tr><td rowspan="3">Three-digit number
7.6%</td><td>3.6%</td><td>A three-digit number plus another three-digit number must be a three-digit number.</td></tr>
<tr><td>3.2%</td><td>Because 100 + 100 = 200, a three-digit number plus another three-digit number should also be a three-digit number.</td></tr>
<tr><td>0.8%</td><td>I am guessing.</td></tr>
<tr><td rowspan="3">Four-digit number
9.8%</td><td>3.7%</td><td>The number will become larger after the addition, so a three-digit number plus another three-digit number gives a four-digit number.</td></tr>
<tr><td>4.3%</td><td>Because 900 + 900 = 1800, a three-digit number plus another three-digit number should be a four-digit number.</td></tr>
<tr><td>1.8%</td><td>I am guessing.</td></tr>
<tr><td rowspan="4">Three-digit number or four-digit number
31.8% *</td><td>15.2%#</td><td>A small three-digit number plus another small three-digit number could be a three-digit number, and a large three-digit number plus another large three-digit number could yield a four-digit number.</td></tr>
<tr><td>9.4%</td><td>Because 100 + 100 = 200, a three-digit number plus another three-digit number can be a three-digit number; however, because 900 + 900 = 1800, the sum can also be a four-digit number.</td></tr>
<tr><td>4.2%</td><td>There are two different selections in this choice, so this is the answer.</td></tr>
<tr><td>3.0%</td><td>I am guessing.</td></tr>
<tr><td rowspan="3">Six-digit number
50.4%</td><td>33.5%</td><td>Because 3 + 3 = 6, the answer should be a six-digit number.</td></tr>
<tr><td>10.4%</td><td>Because thirty-something plus thirty-something equals sixty-something, a three-digit number plus another three-digit number should be a six-digit number.</td></tr>
<tr><td>6.5%</td><td>I am guessing.</td></tr>
</table>

Note. The difficulty index and discrimination power of this item were .317 and .484.

Several misconceptions were found. For example, approximately 50% of the students believed that because 3 + 3 = 6, the sum of 2 three-digit numbers should be a six-digit number, or because thirty-something plus thirty-something equals sixty-something, a three-digit number plus another three-digit number should be a six-digit number.

Item 3. Larry says: I drink 500____of milk every morning.
Which of the following measurement units can best complete the sentence?

Liter (L) 21.3%	12.2%	Compared with the other answers, it is the most rational answer.
	5.1%	An amount of 1 L is very little, so I can drink 500 L of milk.
	4.1%	I am guessing.
Deciliter (dL) 24.2%	11.9%	If compared with the other answers, it is the most rational answer.
	8.4%	An amount of 1 dl isn't much, so I can drink 500 dl.
	3.9%	I am guessing.
Milliliter (mL) 52.0%*	38.2%#	I drink approximately 350 mL of milk every day, so the measurement unit "mL" is rational.
	7.9%	An amount of 1 mL isn't much, so I can drink 500 mL.
	6.0%	I am guessing.
Can't tell 2.2%	1.7%	Without measuring, I can't decide the answer.
	0.4%	I am guessing.

Note. The difficulty index and discrimination power of this item were .504 and .333.

Many students can remember that 1 L equals 1000 mL. However, they cannot estimate how much 500 L is. Therefore, approximately 21.3% of the students believed that "I drink 500 L of milk every morning" is correct.

Item 4. John is 10 years old this year and he is 130 centimetres tall. Under normal circumstances, how tall will he be when he is 20 years old?

130 4.7%	3.4%	Heights of 260, 350, and 170 cm are too large.
	1.3%	I am guessing.
170 45.6%*	20.0%	Heights of 260 and 350 cm are too large, and a height of 130 cm is too small.
	19.8%#	He's only 10 years old and already has a height of 130 cm. After 10 more years, he's sure to become taller, so 170 cm is the most rational answer.
	5.6%	I am guessing.
260 44.8%	9.9%	Because 130 ÷ 10 = 13, he grows 13 cm a year. Therefore, when he is 20 years old, he should be 13 × 20 = 260 cm.
	29.5%	He's grown 130 cm in ten years, and in the next 10 years, he should grow another 130 cm. Therefore, 130 + 130 = 260.
	5.4%	I am guessing.

350 4.6%	3.2%	A 20-year-old adult should be very tall, so a height of 350 cm is the most rational answer.
	1.4%	I am guessing.

Note. The difficulty index and discrimination power of this item were .453 and .264.

Approximately 44.8% of the students incorrectly believed that "because 130 ÷ 10 = 13, he grows 13 cm a year, and therefore, the height at 20 years of age should be 13 × 20 = 260 cm" or "He's grown 130 cm in ten years and in 10 more years he should grow another 130 cm, leading to a total height of 130 + 130 = 260."

Item 5. What is the height from the floor to the ceiling in a classroom? Which of the following statements is reasonable?

300 cm 52.7%*	6.2%	This is what the textbooks say.
	24.8%	Values of 300 mm and 300 m are impossible.
	18.0%#	The height from the floor to the ceiling is approximately equal to the height of two children, which is approximately 300 cm.
	3.7%	I am guessing.
300 mm 9.6%	1.3%	The floor is not very far from the ceiling, so the height should be 300 mm.
	3.0%	Values of 300 cm and 300 m are too large.
	3.0%	The value of 300 cm is too small, and 300 m is too large.
	2.3%	I am guessing.
300 m 33.3%	14.2%	The values of 300 cm and 300 mm are too small, so 300 m is the most rational answer.
	14.2%	It is very far from the floor to the ceiling, so 300 m is the most rational answer.
	4.9%	I am guessing.
Can't tell 4.4%	2.8%	Without measuring, I can't decide the answer.
	1.7%	I am guessing.

Note. The difficulty index and discrimination power of this item were .542 and .434.

Approximately 33.3% of the students incorrectly thought that the height from the floor to the ceiling of a classroom was 300 m. This shows that many students cannot gage the meanings of meter and centimetre.

Discussion and conclusion

The results showed the poor performance of sample students in judging the reasonableness of results. Only approximately one-fourth of the students could apply the number-sense-based method to solve problems. The results are consistent with those of previous studies conducted in Taiwan (Li and Yang, 2010; Yang and Li, 2008). In addition, approximately one-third of the sample students had misconceptions when judging the reasonableness of results. Several misconceptions were found to exist. For example, 11.5% of the fourth graders believed that "I can fit 5000 textbooks into my school bag," and 21.1% of the fourth graders believed that "I can fit 5000 M&M's into my mouth." Approximately 50% of the students thought that the sum of 2 three-digit numbers should be a six-digit number because 3 + 3 = 6, and approximately 21.3% of the fourth graders believed that "I drink 500 L of milk every morning." Approximately 44.8% of students thought that "because 130 ÷ 10 = 13, his height increased by 13 cm a year, and therefore, at the age of 20 years, his height should be 13 × 20 = 260 cm" or "he's grown 130 cm in ten years, and in 10 more years he should grow another 130 cm." These findings show that the sample students have major misconceptions when judging the reasonableness of results. This implies that number-sense-related activities should be introduced in the mathematics curriculum to improve the number sense of Taiwanese children.

In summary, the key findings of this study indicated that the sample students performed poorly in answering questions related to judging the reasonableness of computational results and had high percentages of misconceptions. In addition, more than 15% of the sample students made their reason choice by guessing. It appears that some of the sample students had low confidence when answering these questions. Although the national curriculum in Taiwan emphasises number sense, few activities are found in the elementary mathematics textbooks. To enhance the number sense of students, number-sense-related activities should be introduced in the mathematics curriculum. A major contribution of this study is that it shows that the Web-based two-tier number sense diagnostic test can be used to detect the number sense performance of students through their answer choice and reason choice. The test showed the poor performance of students in judging the reasonableness of computational results, the existence of several major misconceptions, and the use of guesswork by students for obtaining the answer and reasons.

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THEME 3: ASPECTS THAT AFFECT WHOLE NUMBER LEARNING

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Introduction

While theme 2 addresses thinking, learning and development aspects of whole number learning, and refers mainly to the learner's side, theme 3 addresses the conditions that affect whole number learning and includes cultural, social, political and institutional aspects of the teaching-learning activity. In this perspective, awareness of the context where an empirical study is carried out is essential to understand the choices, methodologies and findings reported by authors. This awareness is slowly entering in the international community of mathematics educators (Bartolini Bussi and Martignone, 2013), although it has been highlighted in the past (see Durkheim, 1911, quoted by Mercier & Quilio) and summarised by the sentence "Teaching is a cultural activity" (Stigler and Hiebert, 1999, p. 86) quoted by Inprasitha.

The thirteen papers accepted for this theme address contextual issues from different perspectives. We summarise them as follows.

Language

The contribution of participants from many different contexts offers a unique possibility for first hand reports about the language issues, which may foster or hinder the construction of mathematical meanings.

For instance, Chinese language mirrors, in a transparent way, place value representation (Ni), as do other Asian languages, like Thai (Inprasitha) or Maori language (Young-Loveridge). The consistency of number naming systems with the base-ten system has been hypothesised to assist children in doing well on tasks relevant to base-10 values, such as counting skills and place-value competence. However, according to Ni, this interpretation is difficult because we lack studies controlling for cultural or family processes (e.g., parental expectation, parental assistance, and preschool education) as confounding variables that could also influence children's numerical development. Rather, we have evidence that adult instruction, also in the family context, may play a major role (Ma and Cobb, 1995).

On the contrary, European languages are not so transparent. We may quote the French case (where 92 is said as "four twenty twelve," corresponding to 4 x 20 + 12) that, according to Dehaene (1997) introduces additional difficulties for French speaking children. This French reading is different from the English reading. Peter-Koop et al. report also the German language structure for naming 2-digit numbers, that is based on ones and tens as opposed to tens and ones in the base-10 numeral form (e.g., 26 is read *Sechsundzwanzig,* that is *six and* *twenty* instead of *twenty and six*). Ladel and Kortenkamp report that the situation is even more puzzling when hundreds are involved as a number such as 425 is spoken as *vierhundert fünfundzwanzig*, that is *four hundred five and twenty* in the literal translation. This creates cognitive conflict for German young children as they are introduced to, and learn, these opposing conventions.

In general, place value is not a simple construct for young students in the West: rather than using place value conventions, these languages often use digits to transcribe oral numerals.

The non-transparency of European languages may be related to the late transposition of place value conventions from the far East, through Arabic mediation, which clashed with the original Roman additive representation. Fig. 1 shows the difficulties for early European users, when, in the middle age, the so-called Hindu-Arabic numerals were introduced (Fauvel and van Maanen, 2002, p. 151).

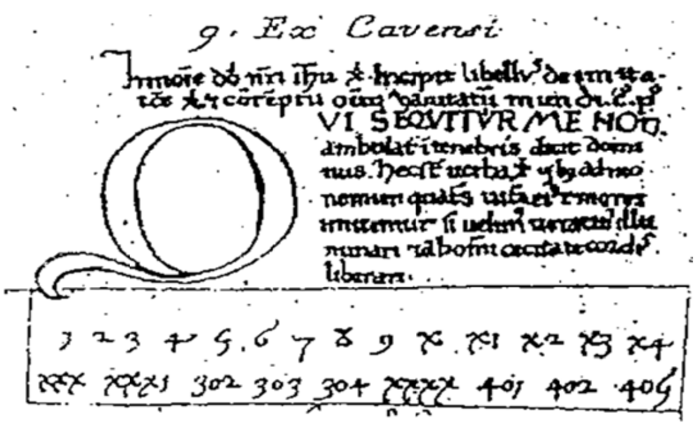

Fig. 1: An ancient Italian manuscript

In one case (Young-Loveridge) the author emphasizes that the advantage of transparency for place value in Maori language is, in some sense, thwarted by the use, in school, of English as the teaching and learning language. This example hints at a more general issue concerning the potential conflicts between everyday language and school language that has been noted in many countries.

Pimm takes an initial cross-linguistic look at the grammar and structures that comprise various sets of number words (cardinal, ordinal, fractional) in twenty different languages, as well as some of the syntactic features with which each language imbues them. One focal concern is to what extent the grammar of the language tacitly conveys information about the nature of these sets of numbers, starting from a concern that the relations between the ordinal and fractional forms and their joint relations to cardinals.

We have discussed above the positive or negative (or null) effects of different languages. Yet even when the language is the same, there is evidence that the change of the institutional context may provoke substantial changes. For instance Mercier and Quilio analyse the differences between primary school education about whole number arithmetic in four French speaking countries/regions (France, Belgium, Quebec and the French Swiss). Here, language is viewed as only one of the variables to be considered, when addressing the functioning principles of education systems.

Tools

There is a cluster of papers addressing different kinds of tools, either traditional, or related to information and communication technologies. The use of tools is reported, predominantly, by authors coming from the Western world (consistent with the report by Borba and Bartolini Bussi (2008) at the Symposium held in Rome to celebrate the centennial of the foundation of ICMI).

In the papers the following tools are in focus:

- The number line (Bartolini), with the historic-epistemological, cognitive and didactical analysis of a very popular teaching aid;

- Tallies and sequences of tallies (finite sequence, of course, eventually empty) as a working definition of natural number with prospective teachers (Hodgson and Lajoie);

- Multi-base arithmetic blocks and arithmetic rack or Slavonic abacus (Rottmann), with the report of practices for special needs education, realised in Germany, drawing on a four-phase-model to support the development of basic computational ideas;

- Cuisenaire rods (Ball and Bass), with the report of a case of instruction that aims to cultivate productive mathematical persistence in elementary students in the context of a challenging problem of whole number arithmetic in the context of a mathematics programme, the Elementary Mathematics Laboratory (EML), for economically disadvantaged fifth graders;

- Tools from everyday life such as a sheet of stamps with regular arrangements in ten lines and ten columns (Inprasitha);

- Multiplicative mini-games in a special case of computer games (van den Heuvel-Panhuizen et al.) played in different formal and informal contexts, with the report of a large-scale cluster-randomized longitudinal experiment carried out in the Netherlands;

- A duo of artefacts (Soury-Lavergne and Maschietto), constituted by a mechanical arithmetic machine (inspired by the instrument designed by Pascal in the 17th century) and its digital counterpart, to enable six-year old students to learn about numbers. The experiment, developed in collaboration between a French and an Italian teacher educator shows the separate conceptualisation processes fostered by the physical and the virtual instruments;

- An intentionally designed virtual manipulative (place-value chart, that runs on I-pads), discussed by Ladel and Kortenkamp, and contrasted with other tools such as multi-base arithmetic blocks with reference to a flexible understanding of place value.

The last two papers address issues relating to the comparison of potentialities of concrete and virtual manipulatives in classroom experiments, and are important given that these discussions are sometimes skipped in the enthusiastic and uncritical presentation of virtual manipulatives as easier and cheaper surrogates of concrete ones (see http://nlvm.usu.edu/en/nav/siteinfo.html).

Teacher education

An overarching issue that encompasses the conditions for the learning of whole number arithmetic concerns teacher education and the effects it may have on students' mathematical working and attitude. All the above papers, in some sense, hint at the importance of teacher education for the effective use of either language or tools.

Two specific programmes for teacher education are addressed within theme 3:

- A programme for pre-service teacher education developed in Canada (Hodgson and Lajoie) that stresses the complementary roles played by mathematicians and mathematics educators;
- A programme developed in Thailand in order to adapt the Japanese Lesson Study to the Thai context (Inprasitha).

Questions for discussion in the working group

Different kinds of papers are collected in this theme, with partial overlapping with issues addressed in the other themes. Some of them address general background issues: the number line in the Western tradition (Bartolini); the two basics (i. e. basic mathematic concepts and basic mathematic skills) in the Chinese tradition (Ni). But most papers report on empirical studies in different contexts (Australia, Canada, France, Germany, Hong Kong, Italy, New Zealand, The Netherlands, Thailand, US). In some cases the empirical studies address special education (Rottman, Bakker) or economically disadvantaged pupils (Ball and Bass).

Some of the questions raised in the Discussion Documents are still open and need to be addressed or deepened during group discussion:

(1) What main challenges for learning whole number arithmetic are faced by marginalized students or, in general, in difficult contexts?
(2) Are there classroom studies on the comparison of different kinds of tools (e.g. concrete vs. virtual ones)?
(3) Which tools used in different contexts exploit traditional activities around whole number arithmetic? It would be interesting to prepare an annotated list of references of tools used in classroom practice, with videos to explain, in more illustrative ways, how the tool is, or might be, used.

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Papers in Theme 3

Bakker M. M., van den Heuvel-Panhuizen, M., & Robitzsch, A. Learning multiplication reasoning by playing computer games.

Ball, D. L., & Bass, H. Helping children learn to persevere with challenging mathematics.

Bartolini-Bussi, M. G. The number line: A western teaching aid.

Hodgson, B. R., & Lajoie, C. The preparation of teachers in arithmetic: A mathematical and didactical approach.

Inprasitha, M. An open approach incorporating lesson study: An innovation for teaching whole number arithmetic.

Ladel, S., & Kortencamp. U. Development of conceptual understanding of place value.

Mercier, A., & Quilio, S. The efficiency of primary level mathematics teaching in French-speaking countries: A synthesis.

Ni, Y. How the Chinese methods produce arithmetic proficiency in children.

Peter-Koop, A., Kollhoff, S., Gervasoni, A., & Parish, L. Comparing the development of Australian and German children's whole number knowledge.

Pimm, D. The ordinal and the fractional: Some remarks on a trans-linguistic study.

Rottmann, T., & Peter-Koop, A. Difficulties with whole number learning and respective teaching strategies.

Soury-Lavergne, S., & Maschietto, M. Number system and computation with a duo of artefacts: The pascaline and the e-pascaline.

Young-Loveridge, J., & Bicknell, B. Using multiplication and division contexts to build place-value understanding.

LEARNING MULTIPLICATIVE REASONING BY PLAYING COMPUTER GAMES

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Abstract

This paper reports about a large-scale longitudinal field experiment investigating the effects of online mathematics mini-games on second- and third-graders' multiplicative reasoning abilities. The study included students in regular primary education ($n = 719$) and special primary education ($n = 81$). There were three experimental conditions: playing multiplicative mini-games at *school*, at *home*, and at *home with debriefing at school*. In the control condition mini-games on other mathematical topics were played at school. For regular primary education, results showed that the mini-games were most effective in the home-school condition, where they promoted both multiplicative skills and insight (significant *d*s ranging from 0.22 to 0.29). In the school condition, an effect was only found for insight in Grade 2 ($d = 0.35$); in the home condition there were no effects. In special primary education, a significant effect was found for the school condition in improving multiplicative fact knowledge ($d = 0.39$).

Key words: mathematics computer games, multiplicative reasoning, primary education, special primary education

Introduction

Computer games are more and more becoming part of primary school mathematics education (e.g., Alexopoulou et al., 2006). The most important benefits of games are their motivational characteristics (e.g., Garris, Ahlers and Driskell, 2002), and their possibility to provide immediate feedback (e.g., Prensky, 2001). Also for students in special education, mathematics computer games are promising educational tools (e.g., Brown et al., 2011). Yet, although meta-analyses did show that in general the use of ICT in mathematics education positively affects learning outcomes (Li and Ma, 2010; Slavin and Lake, 2008), there is still insufficient evidence for the effectiveness of computer games in particular (Bai et al., 2012). The present paper aims to provide such evidence for the domain of multiplicative reasoning (multiplication and division), for both regular and special primary education.

In learning multiplicative reasoning, it is important to develop ready knowledge of number facts (the multiplication tables), and skills in calculating multiplication and division operations. In addition, students need to develop insight in, or understanding of, multiplicative number relations (e.g., Anghileri, 2006; Nunes et al., 2012). They should, for example, have insight into the factors of numbers and the properties of multiplication, like the commutative property (e.g., $3 \times 7 = 7 \times 3$) and the distributive property (e.g., $6 \times 7 = 5 \times 7 + 1 \times 7$). These three aspects of multiplicative reasoning ability – number fact knowledge, operation skills, and insight – parallel the three types of knowledge often distinguished in mathematics education: declarative knowledge, procedural knowledge, and conceptual knowledge (see, e.g., Miller and Hudson, 2007).

Though most of the computer games and other educational software currently used in mathematics education focus on number fact knowledge and operation skills (e.g., Mullis et al., 2012), computer games can also be employed for developing mathematical insight (e.g., Klawe, 1998). The instructional power of games that are focused on insight development is often related to the educational theory of experiential learning (see, e.g., Kebritchi, Hirumi and Bai, 2010). In such games, students can learn new concepts and rules by exploring and experimenting with different mathematical strategies and discovering which strategies are convenient. With experiential learning games, class discussion – also called debriefing – is important to promote reflection on and generalisation of what is learned (e.g., Garris et al., 2002; Klawe, 1998).

Educational games can be played in different settings. Playing in a formal setting at school has the advantage that all instructional aspects of the games can be exploited by discussing them in a lesson. However, playing in an informal setting at home has advantages as well. Besides the benefit of extra learning time (e.g., Honey and Hilton, 2011), playing at home may lead to increased learner control, which is often mentioned as an important motivating factor of educational computer games (e.g., Malone and Lepper, 1987). A possible approach that combines the advantages of playing at school and those of playing at home, is playing the games at home with a debriefing at school (see Kolovou, Van den Heuvel-Panhuizen and Köller, 2013).

Research question

Does an intervention with multiplicative mini-games – either played at school, played at home, or played at home and debriefed at school – affect regular and special primary education students' learning outcomes in multiplicative reasoning; i.e. knowledge, skills, and insight?

Method

Study set-up

To answer our research question we set up a large-scale cluster-randomised longitudinal experiment (see also Bakker, Van den Heuvel-Panhuizen and Robitzsch, 2015a, 2015b). The experiment included three experimental conditions with multiplicative mini-games – playing the games at school integrated in a lesson, playing the games at home without attention at school, and playing the games at home with a debriefing at school – and one control condition in which the students played at school mini-games on other mathematics topics. In the conditions where the games were played at home, the games were presented as a free-choice activity, not as compulsory homework. This was done to maintain the motivating aspect of playing the games.

	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun
Grade 1										*Pretest:* Skills
Grade 2	Game period 1					Game period 2				*Posttests Grade 2:* Knowledge, Skills, Insight
Grade 3	Game period 3					Game period 4				*Posttests Grade 2:* Knowledge, Skills, Insight

Fig. 1: Time schedule of the study

The mini-games – short, focused games that are easy to learn (e.g., Jonker, Wijers and Van Galen, 2009) – were played in four game periods, two in Grade 2 and two in Grade 3, as is shown in Fig. 1 (for special education only the Grade 2 part of the study was performed). In each game period, 8 different mini-games were offered. Before each game period, the teachers were given a manual in which for each game it was described how it had to be treated in class.

A pretest of multiplicative reasoning ability was administered at the end of Grade 1. Posttests on each of the three aspects of multiplicative reasoning ability – knowledge, skills, and insight – were administered at the end of Grade 2 (for the special education students, to reduce test duration, the skills and insight test were combined into a skills/insight test), and at the end of Grade 3 (only for regular education). A description of the tests employed, as well as more information on the interventions, can be found in Bakker et al. (2015a, 2015b).

Participants

The study was conducted in the Netherlands and included schools for regular primary education as well as schools for special primary education. In the Netherlands, special primary education is meant for students with substantial learning difficulties, mild mental retardation, or mild to moderate behavioural or developmental problems.

We recruited 66 regular primary schools and 19 special primary schools. Through a matching procedure on a number of school characteristics, and random assignment, the schools were evenly distributed over the research conditions. For various reasons, such as teacher changes, organisational problems, and problems with computers, some schools dropped out in the course of the research project. Moreover, only those schools in which more than half of the games were treated were included in the analysis. Unfortunately, for the special education schools, due to large drop-out and low intervention fidelity in the home and home-school condition, we could only include the school condition and the control condition in the analysis. Our final sample consisted of 35 regular primary schools (n = 719 students; 112 in the school condition, 202 in the home condition, 78 in the home-school condition, 327 in the control group), and 5 special primary schools (n = 81 students; 40 in the school condition, 41 in the control group).

The mini-games

The mini-games offered in the experimental conditions were mostly adapted versions of multiplicative mini-games selected from the Dutch mathematics games website Rekenweb. Descriptions of all mini-games can be found in Bakker et al. (2015b). As an example, one of the mini-games is shown in Fig. 2. In this game the student makes rectangular groups of smileys and then determines the number of smileys in the group. The game offers practice in solving multiplication problems (either as memorised multiplication facts or, for example, by repeated addition). Furthermore, the game stimulates gaining insight into the relations between multiplication problems; for example, 3 rows of 5 is the same as 5 rows of 3 (commutative property), and if 5 rows of 3 is 15, then 6 rows is 3 more, resulting in 18 (distributive property).

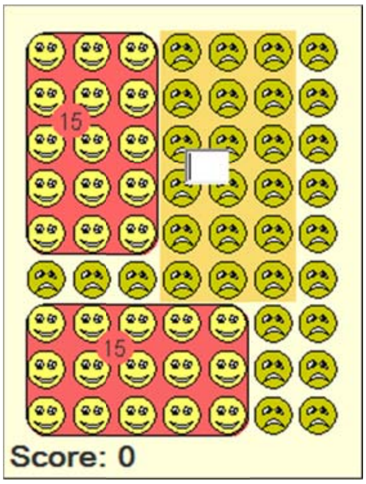

Fig. 2: Example mini-game “Making groups”

Data analysis

The effects of the interventions were investigated separately for the students in regular primary education and special primary education. For both samples, we employed linear regression analyses for each of the aspects of multiplicative reasoning ability, with posttest score as the dependent variable, and pretest score and condition dummy variables as independent variables. Additionally, we controlled for some student characteristics that were found to differ between conditions. Separate analyses were run for the effects of the intervention in Grade 2 (Grade 2 posttest score as dependent variable) and the intervention in Grade 2 and 3 together (Grade 3 posttest score as dependent variable). Missing test scores were handled using multiple data imputation (see Graham, 2009). The clustered data structure (students nested within schools) was accounted for by employing cluster-robust standard errors (see Angrist and Pischke, 2009; this could not be done for the special education students because of too few clusters, i.e., only 5 schools). Because of our directional hypothesis (we hypothesised the games to positively affect learning) we used one-tailed significance tests.

Results

For the students in regular primary education (Tab. 1), the games were found to be effective in enhancing skills and insight, but not knowledge. Specifically, in the home-school condition the intervention had a significant positive effect on both skills ($d = 0.26$ for the Grade 2-3 intervention) and insight ($d = 0.29$ for the Grade 2 intervention; $d = 0.22$ for the Grade 2-3 intervention). In the school condition the games only significantly affected insight, and only the Grade 2 intervention was effective ($d = 0.35$). No significant effects were found in the home condition ($p > .05$). For the special education students (Tab. 2), it was found that the games, played at school, were effective in enhancing multiplicative fact knowledge ($d = 0.39$), but not in enhancing skills/insight.

	Knowledge			Skills			Insight		
Condition	β_{ps}	*SE*	*d*	β_{ps}	*SE*	*d*	β_{ps}	*SE*	*D*
				Posttest Grade 2 (effect of Grade 2 intervention)					
School	0.01	0.24	0.01	0.10	0.24	0.09	0.39*	0.22	0.35
Home	-0.16	0.23	-0.16	-0.04	0.20	-0.03	0.21†	0.15	0.19
Home-school	0.08	0.26	0.08	0.21	0.20	0.18	0.32*	0.19	0.29
				Posttest Grade 3 (effect of Grade 2-3 intervention)					
School	-0.19	0.23	-0.20	0.10	0.18	0.09	0.15	0.19	0.13
Home	-0.05	0.16	-0.05	0.03	0.14	0.03	-0.02	0.12	-0.02
Home-school	0.16	0.13	0.16	0.28*	0.16	0.26	0.24*	0.12	0.22

Note. The pretest score, gender, age, parental education, home language, and general mathematics ability score were included as covariates. β_{ps} = partially standardised regression coefficient of the condition dummy variable predicting posttest score.
† $p < .10$. * $p < .05$. One-tailed.

Tab. 1: Effects of the interventions in regular primary education on knowledge, skills, and insight in Grade 2 and 3 (as compared to the control group)

	Knowledge			Skills/insight		
Condition	β_{ps}	*SE*	*d*	β_{ps}	*SE*	*d*
School	-0.01	0.15	-0.02	0.19*	0.11	0.39

Note. The pretest score, age, and general mathematics ability score were included as covariates. β_{ps} = partially standardised regression coefficient of the condition dummy variable predicting posttest score.
* $p < .05$. One-tailed.

Tab. 2: Effects of the intervention in special primary education on knowledge and skills/insight in Grade 2 (as compared to the control group)

Conclusion and discussion

For regular primary education, our study shows that the most effective way of integrating multiplicative mini-games in mathematics education is by offering them to students to play at home, and debriefing them at school. When the mini-games were offered in this way, they positively affected both students' skills in calculating multiplicative problems and their insight in multiplicative number relations (significant *d*s ranging from 0.22 to 0.29). Also playing the games at school, integrated in a lesson, was found to be effective, but only in promoting insight in Grade 2 ($d = 0.35$). Playing the games at home without attention at school did not affect students' learning of multiplicative reasoning.

The finding that the games were most effective when played at home and debriefed at school can be explained by this intervention having the combined advantage of playing at home (extra time on task, more learner control) and playing at school (debriefing). Playing at home without debriefing was not effective, indicating the importance of debriefing sessions in learning from the games. As proposed by, for example, Garris et al. (2002) and Klawe (1998), the debriefing sessions may have led students to reflect on what they had learned in the games, enabling them to generalise their learning beyond the game context. However, in our study the debriefing sessions may also have served as an encouragement for students to play the games at home. Indeed, in the home-school condition, the games were played more often than in the home-condition.

For special primary education, we found that the mini-games, played at school, were effective in promoting students' multiplicative fact knowledge, but not their multiplicative skills and insights. Yet, although there was no added value of the mini-games for skills and insight, an intervention with mini-games can still be seen as a "safe approach" to be employed as part of the multiplicative reasoning programme in special education, as learning outcomes were not different from those obtained in the control group.

The finding of an effect on knowledge but not skills/insight in special primary education is in contrast with our findings for regular primary education, where there were effects on skills and insight but not on knowledge. Possibly, for the special primary education students, who often are considerably behind in their learning, there was still much to improve in basic multiplicative fact knowledge. Also, multiplicative fact knowledge may, for these students, have been easiest to acquire from the game, because it requires least transfer (see, e.g., Shiah et al., 1994): multiplicative facts occurred in most games in the same way (with the × symbol) as in the textbooks and assessments. Students in regular primary education may have had enough opportunities for automatizing the multiplication tables in the regular mathematics curriculum, leaving room for the acquisition of more advanced types of knowledge. For them, the games were especially useful for acquiring insight, which may be related to the nature of the mini-games used, allowing for free exploration and experimentation (experiential learning).

The finding that the home and home-school condition were not adequately carried out by the special education teachers may indicate that having students playing mathematics games at home by themselves is not in line with the current practices of teachers in special primary education.

In conclusion, our study shows that both in regular and special primary education, mini-games can effectively be used to promote students' learning of multiplicative reasoning. Yet, the two school types appear to differ in terms of the aspects of multiplicative reasoning that are affected by the games, and in terms of the way in which the games can best be offered to the students.

In the course of our research project, it appeared that a large-scale study situated in school practice is hard to carry out. Because of teachers' busy schedules it was hard to find teachers willing to participate in a long-term study, and to motivate teachers in subsequent grades to continue the study. However, we think that conducting this research in real school settings to collect evidence for the effectiveness of mathematics games in (special) primary education was worth the effort. It provided us with knowledge of when mathematics mini-games are useful. Moreover, as the interventions were delivered by the teachers themselves, our results are directly applicable to the school practice.

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HELPING STUDENTS LEARN TO PERSEVERE WITH CHALLENGING MATHEMATICS

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Abstract

"Perseverance," an important psychological construct, matters for mathematics learning because solving challenging mathematics problems often requires a kind of uncomfortable persistence. However, school experience often values speed over persistence, and young learners are rarely guided explicitly to see that perseverance is needed or how to stick productively with a problem. This paper examines a case of a class of 10-year old students, and analyses how they are explicitly helped to learn whole number arithmetic while also developing skills of perseverance and persistence. Three aspects of the instruction comprise the focus: (1) the nature of the mathematical task on which the class was working, (2) the staging of students' work on the problem, and (3) how perseverance – for this problem and beyond – was supported. The paper concludes with several questions that our analysis suggests as important for next steps in trying to understand mathematical perseverance.

Key words: instruction, mathematical perseverance, proof

Introduction: "Sticking With It" in Mathematics

A common finding is that U.S. students tend to give up if they cannot figure out how to solve a complex problem within a few minutes. Stevenson and Stigler (1992) report, from a study comparing U.S. students with their counterparts in other countries, that, on average, U.S. students tend more than others to believe that mathematics depends more on talent than effort. These same students often persist with other challenges outside of school – perfecting a jump shot in basketball, constructing puzzles, or practicing a difficult musical piece. But in school they develop the sense that mathematics is more a matter of talent and speed than persistence and effort. And although they are exhorted to try, rarely are they helped to learn what to do to persevere productively. Further, when they develop a sense that they are not good at "mathematics", they are even less likely to try to work on puzzling problems. Could this persistent cultural assumption be challenged (U. S. Department of Education, 2013)? What might instruction look like that would aim to help students, especially those who have been discouraged in school and who feel themselves not to be "good at" mathematics, to have both the skills and the confidence to persevere in mathematics (Cuoco et al., 1996)?

(How) Might Mathematical Perseverance Be Developed?

In this paper we analyse a case of instruction that aims to cultivate productive mathematical persistence in elementary students in the context of a challenging problem of whole number arithmetic. Of interest in our research are learners who, in average, have not had opportunities or encouragement to succeed in school mathematics, especially with challenging and complex work, and who, by fifth grade, are already less willing to try problems that seem hard or confusing. Our analysis focuses on the role and interaction of three instructional elements: (1) the mathematical task, (2) the teacher's role and practices, and (3) the whole class discussions. The whole number arithmetic task in our study has an unusual feature that is particularly salient with regard to cultivating perseverance: it is mathematically impossible.[1] The class's work to produce a proof is the context for our analysis of what is involved in helping students learning to persevere with a challenging mathematics problem.

Instructional and Research Context

Our investigation of mathematical perseverance took place in the context of a mathematics programme, the Elementary Mathematics Laboratory (EML), for fifth graders that takes place at the University of Michigan every summer[2]. The programme enrols approximately 30 students from a local working class, racially and ethnically diverse, school district. A majority of the students are economically disadvantaged, from a community with a growing number of homeless children[3]. Mathematics achievement in this district, was indicated by students' scores on the state assessment. These data show that approximately 70% of all fourth graders are "not proficient," the lowest level of attainment[4]. Most of the students have gaps in their skills and knowledge and typically have not enjoyed or felt confident with school mathematics. About 75 – 80% of the students identify as black or mixed race, and about 10 – 12% identify as Latino/a. Given the students' past school histories with mathematics, their persistence is a key issue. They tend not to be confident and when they encounter unfamiliar problems, they tend to say that they cannot work on them because they do not understand or have not been taught the material. Therefore, a goal of the programme is to increase students' skills with, and actual experience of, mathematical perseverance.

The Train Problem as a Context for Developing Mathematical Perseverance

One reason for using an impossible problem is that the students assume that all problems in school are solvable, and if they are unable to solve a problem, it

[1] Mathematical problems without solution have a colorful and distinguished history (Suzuki, 2009) in mathematics (for example Fermat's Last Theorem), but we have found two references to their deliberate use with elementary students, in Burchartz and Stein (1998), and Tirosh et al. (2015).

[2] Our research group has been designing and studying the teaching and learning of mathematics in the Elementary Mathematics Laboratory (EML) for over ten years.

[3] Source: *2014 Report Card for Washtenaw County Children and Youth*.

[4] Source: Michigan School Data, *Annual Education Report 2012-13,* for the communities that comprise the EML partner school district.

means that they lack capability or have failed. This experience is intended to build their confidence about the nature of the solution set for a given problem, and is connected to other experiences they have with problems that have one, multiple, or infinitely many solutions.

The Mathematical Task

The basic underlying mathematical task is to find an order in which to list the five numbers 1, 2, 3, 4, and 5, without repetition, in such a way that when subsets of adjacent numbers in the specific list are added together, every number, from the smallest to the largest is possible. For example, one such order is: 2 3 1 4 5. In this arrangement, the largest number, 15, is possible by adding all the numbers together. Six is possible by adding 2, 3, and 1 together, which are adjacent this list. Ten can be made in two ways, using 2, 3, 1, 4 or using 1, 4, 5. But 12 is not possible because the numbers needed to add up to 12 using subsets of 1, 2, 3, 4, and 5 (3, 4, 5 or 1, 2, 4, 5) are not adjacent in this arrangement. Similarly, 14 is not possible.

The actual task given to the students is situated in a story called the "train problem" and makes the class into an imaginary "EML train company" that constructs "trains" to order. This formulation of the problem uses Cuisenaire rods to represent cars on a "train". A "train" is a structure built by laying the rods end to end. The number of passengers that the "train" can hold is determined by summing the number that each car in the train can hold:

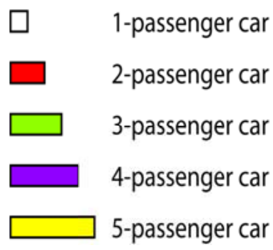

Fig. 1: The small cars

A train composed of exactly one red, one green, and one yellow "car" would thus hold 10 passengers (i.e., 2 + 3 + 5). The problem is formulated with a story about a customer, Mr. X, who seeks a very special train from the train company: Mr. X wants to order a five-car train that uses one of each of the different-sized cars. He also wants to be able to break apart the five-car train to form smaller sub trains that hold fewer passengers. To make things tricky, he wants to be able to form these sub trains only using cars that are next to each other in the larger train. And what he wants is one train that makes it possible to form sub trains for every number of passengers from 1 (just the white car) to 15 (all five cars).

This task is structurally equivalent to the basic numerical version of the task described above, but the context adds features designed to support engagement in this complex problem. One feature is the concrete materials for building the trains. A second is the storyline of a demanding customer who orders this very unusual train, and a "train company" that is trying to satisfy the customer's request.

The Staging of The Students' Work on the Problem

The students' work on the train problem stretched over several days, spending about 45 minutes to an hour each day. The trajectory proceeded through six stages: (1) becoming familiar with the context; (2) collective making sense of and interpreting the problem, and identifying conditions; (3) building and checking trains and recording results; (4) confronting the scope and feeling discouraged and overwhelmed; (5) cutting the problem down to manageable size; (6) proving and becoming confident that no solution exists. The teaching across these stages makes explicit specific practices of tackling and persisting with a difficult mathematics problem. At each stage, these practices were highlighted and labelled, modelled, scaffolded, and rehearsed and refined.

The students were first introduced to the territory with the task below, which designed to help them to uncover some of the key features of the context and to practice reasoning about "trains":

> The EML Train Company makes five different-sized train cars: a 1-person car, a 2-person car, a 3-person car, a 4-person car, and a 5-person car. These cars can be connected to form trains that hold different numbers of people. Try to build some trains. You can use <u>only</u> these five types of cars to build trains, and you can use <u>at most one</u> of each type of car in each train.
>
> What are the different numbers of people that the EML Train Company can build trains to hold?

As the students explored, built, and discussed the trains, they were able to conclude that the greatest number of possible passengers is 15, and that a 15-passenger train requires all of the cars. They also figured out the least number of passengers possible is either 0 (no cars) or 1 (only the white).[5] After a period of working on how to build trains for different numbers of passengers, the students were given the problem with the story of the customer:

> A customer named Mr. X wants to order a special five-car train that uses one of each of the different-sized cars. He wants to be able to break apart his five-car train to form smaller trains that could hold exactly 1 to 15 people. In addition he wants to be able to form these smaller trains using cars that are next to each other in the larger train.
>
> Can the EML Train Company fill the customer's order? Explain how you know.

The teacher helped the students to make sense of what the words in the problem meant, experimenting with what it means to "break apart" a train and to "form smaller trains" with cars that are next to each other that hold other numbers of passengers. For example, in working with one possible train (Fig. 2), students practiced showing that smaller trains that hold 1, 2, 3, 4, 5, 6, and 7 are possible. They were stuck trying to find a smaller train that holds 8 passengers. The

---

[5] It is interesting to listen to fifth graders debate whether a train with no cars is in fact a train.

teacher asked different students to explain how they are sure that they cannot make a smaller train that holds 8 by breaking apart the whole train and, using cars that are next to each other.

Fig. 2: The train

As part of this stage of the work, the teacher and students extracted and recorded the "conditions" of the problem:

1. Only use w, r, g, p, y.
2. Must use all of these cars.
3. Must use each car exactly once.
4. Must be able to form trains for every number of passengers from 1-15 without moving cars around. These smaller trains must be built from connected cars.

Building Mathematical Perseverance

Persevering with challenging tasks is clearly fundamental to skilled use of mathematics to solve problems. Successful problem solvers use a range of strategies that are often not made explicit to others (Schoenfeld, 1985), and the development of the capacity to persist productively is not usually a deliberate part of the school mathematics curriculum.

Developing the capacity to persevere with challenging mathematics certainly requires opportunities to contend with puzzling problems where the solution paths are not obvious. But simply getting stuck in math class does not lead to productive perseverance. What is the nature of instruction that can engage students in productive struggle, rather than frustration? We found that students' opportunities to develop perseverance rested with the nature of the mathematical task, the teacher's role and practices, and the whole class discussions of the work.

For the fifth grade students in the instructional case above, the train problem involved mathematical ideas and skills that were familiar, but the scope was too big to make it feasible to write down all the possible trains (in fact 120 possible trains, each one requiring extensive testing). So although students could begin, they were soon faced with what seemed like an endless search.

As the students worked to solve the train problem, the teacher used a variety of instructional strategies to help them persevere with the problem and to develop the capacity for mathematical perseverance more generally. First, she designed a problem context that required the set the group up to work collectively, instead of creating a problem to be done by the students individually. Students were not left on their own to struggle, but instead contended with the challenging problem together. Efforts made by members of the class were publicly shared and discussed, and successful steps forward were everyone's accomplishment.

Stage of the work	Instructional moves and steps	Learning goal for students	Supporting students' progress on this problem	Supporting students' learning more general practices of perseverance
1. becoming familiar with the context	have students work on a simpler problem that establishes the conventions for building trains	Practice reasoning about trains, practice with explaining	✔	
2. collective making sense of and interpreting the problem, and identifying conditions	engage students in reading and discussing the problem lead students in practicing one possible solution with one train guide students to extract the conditions of the problem	learning strategies for making sense of and interpreting a problem identifying conditions of a problem	✔	✔
3. building and checking trains and recording results	have students work on building trains and making records on their own encourage students to check solutions using the core constraints of the problem	using conditions of a problem to check solutions making records of one's work on a long problem	✔ ✔	✔ ✔
4. confronting the scope and feeling discouraged	allow students to begin to wallow in the enormity of the problem, stimulating desire to find a simpler way through it	using feeling of "stuckness" to seek ways to simplify problem	✔	
5. cutting the problem down to manageable size	consider suggestion that the red and white be on the end refer students to prior work on permutations of three	investigating a conjecture and seeking to prove or disprove it using results of prior work, connect to other ideas	✔ ✔	✔
6. proving and becoming confident that no solution exists	guide students to practice presenting to the customer that his order cannot be filled	linking the imperative to convince someone as an imperative for mathematical proof	✔	✔

Tab. 1: Instructional moves and learning goals across stages of students' work on the train problem

The collective challenge of trying to fill Mr. X's order required frequent group discussions to check possible solutions and to consider strategies. And the context created a situation that readily supported making the work visible. Although the playfulness of the story was obvious, and no fifth grader believed it as "real", the collective engagement and motivation to solve the problem nevertheless created a kind of authenticity.

Tab. 1 summarises the instructional moves and steps used in guiding the students' effort to solve the train problem. Some moves seem more likely to have promoted students' learning of general skills of mathematical perseverance than others.

As she supported the students in developing skills of perseverance, the teacher's instructional moves helped to make explicit the specific things that she and the class were doing as they pursued solution to the problem. One significant move was *naming and labelling* particular tools for mathematical work (Ball, Lewis and Hoover, 2008). A second pattern in the teacher's efforts to make visible the practices specific to persevering with the train problem was the way she *highlighted and underscored* things that she or students did and together unpacked explained what made them helpful. A third technique was to *co-sponsor* the mathematical work. As she saw individual students trying or doing things that could be of general interest or use, she would work quietly with an individual (or sometimes a pair or threesome), helping the student to extend and further articulate the idea or technique so that it could be shared with the class. A fourth instructional move employed regularly by the teacher was to create pauses for *reflection* on progress, questions, stuckness, and new insights.

Taking Stock and Next Steps

While psychologists investigate differences in people's persistence in the face of difficulty (Dweck, 1999, 2006; Duckworth et al., 2007; Mischel et al., 1972), children and adults also exercise varying perseverance across the challenges they encounter. We take the perspective that perseverance is a domain-specific bundle of skills that support confidence in the struggle to succeed. School mathematics is filled with moments of frustration and being stuck, but students too often are not taught practices that enable successful perseverance with complex mathematics.

Important questions are raised by our analysis of this case that merit further study. One centres on our hypothesis that to be helped to persevere with a particular problem helps students to build more general strategies for mathematical perseverance. Because we conceive of perseverance as a bundle of practices, we posit that learning is best situated in actual cases where doing is entailed. If students are helped to work on complex and challenging problems, and succeed at them, does this also set them up to know what to do when they are working alone on another problem? Is the perseverance involved in these students' learning to solve complex problems within whole number arithmetic usable by them when they encounter a challenging geometry problem?

A second issue centres on the role of believing in one's own ability to solve a problem and of motivation. We are arguing that a critical component of perseverance is skill and technique. But does confidence play a role, and if so, what sort of confidence and how does it interact with technique, if at all? Does caring about the goal matter, and if so how does this play out? To what extent are these questions cultural, situated in the contexts of schooling in the U.S.? Might these issues appear and play out differently across national contexts?

A third has to do with the measurement of mathematical perseverance. To study the sorts of questions identified above would require sensitive and valid measures of persistence with difficult mathematics. Developing reliable ways to do this would advance our understanding of efforts to develop perseverance, as well as extend conceptualisation of perseverance itself.

Finally, we became interested in the possibility that perseverance may be a collective as well as an individual capability. Can perseverance be usefully construed as distributed among members of a group and if so, how might this be both cultivated and used?

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THE NUMBER LINE: A "WESTERN" TEACHING AID

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Abstract

This paper aims at discussing a very popular teaching aid, the so-called *number line,* where whole numbers are introduced as labels on unit marks by means of a measuring process and where additions and subtractions can be realised, as operators, with jumps forwards and backwards. Traces of this early approach can be found in the teaching practices of most Western countries, but, surprisingly, not in the most popular Chinese textbooks. A question arises: where does the difference come from? In the following, I review some Western literature to sketch out the analysis of the number line as a teaching aid, from the historic-epistemological, cognitive and didactical perspectives. Later some paradigmatic practices from different countries are presented.

Key words: addition, measuring, mental number line, number line, subtraction

Introduction

The intention of writing this paper arose after a dialogue with Sun Xuhua about the most popular teaching approaches to whole numbers. When discussing uses of the number line in the two cultures (Italian and Chinese) I realised that it is not used in China as much as in the West. This brought me to reflect on how some Western popular approaches are not mandatory, but dependent on culture. As has happened for me in the past (Bartolini Bussi and Martignone, 2013; Bartolini Bussi, Ramploud and Anna Baccaglini-Frank, 2014), noticing this difference was a stimulus towards the reconstruction of the roots of our tradition (from the historic-epistemological, cognitive and didactical perspectives), or, following Jullien (2008), towards the "discovery of our own unthought".

Historic-epistemological perspectives

Epistemological perspective

Hans Freudenthal, a past ICMI President (1967-1970), has devoted several volumes to the epistemological foundation of mathematics education. In the *Didactical Phenomenology of Mathematical Structures* (1983), Freudenthal introduces magnitudes, criticising traditional teaching where measuring is delayed until children are ready to learn common and decimal fractions.

> "The first step in analysing a magnitude, where measuring the magnitude is articulated by the natural multiples of a unit, is possible and desirable at an early age; counting can and must immediately be transferred from discrete quantities, represented by sets, to magnitudes. Modern textbooks start measuring much earlier than tradition allows, but unfortunately this kind of measuring is not yet sufficiently integrated with the operations on natural numbers. <u>The device beyond praise that visualises magnitudes and at the same time the natural numbers articulating them is the number line, where initially only the natural numbers are individualised and named.</u> In the didactics of secondary instruction the number line has been accepted, though it is often still imperfectly and

inexpertly exploited; in primary education it makes progress little by little. […]. It seems to me a disadvantage of the number line that it is so easily drawn and that it cannot be sold together with the textbook as teaching material. […] The number line eclipses the Cuisinaire rods in many respects: The virtual infinity is better expressed by the number line. The number line knows no compulsory scale; number lines on different scales – on the blackboard and on paper – are immediately identified, notwithstanding their incongruency" (Freudenthal, 1983, p. 101, my emphasis).

Historical perspective

The concept of magnitude goes back to Euclid (Euclid, Book 5, in Heath vol. 2 p. 113 ff.). Although the idea of magnitude is more general (and also includes areas, volumes and similar), magnitudes are represented by straight lines in all the Euclid's editions after Heiberg. Interestingly, in Book 7 about the study of whole numbers, numbers themselves are represented as straight lines with no relation with the number size (Euclid, Book 7, in Heath vol. 2 p. 277 ff.). It may seem strange to represent whole numbers as straight lines, in contrast with the Greek tradition of using configurations of pebbles (as in the case of square numbers). Netz (1999) gives an interesting interpretation about the meaning of diagrams and links it to the issue of generality:

> "Often the proof is about "any integer", a quantity floating freely through the entire space of integers, where it has no foothold, no barriers. […] A dot representation implies a specific number, and therefore immediately gives rise to the problem of the generalisation from that particular to a general conclusion, from the finite to the infinite. Greek mathematicians need, therefore, a representation of a number which would come close to the modern variable. This variable […] is the line itself. The line functions as a variable because nothing is known about the real size of the number it represents" (Netz, 1999, p. 268).

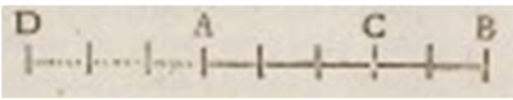

Fig. 1: Wallis' drawing

The first number line with unit marks dates back to John Wallis (1685).

> "Supposing a man to have advanced or moved forward (from A to B) 5 Yards, and then to retreat (from B to C) 2 Yards: if it be asked how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when at A? I find (by Subtracting 2 from 5) that he is Advanced 3 Yards. (Because +5 – 2 = + 3). But if, having Advanced 5 Yards to B, he thence Retreat 8 Yards to D; and it be then asked, How much he is Advanced when at D, or how much Forwarder than when he was at A: I say -3 Yards. (because +5 – 8 = -3). That is to say, he is advanced 3 Yards less than nothing. Which in property of Speech, cannot be, (since there cannot be less than nothing). And therefore as to the Line AB Forward, the case is Impossible. But if (contrary to the Supposition) the Line from A, be continued *Backward,* we shall find D, 3 Yards *Behind* A. (which was presumed to be *Before* it). And thus to say, he is *Advanced* -3 Yards; is but what we should say (in ordinary form of

> Speech) he is *Retreated* 3 Yards; or he wants 3 Yards of being so Forward as he was at A" (ibid., p. 265).

Wallis' number line does not show numerals, but reference to units (Yards). The marks are drawn at regular intervals. The starting point A is the origin of the mark ("zero") whilst the other points are reached with a positive (forward) or negative (backwards) number of steps. Actually "zero" as a label on a number line does not entail the same difficulty and incomprehension raised by zero as either a number or a digit in place value (see below); also the understanding of negative numbers, very difficult until the end of the 18th century, is facilitated.

(Neuro) cognitive perspectives

The mental number line

(Neuro) cognitive scientists study the representation of numbers in a spatial format along the so-called "mental number line," whereby smaller numbers occupy relatively leftward locations (in the case of horizontal representations) or lower locations (in the case of vertical representations) compared with larger numbers. This idea dates back to Galton (1880) and is now focused in many experimental studies (for a short review, Butterworth, 1999). There is evidence in recent studies that blindness alters the direction of mental number line (Pasqualotto, Shichiro and Proulx, 2014), suggesting that the left-to-right organisation can be affected by the enviromental factors and visual perception. Actually a few studies show some correlation between the direction of writing and the direction of the number line (Bender and Beller, 2011).

The number line as a conceptual metaphor

There is a function of the number line in cognitive linguistics. In their programme to understand where mathematics comes from, i.e. what is the cognitive structure of sophisticated mathematical ideas, Lakoff and Nunez (2000) have taken an embodied approach, assuming the motion along a path as one of the grounding metaphor for arithmetic. They argue that abstract mathematical notions have their origins in our specific embodiment and could not have been construed differently. Conceptual metaphors work as projections from a domain (in this case the spatial experience of motion along a path) to another domain (in this case the arithmetic of whole numbers). The idea is not new (see Wallis, above) and was already used by Herbst (1997) who referred to Black's theory of conceptual metaphor (1962), which inspired the modern development of cognitive linguistics, in order to analyse the way of introducing and using the number line in a set of textbooks from Argentina.

A short interlude: towards classroom practices

It is worthwhile to highlight the potential of the number line as a teaching aid. The number line hints at the relationship between whole numbers and magnitudes, initiated in the classical age, and fosters the extension to fractions and rational numbers, by means of measuring. Either real or evoked motion on the line hints at the generation of infinitely many whole numbers, by iterating the action of one step forward. In a similar way, it eases the extension to negative numbers (by iterating the action of one step backward). The so called mental number line is widely accepted by (neuro) cognitive scientists as the mental representation of whole numbers in a spatial format.

Addition and subtraction on the number line are easily interpreted as the opposite of each other (forwards and backwards). This approach to addition and subtraction is consistent with the "counting on" and "counting back" strategies and quite different from approaches involving cardinal numbers (e.g. putting together and taking away). In particular, on the number line, the two addenda of an addition are distinguished (status – operator). Activity on the number line is suitable for both high and low achievers. For the former the challenge is to discover the commutative or associative properties of addition (that seems trivial in the cardinal approach); for the latter (see below) the usefulness is to introduce and reinforce automatic procedures for easy additions and subtractions.

Didactical issues

Reviewing the literature on the didactical use of number line is far too ambitious for this short contribution. Only some approaches are mentioned below in order to represent the variety and the richness of the use of the number line in the mathematics classroom in many countries.

Davydov's curriculum

The Russian elementary programme of V. V. Davydov is the result of the application of Vygotskian theory to school mathematics in order to solve the dichotomies between empirical and theoretical thinking, between arithmetic and algebra, that were (and still are in most countries) typical of primary vs secondary school. A paradigmatic example in this curriculum is given by the number line.

> "The number line arises in Davydov's curriculum from a consideration of simple medicinal dosage calculation using a graduated cylinder which is then tipped ninety degrees to create a horizontal gradient. The elements that are essential to the creation of a number line are also explored thoroughly, including direction, starting point and choice of a unit. If any one of these is unspecified, it is impossible to determine where specific numbers will be located even if the other two elements are provided. When the number line is presented as a ready made representation with these elements in place *a priori*, as is typically the case in the U.S., the arbitrary nature of these determinations remains undetected, since they are usually never explored. When children are presented with a number line such as in Fig. 2, and asked to mark "a + 1" on the number line, they must notice that no unit has been provided and therefore it is impossible to complete this task." (Schmittau, 2010, p. 273)

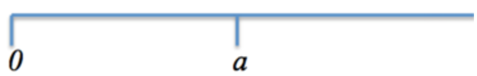

Fig. 2: Davydov's problem

Actually the understanding of unit is problematized in the whole curriculum that aims at introducing numbers by means of measuring rather than counting. Davydov's curriculum was very well known and influential in the West, giving the impression that it was *the* Russian curriculum. Actually it was only *a* curriculum in Russia, and not even the most popular at that time (Ivashova, 2011).

Realistic Mathematics Education: the empty number line

Klein, Beishuizen and Treffers (1998) reported about a new format for the old number line: the empty number line up to 100. They observed that the introduction from the very beginning of a structured number line with ready-made marks for every number fostered only counting and passive reading of the answer on the number line. They therefore opted for an empty number line on which the pupils can draw marks by themselves. They claimed that marking the steps on the number line for the realisation of an addition or a subtraction functions as a kind of scaffolding, as it shows which part of the operation is carried out and what remains to be done. The empty format stimulates a mental representation of numbers and number operations (addition and subtraction). Students using the empty number line are cognitively involved in their actions. These were the hopes, but a further analysis (van den Heuvel-Panhuizen, 2008) carried out ten years later by consulting children about the effectiveness of the empty number line, showed that a rigid application and an improper implementation of this teaching aid worked against the hopes of the designers and suggested the need to further research on this issue.

Other approaches

In most cases the number line has not been used to introduce the measuring aspects involved in whole numbers but rather the sequence of whole numbers and additions and subtractions by means of steps forwards or backwards. The relationships between numbers as labels and their distance from the origin have been seldom addressed. In the paper and pencil context, additions and subtractions are usually represented by small curved arrows, whilst in the context of large number lines traced on the floor, students can be asked to move or even jump from one number to another. Today, there are many software where the same operations are carried out on the computer screen (Ginzburg, Jamalian and Creighan, 2013).

Fig. 3: Addition on a graphic number line

Examples from Italy

A few decades of curriculum development: the number line

Arzarello and Bartolini Bussi (1998) have described the development of research in mathematics education in Italy in the last decades. One of the trends was represented by the strong cooperation between researchers and school teachers for the innovation in the mathematics classroom. An important step in this process was represented by the RICME Project (developed in Rome, under the direction of Michele Pellerey, with a group of teachers including also Emma Castelnuovo). This project supported multiple approaches to numbers, including a measuring approach, and the reference to out-of-school experiences, like traditional games. The *goose game* is a board game with uncertain origins, very popular in Southern Europe. The board has a track with (usually 63) consecutively numbered spaces for the pawns to stand in. Each pawn is moved as many steps as one (or two) dice gives. Each pawn is initially placed in the "starting space" (the "zero" space). The attention to a multiple approach to numbers, to the number line and to traditional games is mirrored in the Italian Standards (MPI, 1985) and, later, in the New Standards (MIUR, 2012), a much shorter document, where some traces of the number line still exist. The number line is used as a teaching aid in most Italian schools. Two examples are reported, both framed by the semiotic mediation approach (Bartolini Bussi and Mariotti, 2008).

Fig. 4: The time tube

The BAMBINI CHE CONTANO Project: the time tube

The time tube is a vertical empty "number line" designed within the pre-school project "Bambini che contano" (Bartolini Bussi, 2013), to complement the standard counting approach with some meaningful experiences of estimation and measuring. The activity was carried out with 4 and 5 years old children in more than twenty schools. The children are familiar with the number line in the form of a monthly calendar. A day-by-day tear-off calendar is also introduced. A cylinder tube of plexiglass with no graduation is gradually filled with small balls made by crumpling tightly each day-sheet: every day, a child tears off the old sheet (yesterday), cramples it very carefully and throws it into the time tube. The *past* goes into the tube, the *present* is visible on the front of the pad and the *future* is still hidden (in the calendar on the wall). If the teacher suggests to mark the level after one month, it may define the unit in approximate way. Guessing games can be played, like "What will the level be on Christmas day?" and the conjectures can be checked some weeks later.

The PerContare Project

PerContare (http://percontare.asphi.it/) is an innovative Italian project, built upon a collaboration between cognitive psychologists and mathematics educators, that aims at elaborating teaching strategies for preventing and addressing early learning difficulties in arithmetic. The Italian school system is totally inclusive, hence one of the aims of the project was to design teaching activities that could have a positive effect on *all* students, including low achievers and dyscalculics (if any). In this project some artefacts have been chosen to introduce whole numbers through multiple approaches. One of the artefacts is the ruler, used from the very beginning of primary school to evoke the number line. In this short report, I am describing the way of fostering the construction of automatic processes in dyscalculic children, by means of the number line. The following is the prototype of a dialogue (one-to-one interaction) between a low achiever (a child during the process of rehabilitation from dyscalculia). The child can read numbers but cannot retrieve from the memory simple arithmetic facts. There is a number line drawn (in analogy with the goose game) as a linear sequence of square spaces numbered from 0 to 10. The pawn to be moved is Tweety. The task is to calculate 4 + 3.

Adult: “Put Tweety on the 4” ;(done); Adult: “Keep Tweety steady and count on 3 with your finger”; (done); Adult: “Read the number”; Child : “Seven”. Adult: “Good job! 4+3=7”.

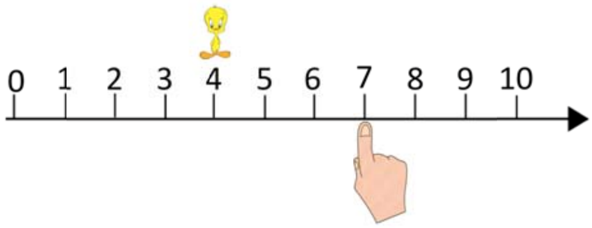

Fig. 5: Moving on the number line

The activity is very guided, hence it might be criticised as the passive use of the standard number line (see above). The aim is, however, not to foster the creative discovery of the potential of the number line but to construct a very simple procedure to be used by the low achiever first in a guided way and then independently, to acquire autonomy in the construction of simple number facts.

Discussion and conclusion

In this contribution some practices have been reported from different Western countries. The historic-epistemological and (neuro)cognitive foundations have been sketched out in a very short way. I do not claim that the number line is or must be a universal teaching aid. The (neuro)cognitive assumption of the mental number line as a spontaneous and natural model has been criticized by Núñez (2011) and related to cultural aspects. The same author however claims that “the number-to-line mapping, although ubiquitous in the modern world, is not universally spontaneous but rather seems to be learned through – and continually reinforced by – specific cultural practices (p. 611)”. For teachers, who may (must) exploit this ubiquity, this seems enough.

But now my question is: *Why not in China*?

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THE PREPARATION OF TEACHERS IN ARITHMETIC: A MATHEMATICAL AND DIDACTICAL APPROACH

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Abstract

This paper focuses on the preparation of pre-service primary school teachers in relation to the teaching and learning of whole number arithmetic. Using as a context teacher education in Québec (Canada), we aim at stressing the complementary role of mathematicians and mathematics educators in such an endeavour. We first present some specificities of our teacher education system. The contribution of mathematicians is then highlighted through the approach used for establishing the foundations of whole number arithmetic. Comments are finally offered on how the didactical component of the preparation of teachers can be integrated in such a context.

Key words: context for the reconstruction of whole number arithmetic, foundations of arithmetic, preparation for teaching elementary arithmetic, role of mathematicians

Introduction

The mathematical preparation of pre-service schoolteachers, in particular of the primary level, has for a very long while been a matter of concern and discussion. Already at the time of the inception of the International Commission on Mathematical Instruction (ICMI), in the early twentieth century, the eminent German mathematician Felix Klein (1849-1925) – who was to become the first president of ICMI – had been presenting a series of lectures mainly intended for secondary school teachers of mathematics (Klein, 1932). Parts of his comments however, especially in the first chapter entitled 'Calculating with Natural Numbers', pertain directly to primary school mathematics and the needed mathematical background and vision with which teachers of that level, in his opinion, should be familiar. The issue of the mathematical preparation of primary school teachers has regularly come back to the forefront over the following decades, as testified for instance by the paper of Rappaport (1958) published right in the peak of the "Sputnik shock" in the United States and proposing a review of research on the preparation of teachers of arithmetic.

The mathematical education of primary school teachers is a complex and multifaceted task requiring the contribution of different actors, and the aim of this paper is to stress the complementary role played by mathematicians and mathematics educators. We present a pre-service framework in which we have for long been involved, with a particular emphasis on Whole Number Arithmetic (WNA), and discuss aspects of our practice, both from a mathematical and a didactical perspective. The contribution of mathematicians is highlighted mainly through the approach used for establishing the foundations of WNA. Comments are finally presented on how mathematics educators can integrate in such a context a didactical component to the preparation of teachers of arithmetic.

Teacher education in Québec

We first briefly survey the evolution of our teacher education system over recent decades. We concentrate here on the province of Québec, education being in Canada a provincial responsibility.

From normal school to university education

Up to the 1970s, teacher education was offered in Québec through a network of normal schools originating from the mid-1800s. The 1960s witnessed a major reform of Québec's whole education system, from primary to post-secondary education, one of whose many outcomes was an increase in the expected level of qualification of teachers. The normal schools were consequently integrated into the university system in the early 1970s, thus ensuring a minimum of 16 years of schooling for teachers. It was also considered that being educated in a context propitious to research would benefit to the future schoolteachers.

As part of an important reform of the teacher education system, some twenty years later, it was stipulated that teacher education must take place within a *concurrent* programme. (See Tatto, Lerman and Novotná, (2009, p. 18) for comments on the notion of concurrent preparation of teachers, as opposed to a *consecutive* model.) The decision to become a schoolteacher is thus taken by the student upon university entrance.

Mathematicians' contribution to the programme of Université Laval

Most components of the teacher education programme at any Québec university are governed by parameters determined by the MEQ (2001). Yet there is still place for some flexibility, so that each university is able to organise with a flavour of its own its programme of study and the environment for the offering of the courses. At the time of the abolition of the normal schools system in the early 1970s, the Department of Mathematics and Statistics at Université Laval was invited to become directly involved in the education of primary school teachers. The Department then reacted positively to this totally new responsibility and this resulted in the creation of two mathematics courses (*Arithmetic* and *Geometry*) specifically designed with primary education in mind and still forming today the core of the mathematical preparation for prospective primary teachers at Laval.

A central aim of these two courses is to allow students to develop a robust and sound vision of the mathematics that they will be working with their pupils. We hope this way to foster their confidence in both their own math capabilities and critical analysis skills, and help them become autonomous in their mathematical (and pedagogical) judgments about primary school mathematics. We consider that prospective teachers need to develop a very specific mathematical expertise, as they are to become *the* experts of the mathematics connected to the education of primary school pupils. In that regard, we wish to present our students with opportunities for significant mathematical experiences, both in theoretical terms and in hands-on contexts. This approach can be seen as related to several research works, such as that of Grossman, Wilson and Shulman (1989), stressing the importance of a sound teachers' knowledge of mathematical content. It also has connections with the famous study of Ma (1999) concerning a "profound understanding of mathematics", as well as with the work of Ball and Bass (2003) about "mathematical knowledge for teaching".

The Laval programme also comprises three mathematics education courses (*didactique des mathématiques*, in French) offered by the Faculty of Education, for a total of five courses devoted to mathematics and its teaching. The presence of math education courses is naturally a standard pattern among universities in Québec or Canada. But the Laval model is somewhat unusual with regard to the role of mathematicians. Bednarz (2012, Tab. 1 and 2) presents a summary of the place given to mathematics courses in programmes for prospective primary teachers both in Canada and Québec. One can see how varied the possible arrangements are – from no math course at all, or one unspecified course but with no attention to the needs of future primary school teachers, to one or even two courses specially intended for primary teachers. In some cases the specific math course is offered by the Faculty of Education, so that in such a context primary teachers will have no direct contact with mathematicians.

Preparing mathematically for the teaching of WNA

A substantial part of the Arithmetic course taught at Université Laval aims at helping prospective teachers to become familiar with the foundations of WNA. When this course was created, in the 1970s, it was at first on a strongly set-theoretic vision that arithmetic was based: as was customary in those "New Math" days, concepts related to sets were considered as more primitive than those related to numbers. The spirit of the times is well captured by the following quotation, from a chapter on 'Primary Mathematics' in a report of an ICMI-supported workshop organised by UNESCO in 1971:

> All modern reformed programmes have introduced the study of sets into mathematical instruction. This topic is perhaps the most visible trait of an actual change in primary mathematics teaching. (…) The use of sets in (…) mathematical instruction varies greatly from one country to another. However there is a universal trend to use sets to develop the concept of cardinal or natural numbers, and the four rational operations on natural numbers. (UNESCO, 1973, pp. 5-6)

In the early days of our course, natural numbers were presented as cardinalities of finite sets, and arithmetic operations were defined in terms of set-theoretic operations. However, it was soon felt that while possibly appealing on psychological grounds or because of the fads of the time, this approach was far from optimal for many of our students. Are sets and their operations really to be considered as more "primitive" than numbers and basic arithmetic operations?

A concrete model for the whole numbers

A major shift occurred in our approach to basic arithmetic when it was decided to restrict sets to the role of a "linguistic" tool for communication, instead of primitive concepts on which the whole arithmetical building should be based. Natural numbers can be captured in a robust way by thinking of them as *counting* numbers, and this vision can be rendered concretely, in a written form, through the notion of a *tally* (or *stroke*). A natural number is "naturally" defined as a sequence of tallies – a *finite* sequence, of course, eventually empty. The set of natural numbers is thus the set comprising all the finite sequences of tallies. And this can be accepted as a working definition with prospective teachers.

Operations on whole numbers can then be introduced in terms of operations on sequences of tallies. For instance the addition of two numbers is defined as the juxtaposition of sequences of tallies (Fig. 1, for the sum of three and four). And multiplication is defined as the replacement of each tally of one sequence of tallies by replicas of the other sequence, eventually presenting the result as a rectangular array (or matrix) of tallies (Fig. 2, three times four).

□□□ □□□□

Fig. 1: Sum of three and four

□□□□
□□□□
□□□□

Fig. 2: Three times four

(It is understood that the two sequences in Fig. 1 are considered to be brought close one to the other so to form a single sequence of seven tallies, while the matrix of Fig. 2 is restructured so to become a sequence of twelve tallies.)

The fundamental notion of equality of two given natural numbers is captured through the fact that the corresponding sequences of tallies are identical, which can be rendered via the establishment, between the tallies forming these sequences, of a one-to-one correspondence – a most natural concept requiring no sophisticated set-theoretic support. This in turn allows to actually *prove* fundamental properties, such as the commutativity of addition: given two arbitrary sequences of tallies, say made of a and b tallies, one shows through a one-to-one correspondence that the order of juxtaposition does not matter.

A crucial point in such an approach to WNA, it should be stressed again, is that natural numbers are fully *defined*, and not simply accepted as a kind of *a priori* notion, v.g., a concept possibly emerging more or less out of the blue jointly with a certain representation scheme – somewhat sophisticated, one must admit:

$$0, 1, 2, \dots, 9, 10, 11, \dots, 20, 21, \dots, 99, 100, 101, \dots$$

(i.e., our usual base-ten positional value numeration system). In other words, we thus avoid the possible confusion between the nature (or essence) of natural numbers, and their actual representation via a numeration system – however important the latter may be in practice. Operations on numbers are just as well *defined* – not taken for granted –, and their properties *proved* – and not simply observed. Such an approach allows to really concentrate on the foundations of arithmetic in a way fully appropriate for sustaining reflections pertaining to primary education. We aim at offering students the opportunity for a personal reconstruction of elementary arithmetic. In our experience, such an approach can be instrumental in helping prospective teachers develop the necessary "conceptual understanding" so to perceive mathematics not as a mere bunch of facts to be memorised, but rather as a coordinated system of ideas. It contributes to the growth of autonomy and critical analysis skills mentioned earlier.

Not surprisingly the approach to natural numbers via tallies has a long history. Strokes or notches on bones, or marks on a wall, are even presented by Mainzer (1991, p. 9) as an "early stone age" vision of counting numbers – Ifrah (2000, p. 64) speaks more prudently of tally sticks as being "first used at least forty thousand years ago". These have obvious links to usual numeration systems, as shown for instance by Ifrah in his discussion of the origin of Roman or Etruscan numerals (2000, pp. 191-197), whose conclusion is that without any "possible doubt", they "derive directly from counting on tally sticks" (p. 196) – see also Ifrah's comments on the "Chinese scientific positional system" (pp. 278 *sqq*). While such marks clearly point to the development of written representations, other related early concretisations of numbers include beads or counters, eventually leading to the use of calculating instruments such as the abacus or the *suan pan* – see the section "From pebbles to abacus" in Ifrah (pp. 125-126).

Representing natural numbers as lists of strokes is also at the basis of the didactical reflection proposed by Wittmann (1975, p. 60), who in turn refers to the work of the logician Lorenzen (1955) as a source for such a constructive (or operative) foundation of natural numbers. This "unary" vision is indeed often encountered in various works pertaining to logic, for instance in Kleene (1952, p. 359) in relation to computability. In a spirit similar to tallies, Courant and Robbins (1947, pp. 2-3) base their study of the laws governing the arithmetic of whole numbers on the use of boxes of aligned dots. (The interested reader will find in Klein (1932, pp. 11-13) a detailed survey of ways of establishing the foundations of arithmetic, including through a set-theoretic approach.)

Mathematical thread for WNA

Space prevents us here from presenting in any detail how the concrete model of tallies is actually used in our Arithmetic course. Suffices it to say that after proposing our students a brief encounter with positional numeration systems in bases other than ten – a somewhat destabilising introduction into the course, but an excellent way of helping them "refresh" their WNA skills and reflect on basic algorithms which they know but not necessarily understand fully –, we move to a more general and theoretic level, where natural numbers and their operations are defined and the basic properties of WNA are proved, as indicated above.

From that point, the Arithmetic course continues in a double direction: introduction of other numerical sets, on the one hand, so to take care of "deficiencies" encountered in the arithmetic of natural numbers – these extensions of WNA can be construed in the spirit of the so-called *Principle of the permanence of equivalent forms* formulated by George Peacock (1791-1858); and seeing natural numbers "in action", so to say, through a problem-solving approach with an aim at exploring situations pertinent to primary education. Typical problems related to WNA that can be used in such a context are discussed for instance in Cassidy and Hodgson (1982) or Hodgson (2004).

Preparing didactically for the teaching of WNA

How could a math educator working with students after the Arithmetic course help them develop a didactical expertise truly anchored to the mathematical competence acquired with the mathematician? Building on the notion of tallies, we present in the following a few possible pathways.

First, it should be stressed that the tally model, which acts in the Arithmetic course as a *context* in the sense of Gravemeijer and Doorman (1999) – that is, an *anchoring point* for the reconstruction, by preservice primary school teachers, of WNA – is still available for the math educator's contribution. This approach can then be used again, either in order to revisit selected mathematical concepts encountered previously, or as a stimulus for launching a didactical reflection.

The experience with the tally model in the Arithmetic course provides students with a strong intuition for the potentialities of one robust and valid tool for the study of WNA. This experience can also serve as a solid basis when working in other contexts, and in particular with different concrete models. Moreover, as the context of tallies is familiar to students, it can be analysed and compared to other situations. This allows students to discuss its relevance when teaching WNA to primary school pupils, and judge its specific merits or limitations.

One of the specificities of the tally model that it may be pertinent to consider in a math education course is that it is neither a positional, nor a grouping, model. Most of the models to which primary teachers tend to turn in their teaching of arithmetic, in particular when considering operations on natural numbers, are positional models. Hence one may wonder what may attract a primary teacher to make use, even if only for a while, of the tally model in her teaching. Such a questioning appears fully pertinent for any prospective teacher, but it is particularly opportune in the Québec context, as our elementary school teachers are expected to bring their pupils to devise their own processes for mental and written computations on natural numbers, before teaching them recognised, standard, algorithms (MEQ, 2001). Grades 1 and 2 pupils, for example, are expected to develop their own mental and written processes to determine the sum or difference of two natural numbers, and should not be taught conventional processes until the beginning of grade 3 (MEQ, 2001, p. 151). Such a stance is certainly not new, nor is it specific to Québec (see for instance Madell, 1985). But it is well supported in the literature, as several studies (such as Fuson et al., 1997, Carpenter et al., 1998 or Gravemeijer and van Galen, 2003) have suggested that children develop a deeper and more flexible understanding of operations through the development of their own processes. These "personal" processes will have a real meaning for them, rather than being standard algorithms that get merely taught, or even "dropped from above".

Research in mathematics education suggests that primary school pupils, when placed in a context where they are lead to develop their own processes, will tend to use by themselves various counting strategies in order to create algorithms that do not (necessarily) rely on properties of our positional numeration system (Madell, 1985; Thompson, 1992). It thus seems highly appropriate to examine, in such a context, what a model like that based on tallies may bring to pupils.

Discussion and conclusion

Expectations towards primary school teachers regarding the teaching of mathematics are huge. Bringing pupils to devise their own processes for mental and written computations on natural numbers before teaching them standard algorithms is, in the case of the province of Québec, only one example of such expectations. As stated by Madell (1985, p. 22) thirty years ago, this type of approach does not make the teacher's job easy, as "teachers must not sit back idly and wait for discoveries"! In fact, teachers need to set up contexts in which their pupils will be able to devise their own processes, and they need to be ready to provide guidance when necessary. Moreover, understanding a child's proposed method of computation can in itself become a highly demanding task for the teacher, as this goes well beyond being merely able to perform each single step of the algorithm (that is, knowing *how* the algorithm works, and not really *why*). In order to be able to understand why a particular method does or does not work, teachers should be able to recognise the mathematical properties and other mathematical ideas used in the process. This specific expertise in turn points to the acquisition of the critical analysis skills that we mentioned above.

While such expectations towards primary school teachers do not make the teachers' job easy, they certainly do not make their preparation any easier as well! In such a context, we see both mathematicians and math educators as having specific and complementary roles to play in helping their students become autonomous in their mathematical and pedagogical judgments about primary school mathematics, and in particular about WNA.

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AN OPEN APPROACH INCORPORATING LESSON STUDY: AN INNOVATION FOR TEACHING WHOLE NUMBER ARITHMETIC

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Abstract

This paper reviews the traditional Thai approach to teaching mathematics and the unsatisfactory results that have thus far resulted from this approach. The author identifies a major cause for this poor performance as being the primary tool used by the teachers in Thai schools, the mathematics textbook, in particular the pedagogical approach that these textbooks follow. This analysis highlights the distinction between the Thai and Japanese mathematics textbooks, where the Japanese text takes a problem solving approach versus the Thai version's computational model. What follows is a proposal on how to adapt both the Japanese lesson study and the structured problem solving teaching approach to the Thai context, and then combine those two elements to form an innovative solution to the problem of teaching mathematics. This solution, the 'Open Approach Incorporating Lesson Study', was the basis for the classroom research that was undertaken. In this exemplar 1st grade students learned to gain meaningful understanding of whole number arithmetic via mathematics activities in the actual class taught by fifth year intern students during the mid-semester of 2013 and 2014 academic years at two lesson study project schools.

Key words: lesson study, mathematics textbook, open approach

Introduction

Teaching and learning whole number arithmetic in schools in Thailand is very crucial and yet has not improved for many decades. The National Test (NT) average score in Year 3 between 2010 to 2013 is 42.04/100 and Ordinary National Evaluation Test (O-NET) average scores in Year 6 between 2010 to 2013 is 41.24/100 (National Institute of Educational Testing Service [NIETS], 2013). This is consistent with international scores such as TIMSS and PISA and has not seen consistent improvement over the last ten years (Chiangkul, 2007; NIETS, 2013; Office of the Educational Council, 2012). The cause for these unsatisfactory results has been attributed, by some, to the teachers themselves and led to the demand for a reform of the teacher's standards in Thailand (Yamkasikorn, 2011). However, Stigler and Hiebert (1999) cautioned that rather than blame the teachers, it would be best to consider the tools that teachers are using in their classrooms. The primary tool of the mathematics teacher in Thai schools is the textbook. IEA results during the 1980s showed that more than 90% of Thai mathematics teachers use the textbook as a tool for teaching: they taught contents appeared in the textbook and let the students do exercises from those textbooks. Most of the exercises and the instruction guidelines in these textbooks still emphasise computation skills and techniques to accomplish those exercises in a short time (Anderson, Ryan and Shapiro, 1989).

The second tool to be considered is the pedagogical approach. Teaching mathematics in Thailand, for the most part, means preparing lesson plans, teaching those lesson plans in their closed classroom, checking the assigned homework, preparing quizzes, and prescribing exercises. To begin, each teacher starts by explaining any new content and giving examples, then allowing students to complete exercises and finally assigning homework. This can be summarised as demonstrating, questioning, describing and lecturing (Kaewdang, 2000; Khammani, 2005; Inprasitha, 2011) as shown in the Fig. 1. These kinds of activities have become a part of their own classroom culture and this is consistent with what Stigler and Hiebert (1999) mentioned, “Teaching is a cultural activity.”

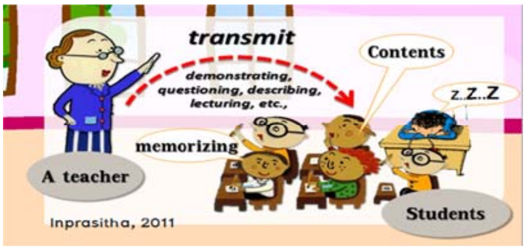

Fig. 1: A traditional teaching approach emerged in traditional classroom culture (Inprasitha, 2011)

Certainly, this kind of teaching methodology cannot respond to the new demand for knowledge and skills in the 21st century, which most countries around the globe are struggling to meet (Levy and Murnane, 2004).

These two elements, the textbook and the pedagogical approach, are the most significant factors that affect the teaching of mathematics in Thailand, and were central to the planning and implementation of this study. The results of which, demonstrate that by implementing an innovative approach to teaching and using contemporary materials, improvements in the depth of both student understanding and performance can be achieved.

Analysis of Mathematics Textbook

Inprasitha (1997) completed the analysis of elementary school level textbooks and found that: (1) the first part of each unit contains routine exercises for drilling computational skill, (2) word problems appear at the end of each unit, and these word problems are routine exercises, (3) most of the word problems in the first and second grades require the students to write symbolic sentences before solving the problem, (4) almost all of the routine exercises and word problems have one and only one correct answer and are strictly formatted. Here are some examples of contents and exercises appeared on the First Grade mathematics textbook.

From the table of content, it is noticed that the sequence of contents starting with teaching number 1 to 5, 0, 6 to 10 in chapters 1 and 2 and go to teach number addition in chapter 3. More details, at pages 1-5, guiding for the way to teach numbers is focusing on “writing” and “reading” “Thai and Hindu Arabic numeral.” In the exercises, it goes the same way it teaches mentioned before.

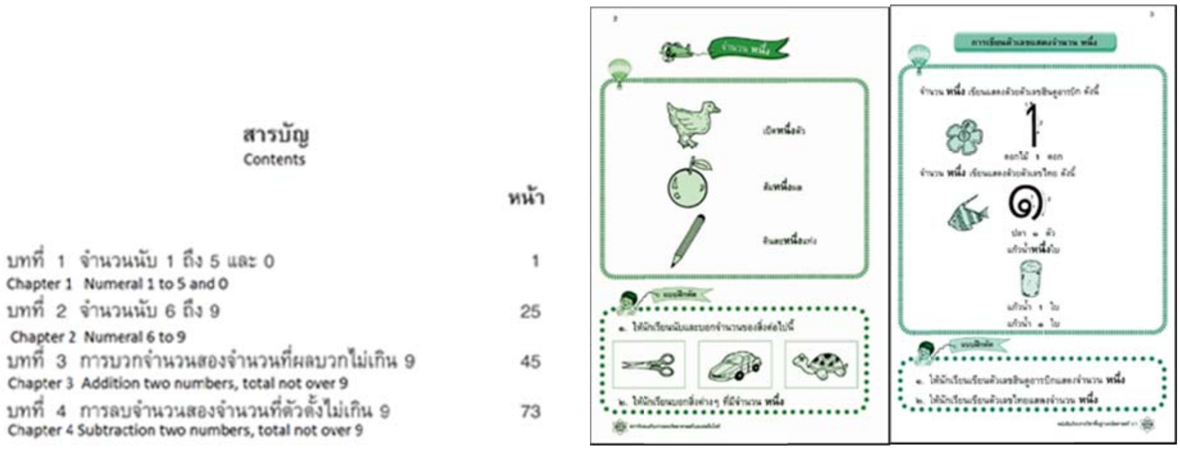

Fig. 2: Excerpted contents and exercises (Extracted from IPST (2008) textbook)

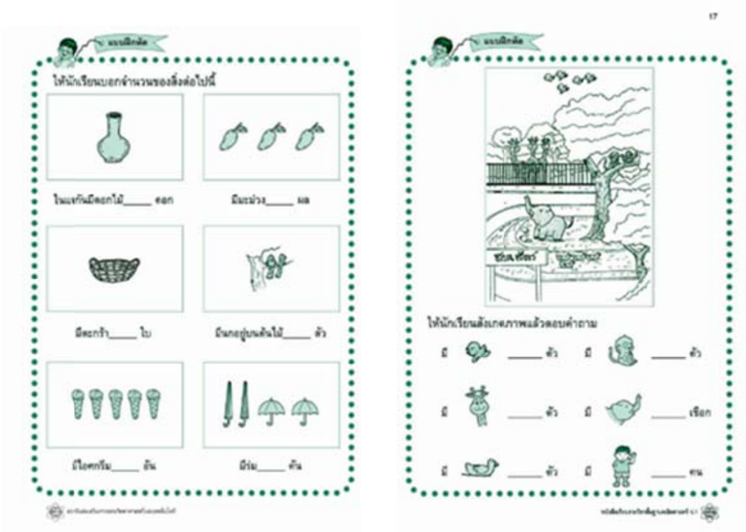

Fig. 3: Exercises for teaching 1, 2, 3, 4, 5 (Extracted from IPST (2008) textbook)

What missing in this textbook are “the multiple representations of numbers”, “concept of one-to-one correspondence”, “number senses”, especially in the situations from the real world, and composing/decomposing of numbers before entering addition.

In contrast, the project textbook (which is translated from a Japanese textbook) covers all of the content that is missing from the Thai version, as illustrated in Fig. 4. It starts with the table of contents categorising the alignment of mathematical domains by colour such as blue for number, pink for geometry, and green for measurement. It also illustrates the connection among mathematics domains using a bold arrow line. The textbook features highlighted colours for first grade kids and covers all of the important subject matters such as multiple representations for numbers (e.g., block, real object, pattern of base 5, and the digit 5 in Hindu Arabic). Furthermore, the concept of one-to-one correspondence appears on page 7.

Fig. 4: First grade Japanese textbook translated into Thai (Gakko Tosho, 2010)

An Open Approach incorporating Lesson Study: An Innovation for Teaching

Before entering the 21st century, there had been many attempts to shift the paradigm for teaching, especially the way to teach mathematics from an emphasis on teacher-cantered to students-cantered approach (Calkins and Light, 2008). In mathematics education, the development of the mathematics teaching approach has been centred on reconciling the following issues: new aspects of mathematics (Polya, 1954; Becker and Shimada, 1997), students' individual differences (Graff and Byrne, 2002), and problem solving as a teaching approach (Polya, 1954; Nohda, 1991; Becker and Shimada, 1997; NCTM 1980; Singapore MOE, 1990, Korea, 1997; Finland, 2004).

For example, during the 1970s, Japan developed a new teaching approach with the emphasis on students' mathematical thinking (Isoda, 2010). This new approach changed the focus from that of the correct answer, or closed problem, to that of teaching to assess the students' mathematical higher order thinking (Shimizu, 1999). An interesting point of this change in the Japanese history of mathematics education is that the teachers consider the assessment first. The problem, that remains, is how to overcome the individual differences especially thinking differences between students (Takahashi, 2006; Mizoguchi, 2008; Miyauchi, 2010).

The new teaching approach has revealed an important step in the teaching process. It was found that by using an open-ended problem at the start of the lesson, teachers were able to focus the student's thinking process on the task at hand. An important point of this teaching approach is all students have their own problems, or the problems are not a given, and this can make the students solve the problems by themselves and drive, or foster, the students to think by themselves (Brown and Walter, 2005).

From these ideas, Inprasitha et al. (2003) has been proposing a paradigm change in the Thai teaching approach from that which was mentioned in the early part of this paper to be an Open Approach incorporating in Lesson Study (Inprasitha, 2011).

An Exemplar of Teaching WNA by Using Open Approach

This exemplar illustrates how 1st grade students learned to gain an implicit understanding of whole number arithmetic via mathematical activities teaching through 4 steps of Open Approach incorporating in Lesson Study (see Fig. 5).

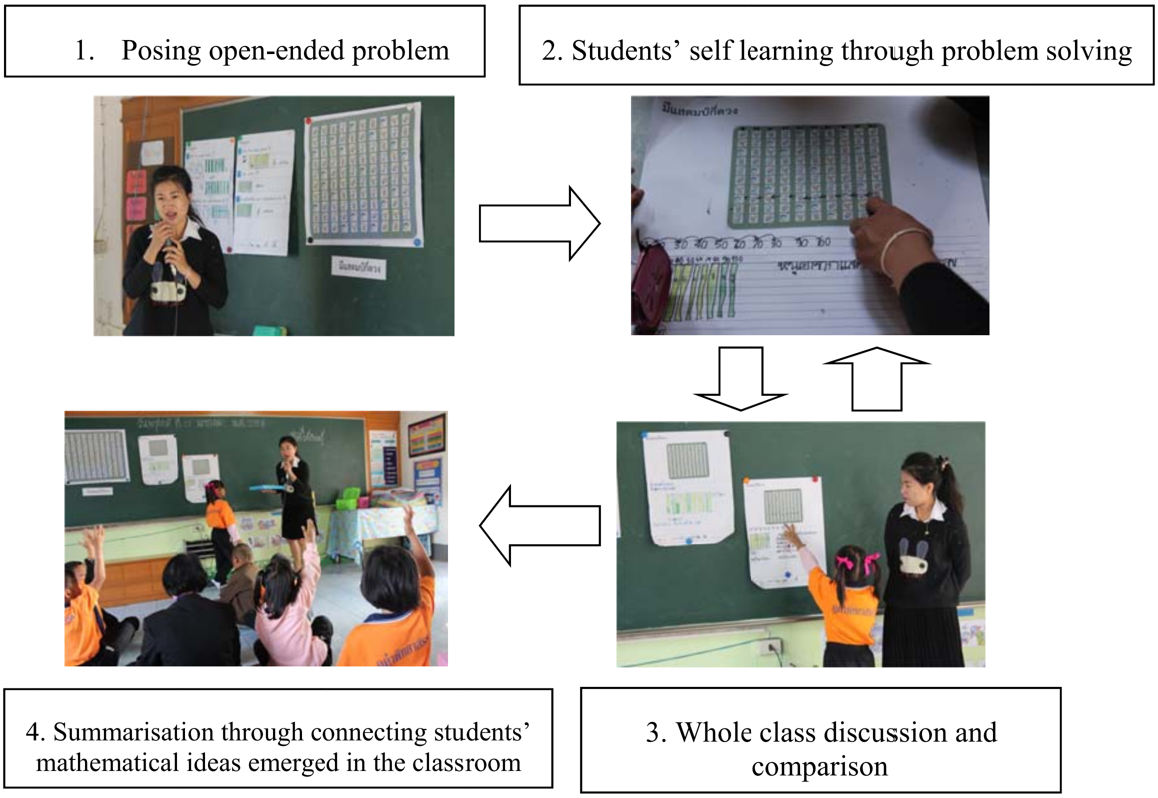

Fig. 5: Scenario of four steps of Open Approach as Teaching Approach

A learning unit was designed within the *"base ten and place values"* of the first grade textbook (Japanese Mathematics Textbook translated and edited in Thai by Inprasitha et al., 2010) emphasising on *how to learn* in order to support students' self-learning through problem solving.

The empirical data below were collected in 2013-2014 academic year from first grade students at Koo-kham Pittayasan School, the first project school in the Northeastern part of Thailand and Thongchai Wittaya School another project school in the Northern part of Thailand.

From Fig. 6, it is noticed that each student uses a decomposing and composing strategy through drawing diagram. After using the diagram the student also uses a bar graph to represent the number more abstract and closer to the meaning of place values, and eventually, they wrote two digit number representing their understanding on place values developed through sequence of diagrams.

Problem Situation 1: Let's make a math story for 7+8.

Fig. 6 a group of students showed a story that *"my father has 7 apples. My mother buys another 8 apples. How many apples are there altogether? They had 15 apples."* They also drew some diagrams and bar graph with two digit numbers.

Fig. 6: Students' ideas of problem situation 1

Problem Situation 2: How many stamps are there?
From the situation, students were asked for:

1) How to solve in many ways?
2) Write down the reasons and explain your solution.
3) Present your work or ideas to the whole class.

Fig. 7 showed students' written works *"I bring out each ten stamps and then order them, and they are increased by each ten. There are all 100 stamps.*"

10 base-ten blocks of ten, there are all 100 stamps.

There are 10 rows and each row has 10 stamps.

Fig. 7: Students' ideas of problem situation 2

From Fig. 7, it is noticed that each student has his/her own way to use the diagram (arrow) to count the number of total stamps by using group of tens. In Thailand, the daily language is consistent with the meaning of place value, like one ten (10), two tens (20) and so on. They also understand the relative of each ten and the total 100 stamps.

Concluding Remarks

After an attempt for a decade, Thailand as a developing country adapting some ideas from developed countries can overcome a long-lasting problem in teaching mathematics, especially the whole number arithmetic in elementary school level. Mathematics classroom has been changed from passive learning to active learning, where the students engaging in with more meaningful mathematics.

Acknowledgements

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DEVELOPMENT OF CONCEPTUAL UNDERSTANDING OF PLACE VALUE

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Abstract

In this article we establish a relationship between the concepts of place value and of bundling and unbundling. Based on results of a qualitative (N=255) and a quantitative study (N=51) with German primary school students we suggest a three-step instruction of these concepts that connects the various concepts and principles in a stringent way. The study has also shown that a large part of the tested students could be classified into groups that either think flexibly with respect to place value or show certain errors that can be traced to linguistic problems and problems that are induced by the material used for teaching.

Key words: flexible understanding, part-whole concept, place value, virtual manipulatives

Introduction

The understanding of place value and the ability to use place value in a flexible way is a necessary requirement for further learning of arithmetic, e.g. efficient arithmetic strategies or written algorithmic. But as place value is already required when exceeding the number range until 9, it is very important to ensure its understanding already in grade 1 and to build up on it in grade 2 and 3.

Hereinafter we are going to present the properties of our numeration system and how these connect to the acquisition of place value concepts. We describe how children develop a flexible understanding of place value and show how they deepen this understanding to a conceptual understanding by relating place value to the principle of bundling and unbundling. Also we show results of a qualitative and a quantitative study of 2nd- and 3rd-graders with respect to place value.

Theoretical Framework

The incomplete developed understanding of place value and the non-insight in the principle of bundling and unbundling states two main reasons of arithmetical weakness (Wartha and Schulz, 2011). Although place value and bundling are closely interrelated it is important for the didactical planning and analysis to distinguish them. Whereas bundling means that elements of a given amount are collected in groups of the same cardinality, place value means that the place (or position) of a digit in a given number provides information about the value of this digit.

In German-speaking countries a particular difficulty is the transposition of places in the oral form. A number such as 425 is spoken as “vierhundert fünfundzwanzig” – “four hundred five and twenty” in the literal translation. This creates an additional problem in the early teaching of whole numbers. Langermann (1912) already states that the parallelism between number system and number words leads to confusion, in particular for "those students who think the most".[6]

Ross (1989) states four properties of our numeration system: the *positional property* (1), the *base-ten property* (2), the *multiplicative property* (3) and the *additive property* (4). To achieve a conceptual knowledge about numbers – a knowledge that is rich in relations – children have to connect their knowledge about these properties to the principle of *bundling and unbundling* (5) that stands as a fifth property (Ladel and Kortenkamp, 2014a).

"The network of the concept of place value grows, if relationships are established to bundling and unbundling when adding or subtracting multi-digit numbers." (Gerster and Schultz, 2007, p. 30, translation by the authors): In the following we will establish the relationship between the concepts of place value to bundling and unbundling in the developmental process of children.

Children develop a general part-whole concept (I) already in early years (Resnick et al., 1991). They experience the additive composition of numbers that follows the additive property of our number system.

General part-whole concept: (I) $P_1 + P_2 + \ldots + P_k = W$	Additive property (4)

On the basis of this general part-whole concept a teacher can instruct children to create "special" parts. The fact that these special parts are multiples of powers of ten is a convention, due to our decimal numeration system (base-ten property). At an early stage this is also the point where the children learn to bundle and unbundle. They learn to switch between different kinds of bundling, e.g. 23 O = 2 T 3 O = 1 T 13 O, etc. The bundle units (O = Ones, T = Tens, H = Hundreds, …) are the multiples of powers of ten. In that way children can develop a decimal part-whole concept (II).

Decimal part-whole concept: (II) $n_k \bullet 10^k + n_{k-1} \bullet 10^{k-1} + \ldots n_o \bullet 10^0 = W$	Base-ten property (2)

The parts (summands), $n_i \bullet 10^i$, in (II) are multiples of powers of ten but n_i is not necessarily single digit. If we actually have a look at different kinds of symbolic number representations, there is only one of them that *needs* single digits for each bundle and hence the positional property to describe an amount in a definite way – the number in standard notation. All others (e.g. numbers indicated by

[6] Original text: „So kommt es, daß der gleichzeitige Gebrauch des Zahlwort- und Ziffernsystems die Schüler zu Anfang sehr oft zu Irrtümern verführt und gerade die am meisten, die am meisten denken“ (Langermann, 1912, p. 47).

bundle units, numerals, numbers in a place value chart) do not need the positional property because there are other indicators (e.g. the bundle unit, words like "-teen", the title column) that give information about the value of the number. In this regard it is only numbers in standard notation that require a continued bundling and hence single digit parts of multiples of powers of ten (III).

Standard decimal part-whole concept:	
(III) $n_k \cdot 10^k + n_{k-1} \cdot 10^{k-1} + \ldots n_o \cdot 10^0 = W$, and $n_i<10$ for all i	Base-ten property Continued bundling; single digit

A major problem is how to get from the principle of bundling to place value. This is the point where the children have to connect their knowledge about the properties of numbers to the principle of continued bundling. Digits in a number do not have different appearances like cubes or bars in base-ten blocks that are used to introduce bundling, neither they have different colours (e.g. following the Montessori-method). It is *only* the place that is decisive for their value.

If we follow the learning process of arithmetical content (Aebli, 1987) the children first of all have to experience place value in an inactive way. In connecting their knowledge about bundling with place value they have to experience that e.g. 1 ten is worth 10 ones, that means by changing the place of a digit or a token in the place value chart its amount has to be multiplied or divided by a multiple of powers of tens (multiplicative property).

While working with tokens in a place value chart there are different "behaviours" (Ladel and Kortenkamp, 2013a). Moving a token from the tens to the ones may mean that the value of this token changes. The children mostly experience this meaning of the action while working with physical material, as this is usually immutable. If they move a token from the tens to the ones, it is still one token but its value changed from a ten to a one and hence the value of the whole number changed its value by -10 + 1= -9. Another meaning of moving a token could be that the token is multiplied or divided by a multiple of powers of ten to keep the same value, thus: it is unbundled or bundled. In this regard place value is connected with bundling. Both meanings are important in understanding place value. To support the connection of bundling with place value the child has to experience the aforementioned multiplication or division. We suggest doing this in three steps (see Fig. 1).

In step one, the child is bundling and unbundling with base ten blocks and learns that there are ones and tens and that 10 ones have the same value as 1 ten. In step two, we introduce the place value chart with the bundling material in the title bar. The amount of ones and tens has to be illustrated by homogeneous counters (or tokens) like tally marks or points. The children learn that if the counters are homogeneous and they want to change the place they have to bundle or unbundle. In that way they connect bundling and place value by bundling in the place value chart. In step three, the children only move the counters and experience the bundling and unbundling by an automatic multiplication and division of the counters. This automation can only be provided by special virtual manipulative. One example of such a VM has been produced by the authors and is available at http://kortenkamps.net/placevalue.

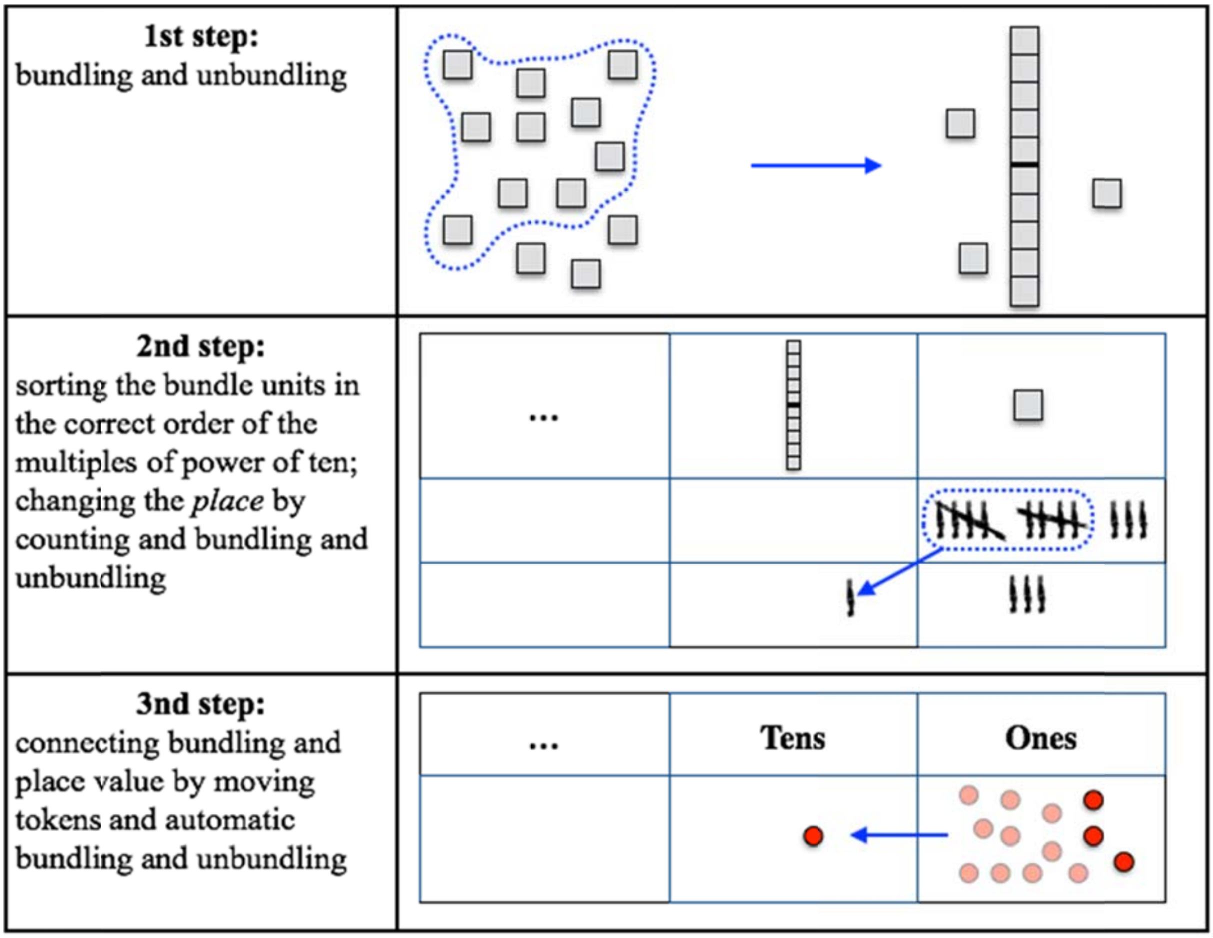

Fig. 1: Bundling and place value: The three steps

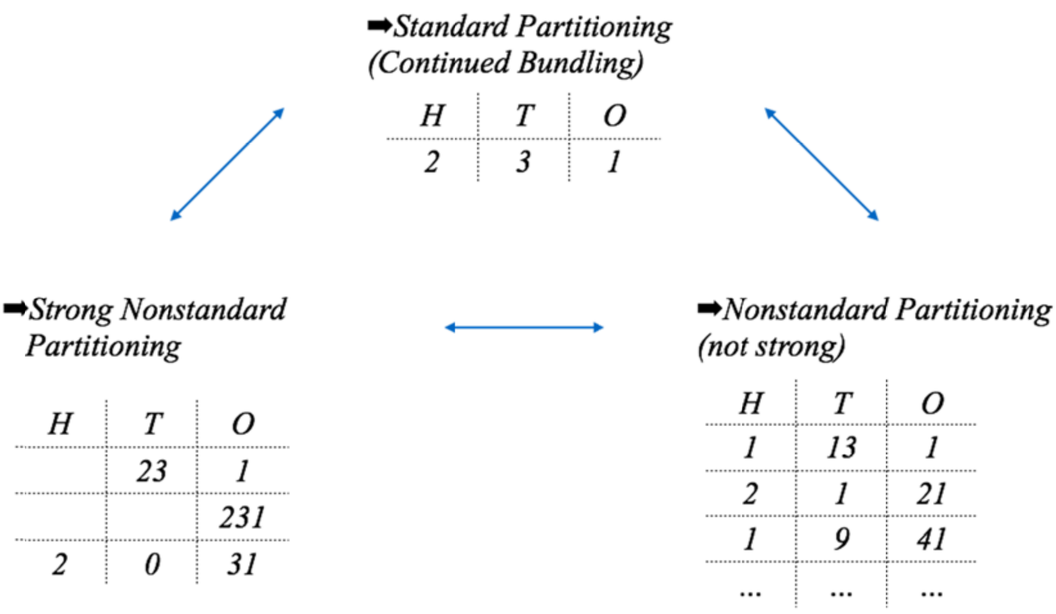

Fig. 2: Flexible decimal part-whole concept

We define the flexible understanding of place value as the ability to switch between different possibilities to split a whole in parts whereupon the parts are multiples of powers of ten (Ladel and Kortenkamp, 2014a). Therefore we have to distinguish between different kinds of partitioning (See Fig. 2). As we could already see in (II) and (III) there are standard and nonstandard partitionings. In the standard partitioning we have a continued bundling with single-digit multiples of powers of tens (e.g. 2H 3T 1O). In the nonstandard partitionings we distinguish again in *strong* and *not strong*. In the not strong nonstandard partitioning it might be necessary to perform bundling and to add bundles in order to get to the standard partitioning (e.g. 1H 13T 1O = (1+1)H 3T 1O), whereas the strong nonstandard partitioning does not need an additional addition (e.g. 23T 1O).

Materials and Methods

For our study we refer to Artefact-Centred Activity Theory (ACAT) (Ladel and Kortenkamp, 2013b). We focus in the following on the main axis of interaction that follows the subject-artefact-object line. The subject, that is the student, externalises its concepts regarding an object, here place value, via an artefact. The artefact itself externalises the object through a suitable representation and visualisation. Through manipulating the artefact, the student can *experience* place value mediated by it. What place value exactly means depends on what we want to teach. In that way we put our knowledge in the artefact, we design it in that way that the children can internalise the knowledge we want them to learn by using it.

The virtual manipulative that was used in our research has been described in (Ladel and Kortenkamp, 2013a) and other places. We are using an iPad-App that enables children to place tokens and to move them between places. Simultaneously the token counts are displayed in the title bar. When moving a token between places, the unbundling and bundling as described above are carried out automatically.

Results

Both a quantitative and a qualitative study have been carried out. In the quantitative study 255 students in grade 3 (age 8-9) from Halle (Saale) and Saarbrücken, Germany, as well as from Luxembourg took part.

The test instrument (Ladel and Kortenkamp, 2014a) consisted of three parts, each to be administered in 10 minutes. In the first part students had to compare numbers given in value-unit notation, for example 2H 5T 3O and 2H 4T 17O. They also had to translate value-unit notation into standard notation, for example 3O 2H 5T into 253. The value-unit notations in both tasks were using non-standard and non-strong partitionings, and the places were not guaranteed to be in the correct order (highest bundle first).

In the second part, children were asked to translate images of tokens in a place value chart into standard notation of numbers and vice-versa. For creating the charts, we always asked for several representations (if possible) of the same number.

The third part was similar to the first part, but using other numbers.

The analysis of students' answers showed that a significant amount of students was not able to solve these problems correctly. We concentrated the analysis on the translation from a standard number into a place value chart picture in the second part. This showed that only 56 students (22%) were able to give several correct representations, 31 students (12%) were showing flexible interpretations but had minor errors (usually wrong counting). 19 students (7%) made mistakes that can be explained by working with base-ten material in another way than the 3-step instruction suggested above. Besides students who were using other symbols (24 = 9%), only one representation (16 = 6%), other arrangements or value changing representations (5 + 4 = 4%), the largest group (69 = 27%) were those who just permutated the tokens in the representation – i.e., they just disregarded place value completely.[7]

While the data suggests that an extensive use of the digital artefact (half of the students had access to it) could help students to better understand the connection between bundling/debundling and place value, further research is necessary to support this finding. In particular, the 3-step programme should be tested over the full introduction of whole numbers in primary school or early mathematics learning. A deeper analysis using Statistical Implicative Analysis of the data is available in (Ladel and Kortenkamp, 2014a).

The results of the quantitative study (see also Ladel and Kortenkamp, 2014b) are in line with the interviews in the qualitative study. Here, we arranged a manual-based interview with 52 children at the end of grade 2 (age 7-8). Children were again asked to compare numbers given in value-unit notation. Also, we asked how many tokens they need to represent 35 in a T/O place value chart. They were given tokens and the virtual manipulative to demonstrate their findings.

The qualitative study revealed four types of typical errors that were used to design the test instrument of the quantitative study. In particular, when comparing numbers in value-unit notation (1) Children just copied the individual digits of numbers from left to right just omitting the bundle units; (2) Children decided that the number of largest bundles is also guiding which number is larger, not taking into account carry overs from smaller bundles – which is only correct in standard partitionings; (3) Children did not compose bundles into a

[7] The remaining 26 answers were unclassifiable.

common number, but took each bundle as a separate number; (4) Children only used the bundle with the largest quantity as indicator for the largest number.

Two other results are connected to base-ten material: When asked to show 35 in the T/O place value chart, they placed 30 (thirty) tokens in the tens column and 5 in the ones column, which actually represents 305. After questioning this, the students confirmed that it does not matter whether there are 30 or 3 tokens in that column – which is an obvious error (and problem) that could be identified again in the quantitative study. Similarly, students claimed that it is necessary to use blue tokens for the ones and red ones for the tens, and only those who already understood the role of tokens were able to abstract from this. Using the virtual place value chart later in the interview, students could experience a value preserving movement of tokens between different places. After this experience, students were questioning their initial findings and came up with a more flexible understanding of place value.

Discussion and conclusion

Place value is a significant problem for German students. In our study we could identify major misconceptions, in particular a missing sense for the importance of places. For spoken numbers, this is not a problem, as the bundling units are integrated into the number words – "-tausend" (thousand), "-hundert" (hundred), "-zig" (corresponds to the English -ty suffix for indicating tens) indicate the value of digits, and the bundle units are not in the usual order of written numbers: 325 is denoted by the German word "dreihundertfünfundzwanzig" (3H 5O 2T).

In our research we could identify typical mistakes and also indicators for a flexible understanding of place value. Also, the quantitative study showed that students having such a flexible understanding made less errors in the other parts of the test instrument. Using a virtual manipulative we have a chance of teaching such a flexible understanding.

In our further research we will investigate whether the three-step programmeme presented above will support students in connecting the concepts of place value and bundling / unbundling and thus help them to think flexibly with respect to representations in a place value chart. Also, we are currently doing further video studies that should help us to further connect the results of the quantitative and the qualitative study.

Finally, a comparison to students in a different cultural setting, in particular one with a different number word system that does not contain the intrinsic difficulties of the German system, would be helpful to identify the cause of the rather high number of failing students in our test. As the exercise test instrument is not language-specific, this could be done easily given a test group of primary school students.

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THE EFFICIENCY OF PRIMARY LEVEL MATHEMATICS TEACHING IN FRENCH-SPEAKING COUNTRIES: A SYNTHESIS

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Abstract

This paper presents the generic results from a report on the efficiency of primary level mathematics teaching in French-speaking countries. The World Bank wanted to know the details of certain classroom practices in these countries, and their efficiency with regard to specific profiles of pupils. However, there are no large-scale empirical studies providing evidence-based results on this question in the French-speaking world. Education and educational systems tend to be more subjects of politics and opinion. We seek to provide some generic results as regards the way educational systems work, as well as some specific results in various French-speaking countries, so as to understand how teachers and pupils interact within the framework of these systems of knowledge transmission.

Key words: classroom practices, conditions of efficiency in educational systems, evidence based good practices, French-speaking countries, primary level mathematics teaching, specific results (countries and populations) on the effectiveness of mathematics teaching

Introduction

The relative efficiency of modern teaching systems have made them almost universal and have led to them becoming the standard means for the transmission of all knowledge, whether it be theoretical, technical or technological. They are the standard of reference in most cultures when trying to transmit knowledge to many sectors of a population, especially when these cultures establish a written description of formal education to transmit knowledge from one person to another (Chevallard and Mercier, 1987).

But the reasons for the efficiency of our systems of education are little understood, as are the reasons for their ineffectiveness for certain kinds of knowledge and skills; we still have little understanding of the influence of possible factors involved in this ineffectiveness, such as certain educational techniques, a certain kind of public, or certain social groups. This renders interventions aiming at their improvement a delicate operation. Teaching systems and their *social ecology* have not been the object of precise research and so remain relatively closed systems on which individual decisions have had no impact: "The education in use in a given society at a precise moment of its evolution, is a set of practices, of ways of doing, and of customs […] They are not, as we believed for a long time, more or less arbitrary and artificial contrivances […] It is vain to believe that we bring up our children as we wish. *We are forced to follow the rules which are reigning in the social background where we live.*" (Durkheim, 1911). Durkheim, at the time when he invented sociology and produced a sociology of educational knowledge, asserted some principles expressing the conditions of efficiency for a system of education.

In the same manner, we try to understand the conditions of efficiency of diverse systems of education, these conditions being considered as *normal* for the societies in which they are established. We then study them as social systems manifesting one or several specific techniques for a generic task, in the same way that systems for the transportation of goods or management systems of retirement pensions vary according to countries and the social groups in these countries. These systems depend not only on rational decisions but also on a complex network of constraints, which determine various types of teaching situations (Brousseau, 1997, Dewey, 1958). Given the social ecology of each specific observed situation, we try to give a synthesis of specific situations where knowledge is transmitted when analysing each case individually. During the term of this investigation we realised that there are no systematic works of experimental design on classroom practices and their efficiency relative to a specified public, and hence no studies producing statistical results of empirical observation. However, we were able to identify some generic results from our research. The functioning principles of educational systems are the main results of research in the *didactics of mathematics* and in the *sociology of education* in the French-speaking world. The conditions for efficiency of this functioning have been confirmed by the results of doctoral works, which are mainly concerned with the precise study of a case described in detail, or the identification of a phenomenon and its influence (see for example Matheron, 2010).

We include within our analyses other fields of existing works, for example that of social disparities with a synthesis on the countries of sub-Saharan Africa (Mingat, 2006); this study made an outstandingly precise analysis of the variables of the problem and the possible political factors involved. It identified peculiarities in the distribution of the internal disparities in a country (according to the genre or the family capital), which seemed to concern both the Sahelian and French-speaking countries. The details are described more exactly in larger publications and sometimes more precisely, as in the French case (Duru-Bellat and Mingat, 1994). It is from classrooms' educational practices that certain questions arise, and for which we will try to find some answers.

The historical context of the educational system in France

Our analysis begins with the historical context of teaching practice in France, which allows us to understand some of the inertia of a system which has left in place structures of thought and action from the past. At the very beginning of the XXth century, education in France had two areas of interest and two systems. Free compulsory education in the "Primary" system and a parallel paying system which catered for the privileged children of big cities, and which led to the entrance examination for university. The latter system was that of "republican elitism" and the primary school was the school of the hard-working people, labourer and worker.

The unification of both systems began in 1959 and ended in 1974. Public primary education became "elementary", whereas public secondary education split into "Middle school" for everybody and schooling became compulsory up to 16 years-old. Nowadays, schools strictly separate different ages into different levels, which hardens the system and leads to low-performing pupils repeating (Prost, 1982). This characteristic of the French system in France and elsewhere in the French-speaking systems of education, (see for example the Report PASEC on Senegal), is that it has resisted the test of time so that, the French “Republican school” has remained an elitist one. The idea of “general access to the best education" produced a "preparatory school for higher education". African French-speaking countries seem to have inherited the problem. This is the means by which these educational systems reproduce social inequality.

Is there any evidence of the effectiveness of mathematics teaching in French-speaking countries?

What are the conditions of validity to enable better decision-making which would direct the system of education toward greater efficiency?

Such are the elements that we will consider, first in known works which deal with the conditions of the production of announced results; such works are readily available and can therefore be submitted for critical discussion. From this analysis we can then present results that could help in decision-making. We are aware that decisions regarding teaching systems are a matter of "moral science"; a science for judgment and action, close to political action, and the related decisions are too often established on the basis of sociology.

From this approach, it will be possible to give some proof of efficiency regarding traditional transmission techniques of elementary mathematical knowledge used by many teachers. We take into account where possible, existing works or their absence, by referring to social situations; we also have to identify the social balances of which every technique is a product; from this we will be able to discern if it is possible to envisage changes, and thus have some possibility of imagining the expected effects, including what will be likely to resist time such as local practices.

What proofs should we retain from classes in French-speaking countries throughout the world, which link the performance in mathematics of pupils in primary classes to the teaching practices in the classroom?

We have thus taken these punctual results as local estimations of the effectiveness of practices in observed classes, whether these estimations be that of a teacher or a researcher having proposed a teaching situation. We have then taken into account the variables that our four “principles” describe, and in-keeping with a style of reasoning familiar in physics, we have considered punctual observations as manifestations of a particular value of these four major variables, the conjunction of which is used as a generic model of teaching systems.

However, we recognise that a social system, in this case for learning, does not function in the same way for all those whose behaviour it organises; we consider social groups as ecological environments for teaching practices; niches which may allow teaching practices to exist, depending on whether they are successful or not. This is how we have interpreted work which includes as contextual elements the family environment, or social and cultural environments where pupils were raised (Mercier and Buty, 2004).

We thus claim to have identified for our generic results, certain styles of classroom practice, as well as some external conditions which determine the success or failure of learning and are the specific conditions of existence of these styles of classroom practice.

Generic and specific results

We shall first present with regard to interactions between teachers and pupils in mathematics classes, the principles which are our proof, that is to say, results which are considered as proven, and which we believe were produced by didactic research in French-speaking regions.

Some functioning principles of education

Principle number 1 states an economic aspect of teaching systems and statements 1.n specify a number of consequences. Principle 2 states a given property of teaching systems and statements 2.n provide a number of consequences relative to the learning properties that these systems produce. Principle 3 states a constraint true of all teaching, which is related to the action of pupils, and statements 3.n provide a number of consequences. Principle 4 states the results of what can be termed "the epistemological position" in teaching research, which is termed didactics in France. The consequences concern the curriculum and will not be treated here.

Principle 1: The efficiency of a modern teaching system is linked to a rationalisation of the teaching of knowledge, which is thus converted into a school discipline and presented according to an established order.

Principle 1.1: Pupils learn not only what is taught explicitly, but also in the majority of cases, what is useful in order to learn this teaching.

Principle 1.2: Study is the link between teaching and learning

Principle 1.3: The teacher begins at the beginning

Principle 1.4: The beginning of a taught lesson has meaning in a society where teaching is organised; this fact should not therefore be considered as the teacher's choice.

Principle 1.5: In practice, the teacher organises the pupils' progress such that they are not expected to know something they have not been taught, and this may be explicitly agreed with them.

Principle 1.6: What is taught indicates and determines what must be learnt, in so far as the necessary conditions to study it, at school or beyond, are present.

Principle 2: The teacher instructs a body of individuals who study as a group; this group is formed by his or her action, as well as the pre-existing social disposition of the pupils toward study.

Principle 2.1: The relationship of the pupils with the knowledge taught cannot be separated from what is learnt as a result of the teaching situation which created the need for that knowledge.

Principle 2.2: The relationship of the pupils with the knowledge taught cannot be separated from situations where they will experience being tested on their learning, and in which it will be necessary to act with the help of their learning.

Principle 3: The different ways of organising study, which the pupils manage together or individually, determines the kinds of knowledge that they can learn.

Principle 3.1: It is not possible to teach explicitly everything that is useful and that must be learnt.

Principle 3.2: Pupils learn independently when the teacher is indicating problems to be resolved during the teaching.-

Principle 3.3: Efficient teachers organise the learning conditions of what needs to be understood implicitly to be able to use the knowledge explicitly taught; to do that the teacher must be aware of their existence and control the conditions necessary to their existence.

Principle 4: Mathematics is a collection of practices mustering symbolic tools in both graphic and written form, which come into existence thanks to gestural and language practices that indicate their meaning and enable them to be controlled.

These principles are at work in every classroom but the way they are expressed is specific to each classroom.

Four different results depending on the country or the citizens in question

Repeating a year

We have compared the results of sociological studies in various countries and cultures, and in this way indicated the importance of the question of repeating a year; it is practised with very young pupils in French-speaking countries where it appears to be linked to time-consuming revision practices. In our opinion, these practices are indicative of a bureaucratic management of the mass of pupils and the teaching of knowledge. Every year it discourages a substantial number of pupils who leave the school system prematurely. It also leads to a significant number of illiterate adults.

Change presupposes abandoning the ideology of republican elitism inherited from the colonial era, as it is no longer what is needed by modern nations.

Chorus learning

In countries where pre-elementary school is religious (koranic), learning is first experienced as learning in a group in chorus. Reading then follows, well before the written and graphic representation activities which enable the autonomous production of texts and graphic work leading to symbolic calculation. Results in mathematics in these countries, particularly in the Sahel region, are significantly lower than in other countries such as Madagascar. This corresponds to our fourth principle which states that mathematical activity is symbolic and fundamentally a form of writing which should therefore be given precedence at the beginning of schooling.

Symbolic and concrete

Language statements accompany symbolic work. In mathematics this phenomena is described with the term “modelling”, and it can give a direction to teaching right from the pre-elementary level; it should be carried out alongside representation practices and the measurement of objects, as well as spatial sense. This is another aspect of the fourth principle.

It appears that the more a society accepts these ideas, the more teachers successfully develop them in the classroom, and this then increases their level of efficiency.

This is because in these conditions, teachers can deal with problems in class that make sense to pupils’ families; they can enable the pupils to invent systems of representation, which they can then appropriate as symbolic objects. The pupils learn that mathematics is a means to find answers to numerous important questions, which is why it is taught to all pupils in a universal educational system.

Positive discrimination

All available studies, either sociological or didactic, show the ineffectiveness of policies of positive discrimination, or even their tendency to be counter-productive. It is possible to make a comparison with observations of the ineffectiveness of policies in the field of special educational needs, which nevertheless work in English-speaking cultures. We should therefore consider this fact as a once again poorly explained consequence of the bureaucratic organisation of teaching in French-speaking countries, which does not enable the smooth management of certain pupils’ progress.

More results

There are two essential questions with regard to the curriculum in French-speaking countries.

The first concerns the decline of work on measuring practices and the use of the decimal system of measurement. Digital resources have rendered such practices obsolete whereas they had been the foundation of the study of decimal numbers, as a filter for rational numbers and a precursor to real numbers. Traditional instruments systematically offered the possibility of reading graduations and sub-divisions and transformed any measure into an evaluation of a length, which requires mobilizing the units of the metric system and interpreting the positional numeral system (Chambris, 2008).

Finally, it is worth noting the considerable time required to learn the uselessly complex names of numbers. International studies show that the high level of success at the beginning of the learning process by pupils from Asian cultures is due to the rationality of the Asian designation of numbers. Any form of simplification on this question would lead to decisive progress for those societies that adopted it, especially when the average length of time in school is five years.

Discussion and conclusion

However we should note that in France, a new style of research in mathematics teaching has emerged with the PIREF, the Incentive Programme of Research in Education in Training. Inter-disciplinary experiments (didactics, sociology, and psychology) on the first year of primary school teaching of mathematics have shown the different levels of efficiency of some teachers with regard to mathematics or reading, which directs attention to their personal relationship with these subjects (Sensevy et al., 2013). They are a reminder of the work of Shulman's students, and in particular those of Ma on pedagogical content knowledge.

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HOW THE CHINESE METHODS PRODUCE ARITHMETIC PROFICIENCY IN CHILDREN

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Abstract

This paper explicates the several features of mathematics education in early years in mainland China that are considered to have contributed to the arithmetic proficiency in Chinese children. Of particular significance are the features of elementary school curriculum, textbooks and classroom instruction, and the cultural values on learning mathematics. These work as a social-cultural system, elements that are instrumental in shaping the achievement of mathematics learning in Chinese children. The content of this paper is developed based on the three previous works by Ni and her colleagues.

Key words: arithmetic, Chinese mathematics education, two-basics

Introduction

Compared to young children in other countries, Chinese young children show impressive achievement in the basic arithmetic, such as in generating cardinal and ordinal number names (Miller al., 2000), understanding of the base-10 system and concept of place value (Fuson and Kwon, 1991; Fuson and Li, 2009), using decompositions as the primary backup strategy to solve simple addition problems (Geary et al., 1993; 1996), and an earlier emergence of linear representation of number magnitude (Siegler and Mu, 2008). The Chinese children's achievement in the basic arithmetic led researchers and educators to have inquired about its contributing factors. One interpretation was attributed to a greater regularity in Chinese number naming system between 11 and 20 and also between 10 and 100. For example, numbers between 11 and 20 are formed by compounding the "tens word" with the "unit word". Thus, the numbers 11 and 12 are spoken as "ten-one" and "ten-two", while 20 is spoken as "two-tens" and 62 is spoken as "six-tens-two." The consistency of the Chinese number naming system with a base-ten system has been hypothesised to assist children in doing well on tasks relevant to base-10 values, such as counting skills and place-value competence (Miura et al., 1994). However, this interpretation has been difficult because none of these studies controlled for cultural or family processes (e.g., parental expectation, parental assistance, and preschool education) as confounding variables that could also influence children's numerical development. On the other hand, several studies have demonstrated that adult instruction mediates the role of the linguistic feature of such a number naming system (Saxton and Towse, 1998; Othman, 2004).

In the following, this paper will show how the cultural factors, including elementary curriculum objectives, textbooks and classroom instruction, and the cultural values put on education and mathematics are instrumental in shaping the arithmetic proficiency in Chinese children.

The Emphasis of "Two-basics" in Elementary School Mathematics

The Chinese mathematics curriculum, which is nationally mandated, has evolved from merely focusing on "the two basics" (basic mathematic concepts and basic mathematic skills) to emphasising mathematic problem solving built on the two basics (Liu and Sun, 2002; Ministry of Education, 2001; Ni et al., 2011). Nevertheless, throughout the evolution of the curriculum, the *two-basics* have remained as the foundation of Chinese school mathematics (Zhang, Li and Tang, 2004). The two-basics approach stresses the importance of acquisition of foundational knowledge, content and skills. Consequently, arithmetic is the most significant part of the Chinese elementary mathematics curriculum because arithmetic constitutes the foundation of the definition system derived from the concept of unit (L. Ma, 2013; X. Ma, 1996). First-grade students are required to mentally perform addition and subtraction using numbers up to and including one hundred (Ministry of Education, 2001). This is achieved by the way that Chinese elementary school mathematics is organised. Ma (2013) characterises the Chinese elementary mathematics with "core-subject structure" in contrast to the US elementary mathematics with "strands structure." Whole numbers and fractions are the core body of the Chinese elementary mathematics curriculum whereas the US curriculum containing many parallel strands (e.g., number and operations, problem solving, measurement, data analysis and probability, etc., NCTM, 2000) juxtaposed but may not being well connected. The emphasis of foundational mathematics in the Chinese elementary mathematics curriculum is also reflected in the recursive way to organise the instruction on whole numbers and operations. For example, the first 30 weeks of mathematics instruction focuses only on arithmetic with the first 10 numerals and their addition and subtraction, followed with addition and subtraction within the first 20 numerals, progressing to addition and subtraction with regrouping between numbers 20 and 100. Along the sequence of the three sections from 0-10 to 11-20 and then to 20-100, the key concepts, such as "the composition of ten," "place value," and "composing and decomposing a higher value unit" are recursively illustrated and repeated practiced, laying a solid foundation for later mathematics learning (Ma, 1999; 2013).

Teaching Materials and Classroom Instruction with High Cognitive Demand

Curriculum materials, especially textbooks and the corresponding teaching manuals are the most important vehicles used to implement the nationally mandated curriculum in the mainland China. The development and publishing of textbooks is closely regulated and monitored by the central government, the Ministry of Education. In the country, there are only a few officially designated publishers who are allowed to develop textbooks and teaching manuals.

Comparative studies of mathematics textbooks indicated that there were high cognitive demands made of students using Chinese textbooks and that there was a link of the cognitive demands to the mathematics achievement of Chinese students (Ding and Li, 2010; Li et al., 2008). One indication of the high cognitive demand is the emphasis on the acquisition and use of mathematical language as the development of mathematical proficiency is inextricably dependent on the mastery of the language for mathematics.

Li et al. (2008) showed that the US teacher preparation textbooks treated the “=” sign and its implication for primary school students as cursory. The expressions to explain the “=” sign often do not focus on equality as symbolising a relation. On the contrary, Chinese mathematics method textbooks and student textbooks highlight equality as a relation by introducing the equal sign in a context of relationships and interpreting the sign as “balance,” “sameness,” or “equivalence.” The equal sign often appears simultaneously with the sign “>” and “’<” to highlight the equal sign representing a relation. The Chinese texts suggest teachers use the one-to-one correspondence concept and procedure to assist students in better understanding of the equal, greater, and less than symbols. The studies (Ding and Li, 2010; Li et al., 2008) indicate that differences in the curricular and pedagogical treatments reflected in the curriculum materials can be a source of the disparity between the Chinese students and the US students’ understanding of equality as a relation.

In addition to the textbooks, teaching manuals are also key tools for Chinese teachers, assisting them to learn about subject matter as well as ways of teaching. In fact, the Chinese teaching manuals are often very useful for teachers because they specify the objectives of teaching and explicitly identify what is “important” and “difficult” in each teaching unit. Specific teaching suggestions are provided for each lesson (Li, 2004). The important points of curriculum content are not only significant for students to learn and master, but also important for teachers, enabling them to develop a profound understanding of mathematics for teaching. For example, for teaching first grade students about subtraction with word problems such as “There are 10 boys and 7 girls in a class. How many more boys are there than the number of girls?”, the teacher manual advises the importance of teaching children to use the one-to-one correspondence principle to compare two quantities. The manual stresses the importance of being able to address and overcome the possible misconception in students to think 10 minus 7 for the word problem as removing seven girls from the group of ten boys.

Such teaching manuals have probably contributed to Chinese mathematics teachers to develop a tendency to follow accurate mathematical expressions in classroom instruction based on their understanding of mathematics knowledge for teaching. In teaching subtraction with regrouping, the majority of the Chinese teachers interviewed in Ma’s study (1999) described the borrowing step in the algorithm as “a process decomposing a unit of higher value instead of saying ‘you borrow 1 ten from the tens place’” (p.8), as advised by the teacher manual on instruction. One third-grade teacher explained why she thought the expression of “decomposing a unit of higher value” was conceptually accurate:

"…. 'Borrowing' can't explain why you can take 10 to the ones place. But 'decomposing' can." (Ma, 1999, p. 9)

Chinese teachers pay close attention to details and expressions of students' responses to answering mathematics questions. Cai (2004) analysed U.S. and Chinese teachers' scoring of student responses. One set of student responses was to the following question: One step needs one block, two steps needs three blocks, three steps needs 6 blocks, four steps need 10 blocks….How many blocks are needed to build a 20-step staircase? One student response was shown as this: 1 + 2 = 3 + 3 = 6 + 4 = 10 + 5 = 15 + 6 = 21 + 7 = 28 + 8 = 36 + 9 ….190 + 20 = 210. Over 60% of the US teachers gave the response 4 points (the highest point) and about 30% of them assigned it 3 points. In contrast, about a third of the Chinese teachers rated it 0 or 1 point and nearly half gave 2 or 3 points. Most of the Chinese teachers were intolerant of the errors. They commented that the answer was correct but two sides of an equal sign should be equal. Student errors such as this regarding the equal sign as "do something" for an answer probably contributes to the late difficulty that students have when learning algebra by treating an algebraic equation not as indicating a mathematical relation but as indicating "do something" for an answer. Therefore, the conceptual acquisition of mathematical representation conventions does not merely mean applying labels to what one perceives. Learning conventions may act as the catalyst for some conceptual changes (Lehrer and Lesh, 2003). It has been shown that the acquisition of the counting conventions may contribute to children's understanding of cardinality and ordinality of numbers. Similarly, the acquisition of the equal sign may positively influence the late learning of algebra in Chinese students.

Cultural Values on Learning Mathematics

The Chinese way of teaching and learning arithmetic to children is also supported by its cultural-social contexts (Ni, 2012; Ni, Chiu and Cheng, 2010). Among others, Chinese mothers believe that their children should master both skills before first grade in order to support academic success in early childhood education whereas US mothers value literacy skills more than mathematics skills in early childhood education, (Hatano, 1990; Stevenson and Lee et al., 1990). In addition, Chinese parents believe that children's trajectories in mathematics achievement are already established early in preschool and tend to persist in elementary school; thus, they think that preschool children who lag behind their peers in mathematics performance tend to fall further behind in elementary school. Hence, Chinese parents, put early pressure on their children to learn, and on preschools to teach, the mathematics curricula. Chinese parents' expectations affect preschool curriculum and instruction. To meet parental demands, preschool mathematics curricula have absorbed mathematics curricula from elementary schools in Hong Kong and major cities in mainland China (Starkey and Klein, 2008; Cheng and Chan, 2005). Urban Chinese children who attended regular preschool and kindergarten education usually can count, add, and subtract 0-20 proficiently before entering first grade (Zhang et al., 2004).

The above discussion is intended to show that Chinese children's mathematics proficiency is a product of the social-cultural system which includes, among others, a nationally mandate school mathematics curriculum emphasizing the two basics, textbooks and classroom instruction of high cognitive demand to serve the two-basic curriculum, and the social context valuing early mathematics education. However, the strengths of Chinese children's mathematics proficiency are accompanied with notable weakness. For example, there could be an inherent problem with the curriculum system in the basic approach to mathematics thinking. Factors such as trial and error, induction, imagination and hypothesis testing are not significant part of mathematic curriculum and instruction (X. Ma, 1996; Wong et al., 2002). Probably as a consequence, for example, Chinese students appeared less tolerant for ambiguity in mathematics classroom (Wang and Murphy, 2004), less willing to take risks when solving mathematic problems (Cai and Cifarelli, 2004). The interest and confidence in learning mathematics of Chinese students was shown to deteriorate over the years as they moved up to higher grades (Liu and Sun, 2002; Ni et al., 2011). However, an analysis of the limitations in Chinese students' mathematics achievement requires a separate paper.

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COMPARING THE DEVELOPMENT OF AUSTRALIAN AND GERMAN CHILDREN'S WHOLE NUMBER KNOWLEDGE

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Abstract

This paper compares the whole number knowledge of 7-year-old Australian and German children and their longitudinal counting development. Children's knowledge was assessed using the one-on-one *Early Numeracy Interview* and the associated growth point framework. The findings highlight that Australian children had greater knowledge of Counting and Place Value ideas, while German children had greater knowledge of Addition and Subtraction strategies. These variations are likely due to the different curriculum emphases in the two countries and differing language structures for naming 2-digit numbers.

Key words: arithmetic strategies, assessment, language structures, teaching strategies

Introduction

Understanding the mathematical capabilities and knowledge of young children is necessary for designing high quality curriculum, assessment and teaching methods that enable all children to thrive mathematically at school. Many studies show that young children are capable of learning and using informal mathematical ideas as part of their everyday lives, but countries vary in the ways that they introduce whole number arithmetic learning to young children. Australian and German early childhood mathematics curricula differ in terms of some content and goals. While in Germany the focus is on understanding numbers from 1 to 20 and on the development of addition and subtraction strategies, in Australia the emphasis is on counting and place value activities and calculation strategies are emphasised later. A key question is how these variations influence young children's learning. In order to explore this question, the authors compared the whole number knowledge of 7-year-old Australian and German children. The children's knowledge was determined using the task-based Early Numeracy Interview and associated Whole Number Growth Point Framework (Clarke et al., 2002) that was first developed in Australia and then translated into German (Peter-Koop et al., 2007).

Gaining Insights About Young Children's Mathematics Knowledge Using the Early Numeracy Interview

It is well established that teachers need access to high quality information about children's mathematical knowledge in order to plan effective instruction and to monitor children's progress. It is also known that formal written tests are limiting in providing this information about young children. For these reasons, the Early Numeracy Interview (Clarke et al., 2002; Peter-Koop et al., 2007) was designed especially for young children, is task-based and interactive, derived from extensive research, and enables young children's mathematical learning to be measured in multiple domains. This instrument was developed as part of the Early Numeracy Research Project (ENRP) (Clarke et al., 2002; Department of Education, Employment and Training, 2001). The principles underlying the construction of the tasks and the growth point framework were to:

- describe the development of mathematical knowledge and understanding in the first three years of school in a useful form and language for teachers;
- reflect the findings of relevant international and local research in mathematics (e.g., Fuson, 1992; Gould, 2000; Mulligan, 1998; Steffe, von Glasersfeld, Richards and Cobb, 1983; Wright, Martland and Stafford, 2000);
- reflect, where possible, the structure of mathematics;
- allow the mathematical knowledge of individuals and groups to be described;
- enable a consideration of children who may be mathematically vulnerable (Gervasoni and Lindenskov, 2011, Peter-Koop and Grüßing, 2014).

The assessment includes four whole number domains (Counting, Place Value, Addition and Subtraction Strategies, and Multiplication and Division Strategies) and three measurement domains (Time, Length and Mass); and two geometry domains (Properties of Shape and Visualisation). Only data for the whole number domains is included in this paper. The whole number tasks in the interview take between 20 to 30 minutes for each child and for the studies described in this paper were administered by classroom teachers in Australia and by pre-service teachers in Germany, who all followed a detailed script. The classroom teachers and pre-service teachers were very competent with using the interview and had participated in associated professional learning. Through-out the assessment interview process, given success with one task, the interviewer continued with the next tasks in a domain for as long as a child was successful, according to the script. The processes for validating the growth points, the interview items and the comparative achievement of students are described in full in Clarke et al. (2002). A critical role for the interviewer was to listen and observe the children, noting their responses, strategies and explanations while completing each task. These responses were noted on a detailed record sheet and then independently coded to

- determine whether or not a response was correct;
- identify the strategy used to complete a task; and
- identify the growth point reached by a child overall in each domain.

The data was entered into an SPSS database for analysis. Of particular interest for this paper were children's growth points in the whole number domains.

The Australian and German Primary School Systems

In Australia children begin school as a whole cohort in February, after the summer holidays, when they are 5 years old (typical ages are from 4.5 to 5.5 years). Australian children are encouraged to complete 15 hours of pre-school in the year before they begin school. This is subsidised by the government. Formal mathematics education begins only when children begin school.

In Germany children begin school at the age of 6 as a whole cohort in August at the start of the school year and after the summer holidays. Most children attend kindergarten prior to school enrolment for at least one year. More typically they attend kindergarten for 3 years between the ages of 3 and 6. Kindergarten education is not compulsory and does not follow a mathematics curriculum.

Whole Number Aspects of Australian and German Mathematics Curricula

The primary school mathematics curriculum in Australia is set by each State and Territory, but follows the framework provided by the Federal Government in consultation with the States. The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2013) focuses on the domains of number and algebra, geometry and measurement, and probability and statistics. The curriculum also incorporates four proficiencies: understanding, fluency, problem solving and reasoning. There is a variety of textbooks used in primary schools, but it is also common for teachers not to use a text book at all, but rather devise their own tasks or draw on a variety of resources, including text books.

Like Australia, the German mathematics curriculum is set by each State following the National Standards (Kultusministerkonferenz, 2005), i.e., the curriculum guidelines agreed to by all States. While there is a clear focus on arithmetic in Grades 1 and 2, other content areas include space and shape, measurement, pattern and structure as well as chance and data. In Germany the vast majority of primary mathematics teachers use one of the major textbooks.

Approaches for Teaching Whole Number Arithmetic

Teachers in Australia use a variety of teaching approaches for whole number arithmetic. One common approach is using problems connected to everyday experiences. It is also common for teachers to encourage the use of manipulatives and pictures for modelling a problem to assist children to find a solution. The use of tokens, blocks and counting frames are customary. Arithmetic racks are used by some. Children are encouraged to work in pairs or small groups to discuss their strategies and solutions. Many teachers use a framework, such as the ENRP Growth Point Framework, to evaluate the development of children's whole number arithmetic strategies, and plan experiences that enable children to replace counting-based arithmetic strategies with basic and derived strategies such as building to ten, doubles and commutativity. Initially children work with whole numbers in the range of 1 to 20 and then expand to increasingly greater number ranges. At this point Multi-base Arithmetic Blocks (MAB) are often used to model the problems and support children's calculation strategies.

The vast majority of German primary mathematics teachers use a mathematics textbook. In Grade 1 the focus is on whole number arithmetic with numbers up to 20. Counting activities, comparing sets, getting to know and learning to write the numerals from 0 to 9 as well as matching numerals to sets is the focus of the first 4 to 5 months of school. After that, firstly addition and then subtraction is introduced with the aim to help children understand the underlying concepts and to increasingly develop and use heuristic strategies based on derived-facts to replace initial counting-based arithmetic strategies. In most classrooms manipulatives such as the arithmetic rack would be used to model addition and subtraction strategies based on derived-facts.

Comparing the Whole Number Knowledge of 7-Year-Old Children

In order to compare the whole number knowledge of 7-year-old Australian and German children, Early Numeracy Interview data was compared for children who had completed Grade 1. Due to the different school starting ages and timing of assessments in the two countries, the selection of the two cohorts for comparison is pragmatic rather than precise. Therefore it is important to be clear about the similarities and differences between the German and Australian cohorts. Overall, we argue that comparing the groups, despite their differences, is useful for highlighting the progression of children's development of whole number arithmetic and of the influence of curricula and teaching emphases. The assessment took place after the second year at school for the Australians and at the end of the first year at school for the German children. The Australian children attended school in the States of Victoria and Western Australia and were assessed after the summer holidays, as is customary, (at the beginning of Grade 2). The German students attended school in the Bielefeld region and were assessed before the summer holidays (at the end of Grade 1). The Australian students were assessed by their classroom teachers as part of the *Bridging the Numeracy Gap in Low SES and Aboriginal Communities* longitudinal study (Gervasoni et al., 2010), and the German students were assessed by pre-service teachers as part of a longitudinal study on children's mathematical development from one year prior to school until the end of Grade 2 (first results of this study are reported in Kollhoff and Peter-Koop, 2014). The assessment data for both groups were analysed to determine children's whole number Growth Points. Fig. 1-4 show the growth point distributions for the whole number domains.

Counting Knowledge

There are some interesting differences noted between the two groups (Fig. 1). Most striking is that twice as many Australian children (60%) could count by 2s, 5s and 10s from zero (GP4), compared with the German children (30%).

It is notable that roughly the same percentage of students from both countries achieved the highest and lowest Counting growth points; the greatest differences exist for the 90 percent of students in the middle of the Counting growth point distribution. Curriculum comparisons between countries show that German students do not typically learn to count by 2s. 5s and 10s in Year 1, but Australian students do. This curriculum variation likely explains the difference.

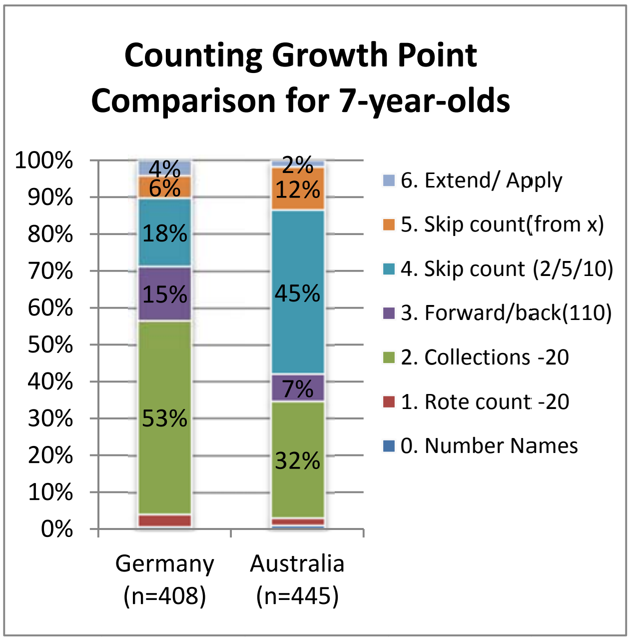

Fig. 1: Counting growth point comparisons for the 7-year-old children

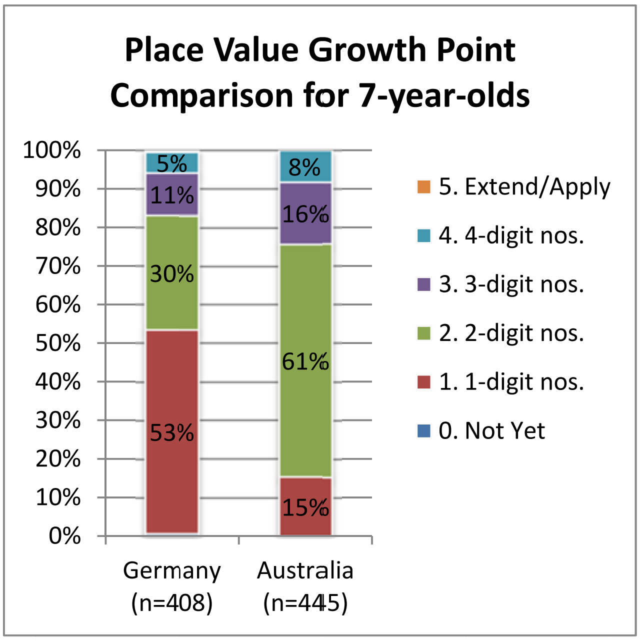

Fig. 2: Place value growth point comparisons for the 7-year-old children

Place Value Knowledge

Fig. 2 shows that there is a large difference between the two groups concerning the percentage of students who understand 2-digit numbers (GP2). This may also be an artefact of the curriculum which in Germany emphasises 1-digit numbers in Grade 1 whereas the focus is 2-digit numbers in Australia. It may also be influenced by the German language structure for naming 2-digit numbers that is based on ones and tens as opposed to tens and ones in the Base-10 numeral form. This creates cognitive conflict for young children as they are introduced to and learn these opposing conventions. This challenge exists only for the number 13-19 in English.

Addition and Subtraction Strategies

In contrast to the Counting and Place Value domains, Fig. 3 shows that twice as many German students (66%) use the *count-down-to/up-to* strategy (inclusive of GP3 to GP6 compared with Australian students (33%). This is intriguing as curriculum guidelines in both countries emphasise the learning of arithmetic strategies in Grade 1. In Australia it is likely that teachers focus more on addition with the use of manipulatives to model a problem and find a solution. Perhaps this is not conducive to students replacing counting-based strategies with basic and derived strategies, nor provides sufficient opportunity for children to construct subtraction concepts.

Multiplication and Division Strategies

Overall, the German and Australian growth point distributions for multiplication and Division Strategies are quite similar (see Fig. 4), except for the larger group of German children using abstract strategies in multiplicative situations. This confirms the trend that German children appear to develop more advanced arithmetic strategies in addition, subtraction, multiplication and division strategies, while the Australian students reach higher growth points in Counting and Place Value. Neither the Grade 1 curriculum in Germany nor Australia emphasise teaching Multiplication and Division strategies. It may be that the children who demonstrate these higher growth points explore these kinds of problems by themselves or with families. This is a fruitful line for inquiry.

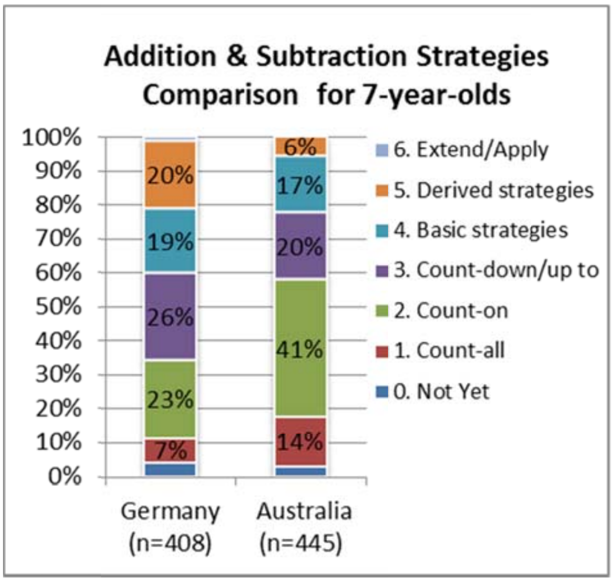

Fig. 3: Addition and subtraction strategies growth point comparisons for 7-year-olds

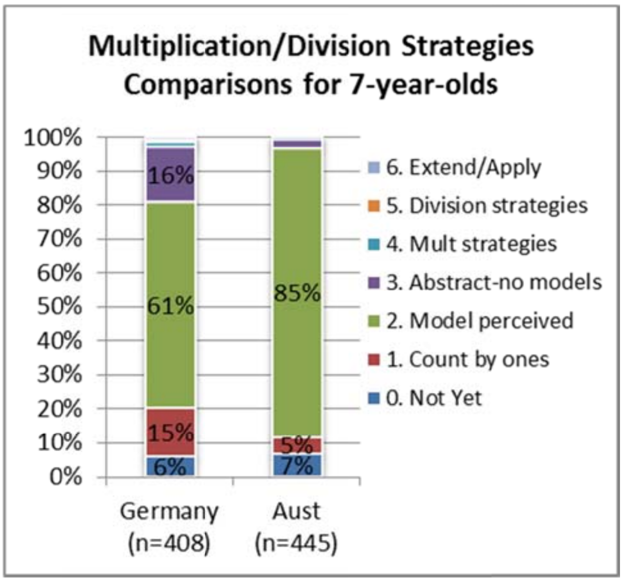

Fig. 4: Multiplication and division growth point comparisons for the 7-year-olds

The Development of Counting Knowledge Over Three Years

In order to gain insight about the longitudinal development of children's whole number knowledge, we traced children's knowledge in the Counting Domain for the two preceding years. The results are shown in Fig. 5 and 6.

The growth point distributions for the 5-year-old children are fairly similar, with the major difference being the number of children able to count 20 teddies (GP2) or count forwards and backwards past 109 (GP3). One year later, as six-year-olds, most German children increased one growth point, but a large group of Australian students increased two growth points (typically from GP0 to GP2, or GP2 to GP4). Nearly half of the German six-year-old children who were attending kindergarten were not yet able to count 20 objects. This type of counting activity is a significant focus of the Australian primary school curriculum, but was not a focus in German kindergartens. The ability of German children to count 20 items changed dramatically after they began school.

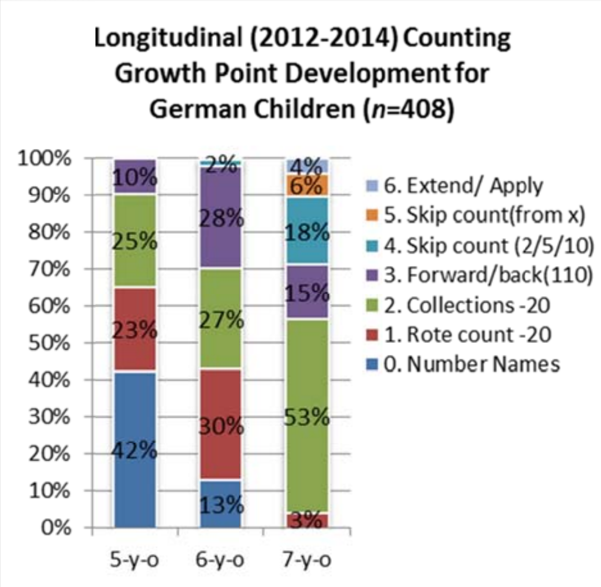

Fig. 5: Longitudinal counting growth point development for German children

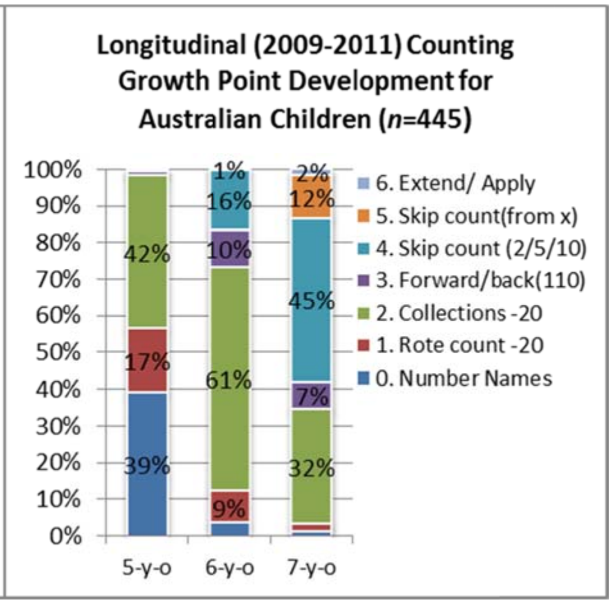

Fig. 6: Longitudinal counting growth point development for Australian children

Discussion and Conclusion

The comparisons between the WNA knowledge of German and Australian children highlight some interesting differences. While it is important to note that the Australian 7-year-olds have spent an additional year at school, overall the Australian children were more advanced in the Counting and Place Value domains, while the German children were more advanced in the Addition and Subtraction domains, with few differences in Multiplication and Division. We hypothesise that these differences can be explained by different emphases in the two curricula and by the challenging German language structures for 2-digit numbers. It appears that the curriculum and language structures for naming numbers influence children's WNA learning and progress. It is also likely that 5-7-year-old Australian and German children's differing opportunities to formally explore WNA at school matters, at least in the short term.

One advantage for the Australian children in this study was that their teachers used the ENI and the associated growth point framework to identify the specific focus for instruction across the school year. This can have the advantage of teachers noticing and then working towards accelerating children's learning if they are not making the anticipated progress. The arithmetic strategies embedded in the growth point framework also provide a useful guide to teachers about the teaching strategies and manipulatives that may be assist children's learning. The framework assists teachers to know when children need to be prompted to use manipulatives, simulation or abstract reasoning to solve the WNA problems posed for mathematical inquiry.

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THE ORDINAL AND THE FRACTIONAL: SOME REMARKS ON A TRANS-LINGUISTIC STUDY

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Abstract

This paper takes an initial, cross-linguistic look at the structures that comprise various sets of number words (cardinal, ordinal, fractional) in twenty different languages, as well as some of the syntactic features with which each language imbues them. The primary question concerns the relations between the ordinal and fractional forms and their joint relations to cardinals. One focal concern is to what extent the grammar of the language tacitly conveys information about the nature of these sets of numbers, starting from an observation that for some languages, such as English, fraction words and ordinals are identical. While the results are necessarily provisional in nature, the paper invites discussion and reflection across a wider range of participants' languages.

Key words: cardinal, comparative linguistics, fraction, language, number word, ordinal

Introduction and background

This paper sits, slightly uncomfortably, in the overlap of mathematics education, history of mathematics and comparative linguistics. In it, I report on the early stages of a larger investigation into number-word systems and structures present in the various languages of the world. By doing so, I suggest *some* of the linguistic knowledge (knowledge that varies by language) young learners need to acquire in order to count and compute successfully, where the verbal (both spoken and written, *within a language*) rather than extra-linguistic notations (such as numerals or symbols for operations) are in view. It also can serve as a placeholder for work in opposition to the increasing hegemony of English and its historical specificities as a language being presumed to be (even relatively) transparent with respect to mathematics. (For more on this, see, for instance, Barton, 2008, or Morgan et al., 2014.)

In particular, it broadens the linguistic focus from solely cardinal words (e.g. one, two, three, ...) to include ordinal words (e.g. first, second, third, …) and fraction words (e.g. half, third, fourth, …) – sets of words that have not been much attended to systematically in the history of mathematics literature on counting words. One instance of this absence can be seen in Karl Menninger's (1958/1969) extensive and authoritative 480 page work *Number words and number symbols*, where there are a mere handful (a hand not even quite full) or references either to ordinal or fraction words. The same is true, albeit to a slightly lesser extent, of the work of historian of mathematics Graham Flegg (1983, 1989, 2007).

There are at least two contemporary reasons for believing that ordinals (and due to the verbal similarity of fraction words to ordinals across languages that is discussed in this paper, fractions) are of greater significance arithmetically than is usually accorded them (not least in conventional school curricula). The first is emerging claims to this end in the neuro-scientific literature (e.g. Lyons and Beilock, 2011, 2013; Lyons et al., 2014), work that is further explored in Sinclair and Coles (2014). The second comes from mathematics education, with one important root in the extensive linguo-pedagogic work of Caleb Gattegno (e.g. 1974) and his intellectual descendants (e.g. Tahta, 1991; Hewitt, 2001), encouraging a view that cardinal counting (what I have elsewhere called *transitive* counting – see Pimm, 1995) may well be overemphasised in early schooling and that issues to do with order, as well as making overt use of the grammatical knowledge of their native tongue(s), may have greater salience in children becoming both numerate and arithmetically competent. There is also a historical reason, which emerges from Seidenberg's (1962) extraordinary paper on the cultural origins of counting, which he sees at bottom as ordinal in nature. For more on this, see Sinclair and Pimm (in press).

In addition, Bartolini Bussi et al. (2014) have recently posed the following question about the language of fractions: 'Why is the denominator expressed in ordinal numbers?' (p. 32). This caught my attention for a number of reasons: I had wondered about this myself; their question caused me to think whether this was a widespread phenomenon, linguistically, or was it limited to a few languages – the question itself does not mention a language; finally, I recalled reading something about it in relation to ancient Egyptian mathematics, but could that possibly have influenced a wide range of languages around the globe? (For more on my response to their piece, see Pimm, 2014.)

The only place I have come across a history of mathematics saying anything remotely about this is in van der Waerden's (1961) book *Science awakening*. Speaking of Ancient Egyptian fractions, he observes:

> Worthy of notice is the verbal expression for $^2/_3$ which means literally "the two parts". The complement, necessary to make a whole out of the two parts is "the third part".
>
> In Greek one also speaks of
>
> "The two parts" $^2/_3$ — "the third part" $^1/_3$
>
> "The three parts" $^3/_4$ — "the fourth part" $^1/_4$.
>
> It presents quite naturally a concrete image: three parts and then a fourth part combine to make the whole. Analogously we can explain our use of the words third, fourth, fifth, etc. In this representation, the fifth part is the last part, which combines with the four other parts to make the unit. Philologically it does not make sense to speak of two fifths, because there is only *one* fifth part, viz. the last. (pp. 20-21)

So van der Waerden is indirectly suggesting that one verbal link between ordinal and fraction arises from a very culturally particular way of thinking about fractions, which increased my curiosity about other languages from further afield. In one of his handful of fraction expression observations, Menninger (1969, p. 79) observes this uncommon naming feature (where a fraction is named in terms as 'the *n*-1 parts' and then 'the *n*th part') as being true of Old Norse as well, and comments: "Quite surprisingly, a number of other languages do the same thing; this of course applies only to fractions commonly encountered in daily life, not to artificial fractions that occur in computations, such as $^{28}/_{29}$". In summary, this paper is empirical in respect to its use of data from different languages to address a question that underlies that of Bartolini Bussi et al (2014) given above, namely *is* the denominator expressed in what they term 'ordinal numbers' across languages and, if not, how is it formed?

Methods

The data discussed in this paper was generated by fairly prosaic means (possibly bordering on the simplistic), namely by identifying native speakers of various languages and then requesting them to produce written tables of cardinals, ordinals and fractions (extending them to the point where the generative pattern becomes regular). In addition, I went back to some of my informants with subsequent specific questions, e.g. concerning any semantic component to the suffixes. I initially invited contributions from the 'departmental sample' (25 doctoral students in mathematics education in the Faculty of Education at Simon Fraser University). To some extent, then, this was an 'opportunity' sample, but one that I enriched as necessary, in order to gain access to a larger range of languages from a broad variety of language groups, and not just from the Indo-European family.

I took educated native speakers of each language as capable informants, especially in this context, as number words are generally regarded as among the most stable linguistic elements available and are frequently used to track other language variation (see Flegg, 1989, pp. 56-61). For the purposes of this paper, the nineteen languages examined with regard to cardinal ordinal and fraction words were as follows: Arabic, Czech, English, Estonian, Farsi, French, German, Greek, Hebrew, Hindi, Hungarian, Italian, Mandarin, Norwegian, Romanian, Russian, Singhala, Slovenian, Spanish and Swedish.

Results

The data are not conducive to presentation in full! A typical submission was like this (Hungarian):

Cardinal:
egy, kettő, három, négy, öt, hat, hét, nyolc, kilenc, tíz, tizenegy, tizenkettő, tizenhárom, tizennégy, tizenöt, tizenhat, tizenhét, tizennyolc, tizenkilenc, húsz

Ordinal:
első, második, harmadik, negyedik, ötödik, hatodik, hetedik, nyolcadik, kilencedik, tizedik, tizenegyedik, tizenkettedik, tizenharmadik, tizennegyedik, tizenötödik, tizenhatodik,tizenhetedik, tizennyolcadik, tizenkilencedik, huszadik

Fractions:
Fél *or* egy ketted, harmad, negyed, ötöd, hatod, heted, nyolcad, kilenced, tized, tizenegyed, tizenketted, tizenharmad, tizennegyed, tizenötöd, tizenhatod, tizenheted, tizennyolcad, tizenkilenced, huszad

I have offered this set first in part because Hungarian is a language structurally unlike all those that surround it geographically (it is closest to Finnish). But also because of a striking feature in the context of this paper: if one looks at the ordinals and the corresponding fraction terms, on the surface (which, as I mentioned, is where I am attending) either the fraction term 'adds' a suffix (-ik) to create the ordinal, or the ordinal 'drops' it to create the corresponding fraction term. Either way, this is an unusual relation between these two sets of words (as we shall later: see Appendix) and can serve as an initial instance of a feature of interest. More generally, what I have done here is to highlight a few features that caught my attention across the data set and I will exemplify them with instances from particular languages.

From the data, it is apparent that, although ordinal and fraction words in English (basically, cardinal+*th*), French (basically, cardinal+*ième*) Spanish (basically, cardinal+*avo*) and Italian (basically, cardinal+*esimo*) are (almost everywhere) the same, ordinals and fraction terms more generally are not identical across the larger set of languages examined. Nevertheless, there was no language I looked at in which the cardinal, ordinal and fraction words were independent of each other – at least, not after the very first few items in each list (e.g., in English, 'second' and 'half' appear to be unrelated lexical items).

Besides Hungarian, the simplest counter-example to the initial conjecture (that ordinal and fraction words are always basically the same) is perhaps German, where a consistent suffix '-te' (later '-ste', due to the fact that the numbers between twenty and ninety-nine are written with the units digit first, e.g. *sieben-und-vierzig* for forty-seven) is added to the cardinal word to form the ordinal and '-tel/stel' to the cardinal (or, equivalently, '-l' to the ordinal) to produce the comparable fraction word. For example, *vier – vierte – viertel* and *zwanzig – zwanzigste – zwanzigstel*. Similarly with Slovenian, where the fraction words are formed by addding '-na' to the ordinal form.

(Before moving on, there is a methodological issue even in talking of 'dropping' or 'adding' 'suffixes'. And that is a presumption that the cardinal words are the base words from which the other forms were subsequently derived. Such a way of framing things might have tacit implications concerning origins and chronological predecessors in the development of the language. I am not making the claim that diachronically (i.e. historically) one form developed before another. And even more importantly here, just because one thing may have preceded another linguistically does not have necessary implications for how it develops within the individual: ontogeny does not always need to recapitulate phylogeny, in Haeckel's memorable phrase. Nevertheless, for ease of description here, and purely by attending to the surface form of the language items, I will continue to speak of 'suffixes', although the case of Hungarian still resonates with me.)

Asian languages are frequently referred to in terms of their regular number word formation (e.g. eleven is 'one ten one' in Mandarin). In Mandarin, too, both ordinal and fractional forms involve the cardinal. My informant observed:

For example, when you want to say ‘first, second, third, ... ’ in an ordinal sense, you would add a character (dì) in front of one, two, three that functions as ‘order’. So for ordinals, you literally say ‘order one, order two, order three, ... ’. On the other hand, for unit fractions, ‘one half, one third, one quarter, ... ’, you would generate expressions that literally mean ‘two parts (fēn) take (zhī) one’, ‘three parts take one’, ‘four parts take one’, etc.

(This connects to Bartolini Bussi et al.’s observations concerning alternate ways of writing numerical fractions, either from the bottom up or from the top down, and its use as a pedagogic strategy in an Italian classroom.)

When looking, for instance, at Norwegian the cardinal root is there in both ordinal (cardinal+*ende*) and fraction (cardinal+*del*) words also, but the endings are different and so the simplest was to get the fraction word from the ordinal, is to go back to the cardinal stem. Moreover, the suffix *-del* is an actual word, meaning ‘part’ or ‘piece’ (and its plural is *delar*, which features in compound fractions). The same word shows up in Swedish fractions: thus, in Swedish, *fyra femtedelar* literally means ‘four fifth-parts’. This connects historically to Menninger’s observation about Old Norse. It also indicates that is several languages (e.g. Norwegian, Swedish, and Mandarin), the ‘suffix’ has a meaning in its own right, namely ‘parts’.

In Farsi, by contrast, ordinals are formed as cardinal+*om* and fractions are cardinal-ordinal hyphenations, e.g. *noh-sizdahom* which is, literally, ‘nine-thirteenth’. In Farsi, then, fraction words are always singular. Notice this means that the shift in English from ‘three fifths’ seen as three of something plural (that Hewitt, 2001, makes so much of from a Gattegno-inspired pedagogic viewpoint with regard to the division of two fractions) to ‘three-fifths’ seen as singular name (‘three-fifths’ is larger than ‘four-ninths’) is not an option in Farsi.

Related syntactic issues

This last point about Farsi raises the thorny question about the grammatical features and functioning of number words in various languages, how stable they are within given categories and how their placement interacts with the mathematical ideas being named. Certain examples throw up the fact that certain categories of number words may vary in form by grammatical gender (e.g. in Hindi ordinals can be masculine, feminine or neuter, whereas all factions are neuter). Most strikingly to me, this issue of syntactic category (and movement across them) has particular relevance with regard to singular and plural (and perhaps other categories in some languages, e.g. the dual in Greek). Reverting to English once more, I can exemplify this with regard to multiplication. *Four* can be seen as an adjective of sorts and thus *four fours* looks like an adjective and a plural noun (*a four*), which would normally take a plural verb: “four fours *are* sixteen”. But given a different set of number words, as yet unmentioned in this piece (what Fowler, 1999, calls *ratio numbers*) namely once, twice, thrice, four-times, ..., results in “four four-times *is* sixteen”.

The grammar of a given language encodes messages about the nature of number words (and hence to what they refer) and also the mathematics at times pushes back against the grammar (see Barton, Fairhall and Trinick, 1998, for instances of this in Maori). There are many features of languages that English does not mark, for instance various forms of grammatical gender and inflection of various types. There are curious features like in Russian number words require the genitive case for the consequent noun being counted and that noun is marked singular for 1-4 and then as plural at five: thus the Russian is, literally, four of house, five of houses. The Russian fraction words are the same as the feminine form of the ordinal (and the same is true in Hebrew). In Greek, fraction words are all neuter, whereas the ordinals can be any of masculine, feminine or neuter.

Each of these features contributes to aspects of the image of a whole number or a fraction that is encoded in any given language: whether the singularity of certain number words past ten (e.g. English's eleven and twelve, deriving from Old Norse *einlief* and *twalief*, meaning, respectively, 'one-left' and 'two-left', after taking away ten; the French base-twenty remnant of *quatre-vingt* for eighty; the German *siebzehn* for seventeen which goes against the written order of 17; and on and on). It is these cultural traces that in subtle and even imperceptible ways shape young children's number worlds. Gattegno's (1974) pedagogic proposal for 'uniforming' the English counting system makes eleven one-ty one. But, done systematically, it would erase some interesting cultural particularities and might serve to cut off parents. (There is a fascinating account of linguistic reaction by Maori parents when the grammar of Maori was perturbed by certain elements of the mathematics register: Barton et al., 1998.)

Discussion and conclusion

Seidenberg's (1962) speculative paper on the origins of counting draws on the connected notions of *myth* and *rite*, described by Lord Raglan (1936/1975), who links them thus: a *myth* "is a narrative linked with a rite" (p. 117), where a *rite* is the corresponding enacted practice. And Bartolini Bussi et al. (2014) in their paper give an instance of where the rite of writing fractions (bottom up rather than top down) caught their attention. But the rites of arithmetic are not part of a language, only the myth is, what is said while a calculation is going on.

My intention in this paper was to draw attention to instances of variation in number words structures across a range of languages, as part of what every native speaker of that language must come to grips with if they are going to count and compute successfully. Gattegno (1974) created his notion of ogden in order to provide a unit of the arbitrary that could compute what memory work was required to acquire a given number-word system. This paper draws attention to the etymology of fraction (and ordinal) names – and how these are different in different languages. With this awareness, we (as teachers) can be more mindful to emphasise aspects of fractions that are perhaps supressed or encouraged due to the particular history of individual languages.

I end with an anecdote, one that I have told elsewhere (in Sinclair and Pimm, in press). Four-year-old Kai is counting small chocolate Easter eggs that he has unearthed and carefully moves them one at a time from the distal pile of the as-yet-uncounted to the proximal pile of the already-counted: "…, nine, ten, eleven, twelve, thirteen". He holds up the next-to-be-counted and asks, "What's this one called?" and is told, "Fourteen". "And this one?" he asks, holding up his next selection. "Fifteen." And then he is off again, both counting and counting eggs in concert all the way to twenty. At a significant level, to count is to name; to be able to count is to know the number names and to be able to recite them correctly, in order. And naming brings number into being: in English, *to tell* means both to say and to count.

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Appendix

In an attempt to summarise some of what I found, here are four diagrams that reflect different relationships among cardinal (C), ordinal (O) and fraction (F) words within a specific language from my data set. The arrows indicate 'adding' a suffix to the previous sets of words to form the new set. Fig. 1(a) captures, e.g., Norwegian, while Fig. 1(b) exemplifies one common relationship (e.g. German): 1(c) is the 'degenerate' case of 1(b) that fits some Western European languages (e.g. English, French, Italian and Spanish), while 1(d) reflects Hungarian.

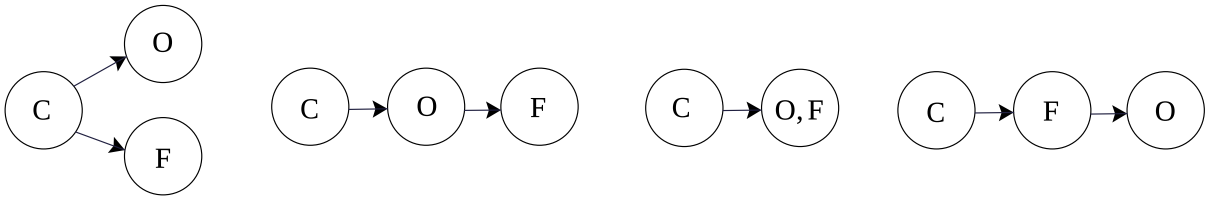

Fig. 1: (a)-(d) Various relationships among number words

DIFFICULTIES WITH WHOLE NUMBER LEARNING AND RESPECTIVE TEACHING STRATEGIES

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Abstract

This paper and the accompanying video introduce a conceptual framework that is used to assist individual students that experience severe learning difficulties in whole number arithmetic. As well as for individual tutoring sessions, this framework can also be applied to classroom teaching and learning. Following a brief description of the organisation and central aims of the intervention programme as well as the used manipulatives, a Four-Phases-Model is presented to support the development of basic computational ideas with respect to the specific arrangement in the case of Ole, the second-grade student featured in the accompanying video. Furthermore, this Four-Phases-Model can be used to document the individual learning process and offers opportunities for the evaluation the intervention programme in future research.

Key words: Dyscalculic pupils, embodied cognition, use of manipulatives

Introduction

Within the framework of the ICMI 23 "Primary Mathematics Study on Whole Numbers" this paper and accompanying video relates to theme 3.3 "Aspects that affect whole number learning" with a focus on children experiencing severe difficulties in learning whole number arithmetic (quite frequently these pupils are labelled as "dyscalculic") and teaching strategies that help to overcome these learning difficulties. We argue that our intervention strategies developed for students with special needs (i.e., with respect to dyscalculia) are beneficial for all students when learning concepts such as place value and addition/subtraction strategies in class. In particular, we focus on the use of manipulatives to foster understanding of mental operations required to solve problems such as 7 + 8 or 15 – 8 beyond the application of counting strategies. These teaching approaches apply to both intervention sessions for children with dyscalculia and classroom teaching in order to prevent long-lasting learning difficulties with whole number arithmetic.

Context of the Accompanying Video

In this section we describe the context of the video provided for the ICMI Study 23 that relates to two different levels: First, the video is embedded in an intervention programme that is part of an elective study in a mathematics teacher education programme. Second, the organisation, the aims and content of a typical intervention session are described.

The interface of pre-service teacher education and an intervention programme for children with dyscalculia

Bielefeld University was one of the first universities in Germany (since the 1970s) providing a "Counselling Centre for Dyscalculic Children" that offers support for families and teachers. Apart from advise in the form of (telephone) consultations and written materials, the centre offers an intervention programme that is free of charge. In order to integrate the centre's work with the primary teacher education programme at Bielefeld University, pre-service teachers conduct the intervention under the supervision of university staff members who are associated with the centre through their research and lectures. Typically, after a first contact with the parents or teacher of a child who is experiencing severe difficulties in early number learning, the child is invited for a diagnostic interview conducted by one of the centre's staff members. This interview is videotaped and serves as a basis for the development of an individual learning plan.

A pair of pre-service teachers who are specifically trained for this task by previous seminars and lectures focussing on mathematical learning difficulties and how the use of manipulatives can help to overcome (and prevent) these difficulties, work with an individual child for one hour every week over a period of about 15 weeks (i.e. the length of a semester in Germany). All intervention sessions are carefully prepared in an accompanying seminar lead by one of the lecturers associated with the centre, i.e., the pre-service teachers write a plan with a detailed description of the approach and the kind of tasks they want to use. This plan is discussed in the seminar (and frequently revised, especially at the beginning of the intervention). Each intervention is video-taped and parts of these videos are also reflected on in the accompanying weekly seminar. These reflections provide the basis for further planning of the intervention.

In total, up to 16 primary children per semester take part in the intervention programme. Generally, they are in grade 2 to 4 – hence, the individual approach that carefully considers individual needs and the content that has already been covered in class. The accompanying seminars for the pre-service teachers typically consist of four teams, i.e., eight pre-service teachers in total, which provides sufficient time for individual planning and reflection.

In addition, the centre has started a support programme for parents in order to help them understand the problems with which their child is struggling. Furthermore, it fosters their understanding of the respective intervention, and offers them communication patterns and strategies on how to support their child's learning in general and specifically in whole number arithmetic, e.g., by helping the child to memorise number facts. The centre's activities also include professional development programmes with respect to the prevention of learning difficulties by appropriate use of manipulatives as well as intervention strategies for children with dyscalculia in a school setting.

Aims and foci of the intervention programme

Children who are accepted for the intervention programme predominantly struggle in three areas. They have not yet managed to develop

(1) a deep understanding of place value (Rittle-Johnson and Siegler, 1998), e.g., the majority of them up to grade 4 cannot tell the difference between the numbers 34 und 43 and frequently claim that they are "the same", because they involve the same digits,

(2) derived-fact strategies for addition and subtraction. They solve respective problems with varying counting strategies, such as *count all*, *count on* and *count down* (Fuson, 1992; Gaidoschik, 2012; Geary and Hoard, 2005),

(3) operational insight and basic ideas ("Grundvorstellungen"; vom Hofe, 1998) that enables them to understand the concept of addition and subtraction (e.g., addition as taking together quantities and subtraction as taking away a quantity from another) including changes between different modes of representation.

Hence, the intervention focuses on these three domains aiming to help the children understand place value, use this concept for calculations, and to offer calculation strategies beyond counting that are based on insights in part-whole schema (Resnick, 1983) and knowledge of respective number facts (i.e., 8 = 7 + 1, 8 = 6 + 2, 8 = 5 + 3, 8 = 4 + 4), so that tasks such as 7 + 8 can be solved as 7 + 3 + 5.

Depending on the individual knowledge and skills (reference point is the diagnostic interview that is conducted before the start of the intervention), the intervention seeks to address the respective content by using manipulatives that model the strategies to be ultimately developed on a cognitive level. Details on the selection of the manipulatives and the specific use of these materials over four distinct phases are described in the following sections.

Suitable Manipulatives for Whole Number Arithmetic

The intervention programme that provides the context for this paper and video acknowledges that all materials and manipulatives that are used to illustrate mathematical concepts need to be learned and understood. Otherwise they cannot serve the intended purpose and rather encourage low achieving children to apply counting activities (Rottmann and Schipper, 2002). Hence, we only use two specific materials that (in our understanding) relate best to the respective mathematical concepts and their desired mental models.

In order to enhance children's understanding of place value we use Multibase Arithmetic Blocks (MAB, see Fig. 1; also called "Dienes blocks", Rittle-Johnson and Siegler, 1998), which stress the cardinal understanding of number. The MAB allow concrete bundling activities (10 minis make a long, 10 longs make a flat, 10 flats make a block) and provide a suitable representation of the place value of a numeral and the relations between the ones, tens and hundreds.

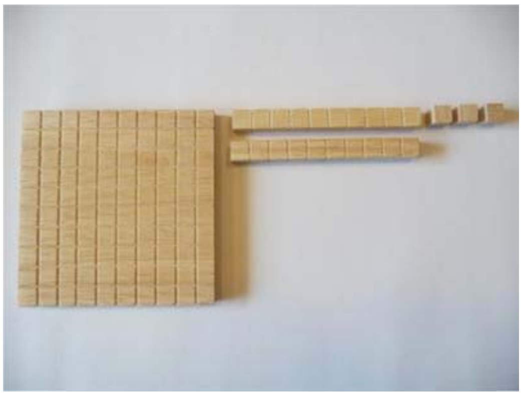

Fig. 1: MAB

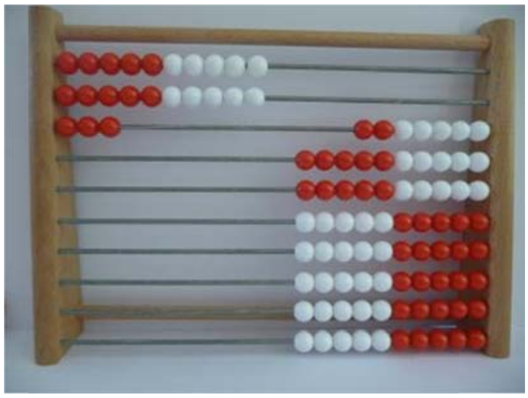

Fig. 2: Arithmetic rack

To prompt the replacing of counting-based calculation strategies with derived fact strategies, the arithmetic rack is used (see Fig. 2). Using the arithmetic rack has the advantage that a number can be either recognised quasi-simultaneously or produced and displayed by setting the elements using the structure of the arithmetic rack. As a convention, it is predefined that numbers are set on the left side of the material. Additive operations are equivalent to panning to the left, and subtraction operations are equivalent with panning to the right.

Conceptual Framework for Intervention as well as Classroom Instruction

The conceptual framework that informs the intervention programme is based on "Grundvorstellungen" (vom Hofe, 1998), i.e., basic ideas and related individual images of a mathematical concept, such as addition or subtraction. This means that concrete actions or pictures can be rebuilt according to mental operations and mental images by using suitable manipulatives. With respect to high-achieving children, this process of *developing mental tools* frequently occurs without special support of the teacher. Children with dyscalculia, i.e., those who are extremely vulnerable with the learning of whole number arithmetic, in contrast, tend to use the material as a counting aid or do not manage to develop strategies beyond the use of the concrete objects (Rottmann and Schipper, 2002). Frequently, they cannot describe their actions, which can be interpreted as they do not understand what they do and how this relates to a computation strategy that does not draw on counting. Gervasoni (2013, p. 2) states that "through experience, we learn to simulate the action of using and seeing mathematical models to calculate without actually moving or seeing". However, this implies that pupils are provided with more or less extensive experiences in their mathematics classrooms (or during an intervention) to master the shift from manipulating with materials to mental calculations. Verbal descriptions of the concrete actions and their link to the computational level support this process.

Hence, the intervention programme that guided the video example follows a *Four-Phases-Model* (Wartha and Schulz, 2012) that acknowledges the need for verbal descriptions when using concrete objects/manipulatives and a transition phase from manipulating with material to mental operations that activate a mental concept which allows the child to imagine the actions required in order to solve an addition/subtraction problem.

This model is based on initial ideas of Bruner and the further development of Bruner's theory by the Swiss psychologist Aebli (1976). Bruner (1973) distinguished three types of representational systems: the inactive, the iconic and the symbolic representation. While the inactive representation is based upon actions, the iconic representation comprises both, pictures and mental images. The symbolic representation involves mathematical symbols (as written numbers or operation symbols) as well as language. Bruner strongly links learning processes to translations of one representational system into another. And Aebli in addition describes gradual internalisation processes from enactive to mental actions, which focus on the transition from one representation to another.

With emphasis on verbal descriptions of enactive and mental actions, the Four-Phases-Model stresses the relevance to assist the development of mental images by a gradual and systematic removal of the manipulatives.

Phase 1	**Concrete usage of manipulatives and verbalisation of operations** Teacher and child actively use the material and verbally describe their operations and their meaning. When the child is confident in working with the material, the child takes over and verbalises the operation itself.
Phase 2	**Verbal description of the imaginative use of the manipulative in sight** With the manipulative in sight, the child describes the operations on the manipulative to the teacher or a fellow student who performs the according operations following the child's descriptions.
Phase 3	**Verbal description of the imaginative use of the covered manipulative** With the manipulative covered by a screen/shield, the child describes the operations on the manipulative to the teacher or a fellow student who performs the according operations following the child's descriptions.
Phase 4	**Verbal description of the mental operation** The child verbally describes the operations without the manipulative being present in any form other than the child's imagination. The tasks are given in a symbolic representation.

Fig. 3: Four-Phases-Model to support the development of basic computational ideas

It is important to understand how much time an individual child requires to get ready to move on to the next stage. As soon as problems occur, one would move back to the previous stage and continue from there. During the intervention programme it usually takes a minimum of 10 to 12 weeks (most frequently even longer) before the children attain the final stage (Phase 4).

The Case of Ole

The accompanying video shows key steps of the learning of derived-fact strategies for addition and subtraction of Ole in the context of the intervention. Because of his substantial problems with respect to whole number arithmetic, Ole was registered for the intervention programme by his parents. The initial diagnostic interview was conducted in September 2012, when Ole was in grade 2 of a primary school in Bielefeld, after he was relegated to grade 1 in January 2012. In this first interview Ole showed difficulties with counting backwards and solving addition and subtraction problems with numbers up to 20. He did not show any use of derived-fact strategies, but used his fingers or the arithmetic rack for counting.

Ole was unable to solve tasks like 15 – 14 without the use of manipulatives and his use of them showed that he was counting each individual element, but moved them in bigger units. His operations with the material did not comply with the conventions for using the arithmetic rack, because he set up the minuend or the first addend respectively on the right (not on the left) side of the arithmetic rack.

The intervention started in October 2012. First, the focus of the intervention was on learning to use the manipulatives (in this case the arithmetic rack) and developing an understanding of its structure (e.g., by subitizing and displaying numbers by using bigger subunits). Both, the accompanying video excerpt as well as the overview of the different phases of the solution processes during the entire 13-week intervention (see Tab. 1) are focussing exclusively on addition and subtraction tasks of the type '2-digit number plus/minus 1-digit number'. The intervention seeks to enable the children to successfully apply the strategy „bridging tens" (e.g., 28 + 6 by 28 + 2 = 30 and 30 + 4 = 34), which is a universally applicable – and hence appropriate strategy for those types of addition/ subtraction problems (Foxman and Beishuizen, 2002).

Through the course of the intervention, Ole learned to describe his use of the arithmetic rack (Phase 1, week 4 of the intervention). In order to foster the development of mental images Ole was asked to verbally describe the use of the manipulative without performing the actions himself. The concrete manipulation was replaced by observing the pre-service teacher`s manipulations (Phase 2, week 5 of the intervention).

Obstacles in the transition from one phase to the next are caused by a premature progress to the next phase before the current stage is sufficiently mastered. In the accompanying video Ole shows problems at the beginning of Phase 3, because he has not yet deepened his understanding of part-whole relationships and therefore does not subdivide the second addend into appropriate components in cases when the manipulative is fully covered (transition from Phase 2 to Phase 3, week 5 of the intervention).

Moving backwards to Phase 2 was necessary before Ole was able to visualise the respective operation and to describe it without direct view of the arithmetic rack (Phase 3, week 5 of intervention). At the end of the intervention Ole successfully built up a mental model of the operation and was able to activate it when solving an addition/subtraction problem (Phase 4, week 13).

Tab. 1 provides an overview of the whole intervention. The focus was on tasks of the type ‘2-digit number plus/minus 1-digit number’. For each of the given problems it was analysed in which of the four phases this task was dealt with by Ole and whether he arrived at the correct answer. The analysis shows that Ole clearly struggled with Phase 3 (i.e., when the manipulative is shielded), only 63% of his answers are correct. The substantially higher success rates in the first two phases suggest, that the use and presence of manipulatives foster his solution processes, while it seems irrelevant whether Ole actively uses the manipulatives (93% success rate in Phase 1) or rather observes them (89% success rate in Phase 2). Of crucial importance in this respect was obviously the opportunity to apply counting strategies when determining given representations of numbers in the first six weeks of the intervention.

	Week of intervention													
	1	2	3	4	5	6	7	8	9	10	11	12	13	% success rate in phase
Phase 1	3/0	2/1	2/0	4/0	1/2	13/0	12/0							93
Phase 2				1/2	1/0	5/1	2/0	4/0	6/0	6/0				89
Phase 3					0/2	3/2	5/5	8/2	8/3	5/5	3/5	6/1	4/0	63
Phase 4											2/0	2/1	4/2	73
no expla-nation			0/1								4/0	1/1	2/0	80
% success rate per week	100	67	67	71	33	88	79	86	83	69	62	75	83	

Tab. 1: Overview of the four different phases of the solution process – provided are percentages with respect to the success rate within a phase as well as within an intervention unit

For each of the 13 intervention units (each 60 min) Tab. 1 shows the number of tasks that Ole solves correctly in the respective phase (first number) and the number of wrong solutions or solutions with the help of the teacher (second number). Below the four phases derived from Fig. 4 we have included a category “no explanation” to record the cases Ole gave the correct answer without any further explanation.

Implications for the Classroom

The Four-Phases-Model explained in this example has been successful in assisting the learning of whole number arithmetic for students who experience severe difficulties in while number arithmetic. However, this method used in one-on-one sessions can easily be transferred to regular mathematics classrooms. This was demonstrated by longitudinal studies conducted by the Counselling Centre. It was found that students working in pairs provide a good opportunity for the students to make up computation tasks that are solved with the concrete materials first, then with covered materials and finally mental operations without the material.

It is highly important that the classroom teacher acknowledges that each student requires an individual amount of time and practice at each phase. The teacher needs to make sure that each student is provided with sufficient time and opportunity for practice to enable deep understanding.

Implications for Future Research

While the Four-Phase-Model described in this paper originally provided the background for the planning, implementation and evaluation of an intervention programme for dyscalculic primary students, it also offers research perspectives with respect to the longitudinal evaluation of the intervention. With respect to the monitoring and analysis of the effect of intervention programmes based on the Four-Phase-Model numerous (research) questions arise, e.g., *What kind of verbal interaction supports internalisation processes effectively? Are there examples for the successful skipping of single phases?* More detailed analyses that focus on the transition from one phase to another as well as the verbalisation of the enactive as well as the mental use of manipulatives seem to be of crucial importance for the further development of intervention.

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NUMBER SYSTEM AND COMPUTATION WITH A DUO OF ARTEFACTS: THE PASCALINE AND THE E-PASCALINE

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Abstract

We are using a duo of artefacts, constituted by a mechanical arithmetic machine and its digital counterpart, to enable six-year old French students to learn about numbers. The experiment shows the separate conceptualisation processes involving numbers as sign of a quantity and number sequences on one side and recursive addition, computation and its effect on the decimal code for numbers on the other side. The duo of artefacts enabled the design of situations that required these processes to be connected. We observed how students and teachers used the duo and discuss the results concerning the conceptualisation of number.

Key words: duo of artefacts, e-pascaline, number system, pascaline

Teaching number decimal system and computation

One of the aims of the first year of compulsory education for 6-year old French students is to learn the decimal system of writing numbers and to use it to perform computation. In their review of studies about whole numbers, Nunes and Bryant point out the key question: 'how do children come to understand that any number in the counting sequence is equal to the preceding number plus 1?' (2007, p. 4). We reformulate this in a more general way by asking how children connect what they know about numbers, number sequences and the manipulation of quantities with what they know about computation, performing addition or subtraction within the number system. The question also concerns the representation of number, taking into account that the number sequence is often learnt as an oral sequence, while the number system is a symbolic written system.

In France, before they are six, students begin to learn the numbers up to 30 at "école maternelle". For these children, number is used to indicate a specific characteristic of a collection, which is the quantity of its elements, or a position in a list (Margolinas and Wosniack, 2014). Comparing numbers involves going back to the collections and operating on their objects. The number name is a label that signals the number of elements. These names are ranked and can be recited in a given order. Children count by using the oral list of number names, which turns to be an action on the objects of the collection. They may even perform some kind of addition, which is, in fact, the union of two collections of objects. Once the two collections are unified, children can determine the number of its objects. Even if they use digits to write a representation of numbers, they use these as an icon and do not manipulate the decimal number system.

When children start "elementary school", during the first year of compulsory education, they have to learn the decimal number system used to represent numbers up to 100. The aim is that they understand how this system is linked to the collections of objects they have used to count, in a more precise manner than just as different names for different kinds of collection, and that one can operate on numbers by operating on their digits. They have to build the relationship between successive numbers in the number sequence; for instance the fact that the number next in the list is the previous one plus one unit.

We are investigating the use of a duo of artefacts, with a physical machine (the pascaline) and its digital counterpart (the e-pascaline), in the learning of numbers and computation. Following Italian research about mathematical machines (Maschietto and Bartolini Bussi, 2009), we assume that the physical machine enables the action-perception loop linking eyes and hands which is important for mathematical conceptualisation (Edwards, Radford and Arzarello, 2009; Kalenine, Pinet and Gentaz, 2011). However, using only a limited range of physical material may also lock students into procedures that require the presence of the physical artefacts, even when the didactical sequence and the teacher try to take this possibility into account and to facilitate the transfer and generalisation of procedures. Therefore, we have extended the physical artefact by a digital version of the machine that enables students to use their procedures in another context (Maschietto and Soury-Lavergne, 2013).

The aim of this paper is to describe how the duo of artefacts offers students a way to learn about the number system by solving problems that require flexibility in moving between number writing and computation. We have designed the e-pascaline in order to enlarge the mathematical experience of the students and to make complementary activities with the two kinds of artefacts. This paper discusses a French teaching experiment carried out in ordinary classrooms with voluntary teachers (Soury-Lavergne, 2014).

Materials and methods

A duo of artefacts: the pascaline and the e-pascaline

The pascaline is an arithmetic machine composed of gears analogous to the famous machine, called Pascaline, invented by the French mathematician Blaise Pascal in 1642. It is a crucial tool in the history of European mathematics because it represents the first example of addition performed independent of the human intellect. When the pascaline is introduced in the classroom, the reference to Pascal and his motivation for the construction of the machine plays an important role in the vision of mathematics as a cultural product. It provides a symbolic representation of the whole numbers from 0 to 999 and enables arithmetic operations to be performed. Each of the five wheels has ten teeth. The digits from 0 to 9 are written on the lower yellow wheels, which display units, tens and hundreds from the right to the left (Fig. 1 shows the pascaline displaying the number 122). When the units wheel (respectively the tens wheel) turns a complete rotation clockwise, the right upper wheel (respectively the left upper wheel) makes the tens wheel (respectively the hundreds wheel) go one step forward. The jerky motion of the wheel supports the recursive approach to number as it rotates one tooth at a time, adding 1 according to clockwise rotation. Anticlockwise rotation, corresponding to subtract 1, allows the operations of addition and subtraction to be linked as inverses of each other.

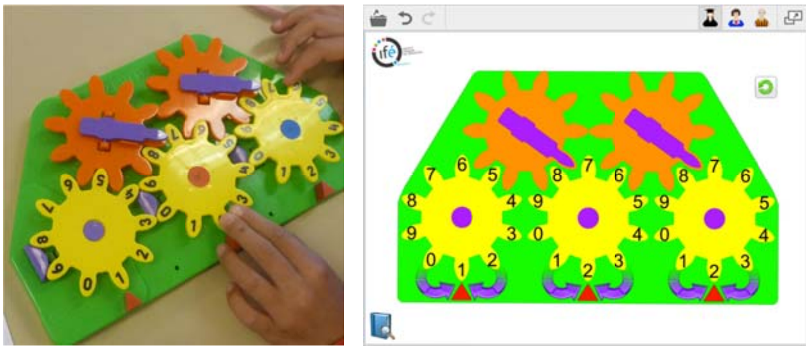


Fig. 1: The pascaline (left) and the e-pascaline (right) are displaying the number 122

Addition is performed by two different procedures which both start from displaying the first term on the pascaline. The iteration procedure consists in repeating the operation of pushing the units wheel, one tooth at a time clockwise, until the number of clicks correspond to the second term in the sum. For example, adding 26 by iteration takes place by the user clicking 26 times on the units wheel). Then, the pascaline use is based on a ‘counting on’ process, which is relevant for linking the knowledge of part-whole with the counting sequence (Nunes and Bryant, 2007). The decomposition procedure in contrast consists in pushing each of the three wheels by a number of clicks equals to the corresponding digit of the second term. For instance adding 26 by decomposition occurs when the user clicks 6 times on the units wheel and 2 times on the tens wheel. The iteration procedure is based on the quantity represented by the number while the decomposition procedure is based on the decimal coding of the number and the signification of the digits. The evolution of students’ procedures hence indicates an evolution of the mathematical signification associated with the digit code of a number.

We have designed the e-pascaline, a digital version of the pascaline (Fig. 1, on the right), to build a complementary duo of artefacts, in which each component adds value to the other (Maschietto and Soury-Lavergne, 2013). The e-pascaline is not a simulation of the pascaline, as close as possible to the model, but is rather a separate artefact which is close enough to the physical one to enable students to transfer some schemes of use, but also different enough (in appearance or in behavior) to reduce components that have inadequate semiotic potential for mathematic learning. The e-pascaline is a component of a Cabri Elem e-book developed with the Cabri Elem technology (Mackrell, Maschietto, and Soury-Lavergne, 2013). Later in the paper, two e-books will be presented.

The didactical sequence

The sequence is composed of four teaching units. The first concerns the exploration and use of the pascaline alone, while in the other units students and teachers use both the pascaline and the e-pascaline. Each teaching unit contains tasks for students working in small group with the pascaline or the e-books, individual tasks, and collective discussions with the teacher. The activities of the units were planned within the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008) the Instrumental Approach (Rabardel and Bourmaud, 2003) and the Theory of Didactical Situations (Brousseau, 1997). In particular, the activities in the e-books were constructed by considering the didactical variables in different situations and the different kinds of feedback that the software enabled to be introduced (Mackrell et al., 2013).

Analysis of specific tasks: addition and writing numbers under constraints

We analyse the e-books "adding with the e-pascaline" and "counting the e-pascaline clicks" which are linking computation and number writing.

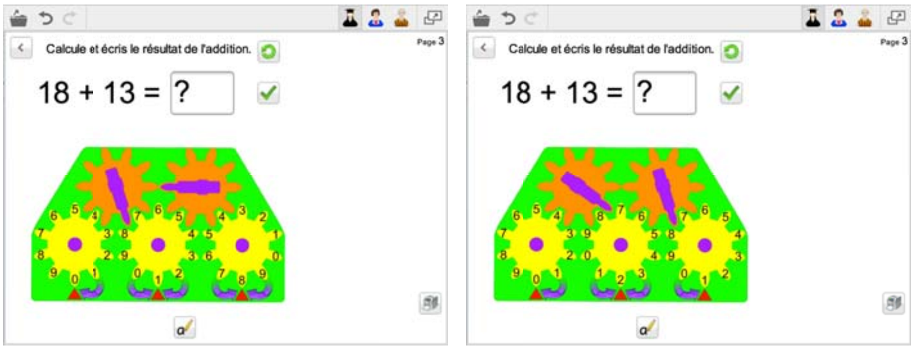


Fig. 2: On the left, the first term 18 is written. On the right, the second term is added, by using the units wheel. After 3 clicks, the adding units arrow disappears.

The e-book for addition takes into account the crucial passage from the iteration procedure to the decomposition procedure, corresponding to the transition from a procedure based on the quantity represented by the number (adding by counting one by one) to a procedure based on the decimal system. It is a delicate passage because most six-year old students apply the iteration procedure even with large numbers. The e-book consists of three pages with the same structure and components (e-pascaline, reload button, evaluation button, arrows for navigation…). The differences from one page to another concern the size of the proposed numbers for addition (up to 30 in pages 1 and 2, up to 69 in page 3) and the type of feedback given by the e-pascaline in response to students' procedures. We have implemented feedback to compel the evolution of students' procedures from iteration to decomposition. To be precise, the e-pascaline wheels turn in the direction of the arrow when the user clicks on one of the two arrows beside the red triangles (Fig. 1). When the addition arrow is not displayed, the user cannot add units. The possibility of hiding the arrows is used to force the students to choose a different wheel from the units one. In the first page all procedures are possible, to support appropriation and devolution of the task, while in the following pages, the unit wheel can only be used the number of times equal to the sum of the unit digits of the two terms. For example, to add 18 + 13 (Fig. 2), the user can click 8+3 times on the units wheel before the addition arrow disappears. The iteration procedure, which needs 13 clicks on the units wheel, is hence no longer possible. In such a way, the student has to look for another strategy to perform the addition.

The e-book “Counting the e-pascaline clicks” contains a task, which consists in minimising the number of clicks required to write a number on the e-pascaline. This task appears to only concern the writing of numbers, but in fact, it requires the exploration of different ways of reaching a number through combinations of additions and subtractions. Starting with the e-pascaline displaying 0, there are three possible procedures to display a given number on the e-pascaline. Let’s consider an example. The number 17 can be written by iteration (17 clicks) or by decomposition (8 clicks), but the minimum of clicks is obtained by a computation 20-3 (5 clicks). This third strategy requires knowledge about the decomposition of numbers and also for the students to change their point of view on the problem and to move from writing the number to computing it.

Experimental setting

The didactical sequence has been developed within a French project gathering teachers, researchers and teacher educators <http://ife.ens-lyon.fr/sciences21/ressources/sequences-et-outils/pascaline-CP>. It has been experimented during the last school year by a team of eight teachers who did not participate in the initial project. During the twelve week experiment, we were able to directly observe working sessions with two classes of six-year old students. We also collected information from the teachers through interviews and written reports throughout the experiment. The results refer either to direct observations of students’ behaviours (with teachers Stina and Nelly) or to the teachers’ reports and interviews (Stina and Cleo).

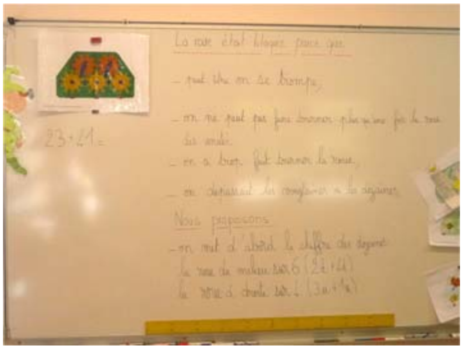
Results

In the teaching sequence, students first worked on addition with the pascaline alone and then used the e-book. In almost every teaching experiment with the pascaline, students’ initial strategies to add two terms below ten were: (i) the two terms were written on two separate wheels and the result was expected to appear on the third wheel; (ii) the addition was done by mental calculation and the result was written on the pascaline. With the first strategy, analogous to the use of a calculator, the user transferred the main part of the work to the pascaline. With the second one, the user performed the main part of the work. The iteration procedure appeared only after teacher interventions and discussion about the two previous strategies.

Adding numbers with the pascaline and the e-pascaline

In Cleo’s class, the iteration procedure appeared after she suggested using the units wheel alone. Then, with terms over ten, mistakes did not increase enough to make the students look for another strategy, even when the teacher suggested that they did. Only one of Cleo’s twenty-three students found the decomposition procedure. When Cleo first set the addition e-book to the students as an individual activity, they still had the possibility to use the pascaline to do addition. Many of them hence used the pascaline, on which the iteration procedure was possible. Then, Cleo compelled the students to perform addition using the e-book. The analysis showed: (i) the students' resistance to changing their strategy; (ii) that the cognitive processes of identifying the decomposition procedure and manipulating symbolic writing are complex. For some students, there was no evidence of the decomposition of the written number into units and tens for doing addition.

In Stina's class, pairs of students used the e-book "addition" on laptops. They used the iteration procedure and were completely unable to progress when the arrow disappeared. The session after, Stina transformed this incident into an opportunity to understand the functioning of the e-pascaline and to share ideas regarding solutions. She led a collective discussion and summarised it on the whiteboard (Fig. 3).



The wheel was blocked because:
- maybe we are wrong
- one cannot turn the wheel more than one unit
- we have turned the wheel too much
- we overcome the twenties or the tens

We propose:
- to first put the tens digit

Fig. 3: Students' explanations for the missing arrow and solutions to compute 23 + 41

She then asked her students to look for additive decompositions of numbers. They worked in small groups and had to write 23 and 41 in different ways, using the pascaline to check. They obtained the following decompositions for 23: 20 + 3; 13 + 10; 10 + 10 + 3; 10 + 5 + 5 + 3; 10 + 10 + 2 + 1; 5 + 5 + 5 + 5 + 2 + 1 and even 11 + 12 and 14 + 9.

When we observed her students, most of them were able to use the two procedures and to compare their efficiency.

Writing a number on the pascaline with a minimum of clicks

The e-book "Counting the e-pascaline clicks" represented a challenge for every student. At the beginning, Cleo's students considered the task easy to perform because of the way it was formulated; it was about writing numbers. The condition of minimising the number of clicks transformed the easy task into a real problem, however. The e-book functioned well in provoking the use of the decomposition procedure. Finding additive or subtractive decompositions of a number were difficult for her class, however. Hence the children adopted other strategies, such as controlling the position of the digits, and reading them near but not above the red triangles.

When Stina's students were first exploring the e-book, they did not succeed in finding the right solutions (indicated by obtaining "smileys" when evaluating their solution). At the next session, Stina raised the problem and asked them to answer the question *"Why haven't we succeeded in obtaining 'smileys'?"*. She also used an intermediate task, with the pascaline, to make pupils find decompositions of numbers. With this intermediate situation, she introduced the solution by asking students to complete a partial decomposition with a subtraction, such as 28 = 30 – … . After this episode, most of her pupils were able to solve the problem.

We directly observed the first use of the e-book in Nelly's class and we made two important observations. First, students didn't use the iteration procedure. They directly used decomposition, starting with the tens digit. This meant that they reproduced on the e-pascaline the spatial organisation of the digits in the written number. Moreover, two numbers, 9 and 19, provoked different procedures, although the successful procedure asked for computation. Students failed to write 9 with the minimum of clicks while they succeeded with 19, first writing 1 on the tens wheel and then turning the units wheel one click in the anticlockwise direction, after having observed that the tooth with the digit 9 was close to the red triangle. They finished by adjusting the tens wheel (one click) when they observed that it had returned to 0. Their procedure could be represented by the computation 19 = 10 – 1 + 10. The fact that they didn't do this for 9 illustrated that they are not in the process of computing but in the process of writing the number (and adjusting the wheels if needed). Their procedure is not equivalent to 19 = 20 – 1, which could have been transferred to 9.

Discussion and conclusion

We have elaborated the duo of artefacts and the e-books to build didactical situations that require the evolution from the iteration strategy to the decomposition one. With the "minimum of clicks" e-book, a third strategy requires computing. It is worth remarking that on one hand, the addition e-book explicitly required computation, but, as in any process of computing, it requires taking into account the way numbers are written with digits and not just adding units one by one (Nunes and Bryant, 2007). On the other hand, the "minimum of clicks" e-book explicitly required writing numbers, but a successful strategy required the students to compute. Both situations were rather difficult for students and successes resulted from teachers' interventions. They revealed that the concept of number, its properties and the signification of its digits code are not stable. They compel the students to make connections between the number designing a quantity and the number represented by its digits code. This relationship frames the fundamental conceptual understanding of whole numbers. Another example of this relationship is the connection between two successive numbers in the sequence, the operation +1 and its effect on the digits of the numbers codes.

In this learning process, both artefacts of the duo played a crucial role. The pascaline was used to produce a sequence of clicks that can be counted. Meanwhile it displays numbers code and ask for operating on these codes. The e-pascaline provided different constraints and feedback that led students to change their strategies and to deal with complementary conceptualisation of numbers.

Further observations are needed to deepen our understanding of the different aspects of numbers that are developed by students while using the duo of artefacts. We have planned to conduct further experiments in France as well as in Italy, to record students' strategies with the pascaline and the e-pascaline and to address some cultural issues.

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USING MULTIPLICATION AND DIVISION CONTEXTS TO BUILD PLACE-VALUE UNDERSTANDING

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Abstract

The paper describes a study with five-year-old children to explore how multiplication and division problems helped them to develop early place-value understanding. Two teachers taught a series of focussed lessons over two four-week periods. The children solved problems using familiar materials grouped in twos, fives, and tens. By the end of the instructional period, virtually all children knew that two fives make ten; the majority could work with tens. Half of them could add tens and ones, fewer partitioned tens, and few could work with multi-unit processes. We propose a 5-level framework that describes developmental progressions in children's awareness of groups of five and ten as building blocks for place-value understanding.

Key words: division, multiplication, place-value understanding, primary/elementary

Introduction

Whole Number Arithmetic (WNA) continues to have a prominent place in most school mathematics curricula. A key aspect of WNA is the numeration system, where each digit in a multi-digit number has a different value according to its position within the numeral. Understanding place value requires students to be part-whole thinkers so they can partition numbers into different-sized units. Typically, mathematics in the early years of school focuses on counting, and this tends to be within the context of addition and subtraction. Place value is usually introduced as part of addition and subtraction with multi-digit numbers, before children have experienced meaningful multiplication and division. It is not until children have been at school for more than two years that multiplication and division become the focus of mathematics instruction.

Increasingly mathematics education researchers recognise that place value is inherently multiplicative (Askew, 2013; Bakker and van den Heuvel-Panhuizen, 2014; Nunes et al., 2009). Ross's (2002) work identified four key major properties of place value, including: positional, base-ten, multiplicative, and additive. It has been suggested that experiences with multiplication and division may be important in helping children develop a deep and connected understanding of place value (e.g., Askew, 2013).

Because place-value understanding is inherently multiplicative, it is far more complex than additive thinking (Clark and Kamii, 1996; Vergnaud; 1994). In contrast to additive thinking, where quantities of the same kind are manipulated (one variable), multiplicative thinking involves working with two variables (number of groups and number of items per group), and these are in a fixed ratio to each other, in a many-to-one relationship (Nunes et al., 2009). For example, a problem about four monkeys each with five bananas involves a 5:1 ratio between a monkey and its bananas. This many-to-one ratio must be strictly maintained to work out that four monkeys would have 20 bananas altogether. A division problem such as the number of boxes needed for 30 cupcakes if each box holds five cupcakes, requires the decomposition of 30 into groups of five (quotitive division). According to Vergnaud (1994, p. 47), 'multiplication and division are only the most visible part of an enormous conceptual iceberg' (the multiplicative conceptual field), that includes fractions, ratios, proportions, and measurement – all concepts involving proportionality.

Evidence clearly shows that quite young children are able to solve multiplication and division problems, although their strategies may differ from those of older children and adults (e.g., Bakker and van den Heuvel-Panhuizen, 2014; Blote, Lieffering and Ouwehand, 2006; Squire and Bryant, 2003). It makes sense for teachers to capitalise on that prior knowledge in the mathematics classroom.

More recently it has been argued that the development of number sense has an important spatial dimension (e.g., Papic, Mulligan and Mitchelmore, 2011; Thomas et al, 2002; van Nes and de Lange, 2007). A spatial structure is about the relationship between elements of a pattern, which has regularity in terms of number or space, including shape, spacing, or alignment.

Research on children's awareness of mathematical pattern and structure (AMPS) has shown the importance of students developing an awareness of structural relationships in mathematics (e.g., Mulligan, 2011). Low level of AMPS is associated with poor visual and working memory. Mulligan found that students with low AMPS tended to "rely on superficial unitary counting by ones" (p. 36), and did not develop efficient and flexible strategies for solving problems. AMPS also impacts on the development of measurement concepts and proportional reasoning. Mulligan's work on promoting awareness of pattern and structure is consistent with other research on the importance of helping children develop knowledge of place-value structure (Cobb, 2000; Fuson, Smith and Cicero, 1997; Thomas, Mulligan and Goldin, 2002). Many of the tasks used to assess AMPS involve the presentation of structured groups of objects for which shape, spacing, and alignment are important aspects of the structure. Children are asked reproduce displayed patterns by drawing them on paper (Mulligan, Mitchelmore and Stephanou, 2015). Mulligan's (2011) work on students' awareness of mathematical pattern and structure show the importance of constructing and representing composite units (multiples) and unit iteration (unit of repeat).

Recent research on so-called "groupitizing" has shown that grouped arrays can be quantified more quickly than ungrouped arrays because children can capitalise on the grouping structure to quantify objects in a display (Starkey and McCandliss, 2014). The advantage of structure becomes increasingly marked with grade level. A growing awareness of number composition in terms of part-whole relationships among the quantities accounts for the improvements in performance with age. This is consistent with research showing that intervention focused on enhancing children's awareness of pattern and structure leads to improvements in mathematics achievement (Mulligan, 2010, 2011).

A key feature of place-value development is the shift from a unitary (by ones) way of thinking about numbers to a multi-unit conception (e.g., tens & ones). The recent work on pattern and structure includes familiarity and use of structured groups of ten (ten-frames consisting of two rows of five) in the assessment of AMPS (Mulligan et al., 2015). Children with high AMPS construct multi-digit quantities quickly using structured material (ten-frames).

Research comparing the place-value understanding of children whose languages vary in the transparency of their decade-based structure for the "teen" numbers has found that children with the most transparent language structure (e.g., Korean, Japanese) have better place-value understanding than those with irregularity (Miura et al., 1993). Most of this research has focused on children from Confucian-heritage countries such as Japan and Korea. However, there are other less well-known languages that also have transparent decade structure, such as the Māori language used by some indigenous New Zealanders.

Overemphasis on counting in the context of addition and subtraction has detracted from an important idea of the composite unit or the notion of multiplicative or additive thinking (Behr et al., 1994; Lamon, 1996; Sophian, 2007). Although many teachers encourage children to skip count by twos, fives, and tens, links are not always made between these number-word sequences and the groups they represent in meaningful multiplicative contexts. A foundational idea underpinning all of mathematics learning is the concept of the unit, and this is the focus of much research on topics such as proportional reasoning and measurement (Mulligan and Mitchelmore, 2013). According to Behr et al. (1994, p. 123), a hidden assumption underpinning primary mathematics is that "all quantities are represented in terms of units of one". Thus the idea of equal groups or composite units leading to multiplicative thinking is not linked to that learning.

The New Zealand Number Framework is embedded in the primary mathematics curriculum and this is linked to the expectations outlined in the Mathematics Standards (Ministry of Education, 2008, 2009). Expectations for the first two years of school are specified in terms of increasingly sophisticated counting strategies to join collections together. After three years, it is expected that children use so-called "part-whole strategies that utilise number properties.

The Study

This exploratory study was set in an urban school (medium SES) in New Zealand. The participants were 35 five-year-olds (21 girls & 14 boys) in two Year 1 classes. The average age of the students was 5.4 years at the start of the study (range 5.0 to 5.8 years). Children came from a diverse range of ethnic backgrounds, with about one third Māori (the indigenous people of New Zealand), one quarter European, one quarter Asian, and the remainder including African and Pasifika (Pacific Islands people). Children were assessed initially using an individual diagnostic task-based interview. The interview was completed again after the second 4-week teaching block (six months later). Tasks included: word problems involving addition, subtraction, multiplication, and division, subitizing, known facts, counting sequences, and place value.

Two series of 12 focused lessons were taught in May and August. Children were introduced to groups of two, using familiar contexts such as pairs of socks and shoes. Multiplication was introduced using simple word problems, such as:

> Three children each get 2 socks from the bag. How many socks do they have altogether?

Once children were familiar with groups of two, fives were introduced using contexts such as gloves (five fingers). Tens were introduced using egg cartons that held exactly ten eggs. For example:

> There are 20 eggs. Each carton holds 10 eggs. How many full cartons are there?

Later problems included numbers that were not multiples of ten, resulting in 'leftover' ones (i.e., the remainder).

> There are 23 chocolates. Each tray holds 10 chocolates. How many full trays are there?

Lessons began with the whole class solving a problem together. The teacher recorded children's problem-solving processes (e.g., drawings and number sentences) in a "modelling book" (a blank scrapbook). Following whole-class discussion, children completed a problem in their individual project books. These problems used the same context and language as the class problem, with a range of numbers to cater for varying abilities.

The Framework

Tasks related to place-value understanding and groups of ten were selected for analysis. Individual profiles were constructed by putting tasks in descending order of difficulty and grouping tasks according to similarity. Students' totals were then ordered to reveal a hierarchical pattern of acquisition (See Appendix – Tab. 1). The easiest task was knowledge of ten as two groups of five (quinary), while the hardest was working with multi-unit processes such as division by ten with remainder.

Results

Initially, only about half the students knew that ten is two groups of five (see Fig. 1). Some were starting to quantify two or three groups of ten and combine tens and ones. Only three students could halve 20, and this was the extent to which they could partition tens within whole decades. Final assessment after intervention showed marked improvement on the selected tasks.

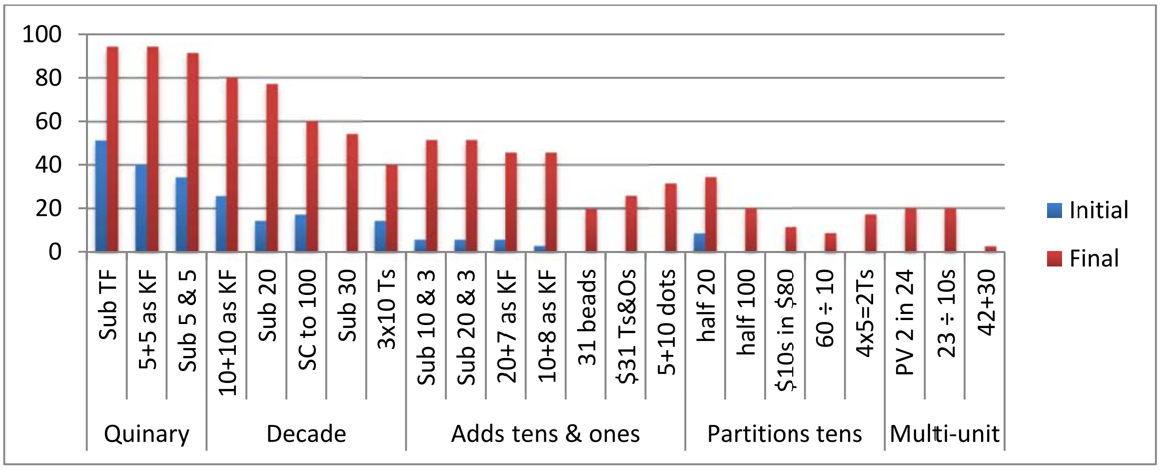

Fig. 1: Percentages of children succeeding on each task: Initial and Final

After the lessons, nearly all students knew that ten is two fives (Level 1). Successful performance on decades ranged from 40-80% (Level 2), 20-50% could add tens and ones (Level 3), and 9-37% could partition tens (Level 4). Only seven students identified tens within multi-digit numbers beyond the 'teens' (Level 5). The two students who could show 2 in 24 and divide 23 into 10s were the only ones who could explain the meaning of 2 in 24 as "two tens".

Discussion

Literature on the development of place value focuses on groups of ten without acknowledging the role of structure within ten, whereas the quinary structure (e.g., 2 rows of 5 dots) has been emphasised by some writers (e.g., Mulligan, 2010, 2011). This study supports Mulligan's (2011) assertion that the quinary structure is foundational for developing base-ten understanding. By the end of the study, all students were successful on at least one quinary task, but for some this was all they could manage (see Tab. 1). This finding is consistent with Mulligan's (2011) point about the importance of explicit features of structural development, including unitising, congruence, and collinearity. Awareness of mathematical pattern and structure (AMPS) is necessary for learning mathematical concepts. Moreover AMPS is multiplicative structure based on grouping and spatial visualisation groupitizing Starkey and McCandliss, 2014). Low levels of AMPS are explained by poor visualisation skills and visual memory, but intervention using the Pattern and Structure Mathematics Awareness Programme (PASMAP) can address this (Mulligan, 2011).

The analysis of performance taking ethnicity into account showed that Asian students performed better than the other two groups (see Tab. 1). This finding is consistent with research showing more advanced place-value understanding for children from Confucian-heritage cultures (e.g., Miura et al, 1993). Māori students did not perform as well as either of the other groups. Although the counting words used in the Māori language have a transparent decade structure, only children who are taught through the medium of Māori develop the fluency to speak and think in the Māori language. In reality, many teachers and students learn Māori as a second language, rather than being truly bilingual. The majority of Māori children are educated in mainstream (English) classrooms and experience only limited Māori language at school. This could explain why the Māori children in our study did not perform as well as the other two groups.

Although some curriculum documents suggest that basic facts should initially be restricted to small sums, we found that the children were more successful with 5 + 5 and 10 + 10 than with tasks where the sum was five or smaller. This may be a result of the salience of five fingers, and the early emphasis on numbers to 20. In the final assessment, some children visualised two groups of five bananas as a group of ten, then added this to the other two groups, finally adding the two tens.

The study showed that five-year-olds can work with multiplication and division problems using familiar contexts (e.g., fingers in gloves, eggs in a carton) and materials to work with fives and tens. This contrasts with Thompson's (2000, p. 291) claim that place value "is too sophisticated for many young children to grasp". It also challenges the many curricula that introduce place value before multiplication and division. The study has some important implications for teachers who could support the place-value understanding of their students by providing meaningful multiplication and division. Further exploratory studies are needed to focus on refining the framework, and explore other ways to support place-value ideas.

Acknowledgements

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Appendix

Tasks	Initial Total n=35	Final Total n=35	Initial Ma n=13	Initial As n=9	Initial Oth n=13	Final Ma n=13	Final As n=9	Final Oth n=13
1. Quinary (10=2 fives)								
Subitizes 1 full ten-frame	51	94	31	56	69	85	100	100
5 + 5 as a Known Fact	40	94	31	56	38	85	100	100
Subitizes 2 dice patterns of 5 dots	34	91	31	44	31	85	100	92
2. Decade (groups of ten)								
10 + 10 as a Known Fact	26	80	23	33	23	54	100	92
Subitizes 2 full ten-frames	14	77	0	22	23	69	89	77
Counts by 10s to 100 verbally	17	60	0	33	23	54	78	54
Subitizes 3 full ten-frames	0	54	0	0	0	54	78	38
3 rows of 10 by 10s or Known Fact	14	40	0	0	0	38	67	15
3. Adds tens & ones								
Subitizes 1 ten-frame & 3 single dots	6	51	0	0	15	31	78	54
Subitizes 2 ten-frames & 3 single dots	6	51	0	0	15	46	78	38
20 + 7 as a Known Fact	6	46	0	0	15	23	89	38
10 + 8 as a Known Fact	3	46	0	0	8	23	100	31
Show 31 beads by 10s & 1s	0	26	0	0	0	0	56	31
Get $31 by $10 notes & $1 coins	0	26	0	0	0	0	44	23
Dot strips 5 + 10 as Known Fact	0	31	0	0	0	8	78	23
4. Partitions into tens								
half 20	9	34	0	11	15	23	56	31
half 100	0	20	0	0	0	15	33	15
$10 notes for $80	0	11	0	0	0	8	22	8
60 sticks in 10s	0	9	0	0	0	8	11	8
4 groups of 5 = 10+10 or 2Ts	0	17	0	0	0	23	11	15
5. Multiple units								
PV for 2 in 24	0	20	0	0	0	23	22	15
23 eggs ÷ 10s	0	20	0	0	0	8	44	15
42 + 30 sheep	0	3	0	0	0	0	0	8

Tab. 1: Percentages of students who were successful on tasks at each Framework level for progressions in place-value development

THEME 4: HOW TO TEACH AND ASSESS WHOLE NUMBER ARITHMETIC

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Introduction

Theme 4 addresses general and specific approaches to teach WNA, an area that has been given particular attention in mathematics education research for the last decades. In the scientific community all over the world, we can see a great variety of general approaches to teach and assess WNA. This underlines the importance of theme 4.

Current research in the issue of teaching and assessing whole number arithmetic needs to consider theoretical and methodological frameworks that can capture the complex relationship between whole number learning, teaching and assessing. The outputs from studying teacher practices have important consequences for (not only) primary mathematics teacher education. Research in this area often focuses on teacher knowledge and its development and uses mainly qualitative methodological approaches. While there is wide attention to teaching and assessment in the mathematics education field, one of the difficulties in relation to WNA is that while the work on teaching WNA is located squarely in mathematics education, the work on assessment has tended to be dispersed across mathematics education, studies in psychology and in neuro-cognitive literature. Thus, one of the aims of this theme and of the ICMI study conference more broadly is to bring these diverse perspectives into conversation.

Attention to how teachers promote the development of pupils' metacognitive strategies during the learning of WNA can be seen as the background question for all contributions to theme 4.

The fourteen papers accepted for this theme address the issues of teaching and assessing WNA from different perspectives. The contributions by participants from many countries offer a unique opportunity to compare and contrast different approaches to teaching and assessing WNA.

Approaches to teaching elements of WNA

Learning of mathematics is based on inventing new solutions to new problems (from the student's perspective) and not only on mere reproduction of algorithms. One of the manifestations of learning is the student's ability to come up with original solutions to new problems (Sarrazy and Novotná, 2013). Many tasks that teachers use when teaching WNA require reproducing algorithms; they do not support pupils' independence and creativity. Looking for effective approaches to teaching mathematics at all levels, but especially at the primary level, is one of the crucial tasks that mathematics education faces.

The papers focusing on approaches to teaching of WNA can be grouped around three sob-topics: the nature of 'good' teaching of specific content in different contexts, teachers' interactions with and responses to learners within their teaching, and possibilities for working for teaching development related to WNA on a larger scale.

Nature of 'good' teaching of specific content

Askew's contribution is a case study of one teaching episode dealing with place value in Grade 2 in South Africa. The author argues that within a context focused around whole class teaching it is still possible to engage learners with mathematics in ways that go beyond merely re-producing procedures demonstrated by the teacher.

Lin focuses on teaching the structure of standard algorithms for multiplication with multi-digital multipliers via conjecturing. She confirms that conjecturing is one of effective instructional approaches to teaching multiplication with multi-digit multipliers. Cao, Li and Zuo present the Chinese tradition of teaching mathematics that influences Chinese curriculum and classroom practices. They show the characteristics of the Chinese approach from the perspective of content, organisation, the arrangement of teaching, ways of presenting, and cognitive demand level with a special emphasis given to multiplication tables.

Teachers' interactions with, and responses to, learners within their teaching

The teacher is the actor who offers pupils the opportunities to develop their understanding and creativity (Sullivan and McDonough, 2002). Ekdahl and Runesson examine shifts in the nature of responses of three South African Grade 3 teachers to pupils' incorrect answers when teaching the part-whole relationship in additive missing number problems, and discuss consequences.

Barry, Novotná and Sarrazy show that the knowledge of variables that determine the difficulty of an additive problem differs considerably from one teacher to another. They justify that the roots of these differences are neither in the teaching experience nor teacher education but in the differences in pedagogical beliefs.

Possibilities for working for large-scale teaching development related to WNA

Brombacher reports on a nationally based research activity conducted in Jordan aimed to improve performance in early grade mathematics. He suggests that deliberate and structured daily focus on foundational whole number skills can support the development of children's ability to do mathematics with understanding.

Approaches to assessing and testing of elements of WNA

It is documented in the literature that there are significant differences in pupils' accuracy and speed in recalling basic number facts and strategies in solving whole number tasks. There is diversity in approaches to test WNA not only in different cultural and educational settings, but even in the same environment.

Looking at this diversity on a small-scale, Pearn's study compares the reactions of the Grade 4 teachers in one school to their pupils' results on a WNA test. This paper follows up Pearn's previous work on assessment approaches published in 2007 (SEMT 07) and 2009 (MERGA).

Zhao, Van den Heuvel-Panhuizen and Veldhuis present an exploratory study of the use of classroom assessment techniques (CATs) by primary school mathematics teachers in China when assessing WNA. They discuss challenges that the participating teachers faced when they were supposed to implement classroom assessment techniques and assess the results.

The challenge of meeting each pupil's learning needs is highlighted by Gervasoni and Parish. They present the results of one-to-one assessment with nearly 2000 Australian primary school pupils and demonstrate the complexity of classroom teaching.

Curriculum

Curriculum as a guide for learning covers a range of modalities from national obligatory curriculum to individual school or group curriculum, from explicit to hidden curriculum, from a framework for teaching to a detailed description of prescribed teaching strategies, materials, textbooks etc. It strongly influences what happens in teaching. Research focusing on curriculum has a long tradition; see e.g. (Hamilton, 2014).

In the context of globalisation and competition, there is substantial attention to seeking worthwhile curriculum practices and educational policies in order to establish effective school systems for K-12 schooling. Tensions between top-down 'adopting' or 'adapting', and 'bottom-up' locally responsive curricula have a long history in curriculum studies. Wong, Jiang, Cheung and Sun argue that no education system can provide Macao a ready-made curriculum model. They introduce Macao's 15 years of experiences of primary mathematics education, after the official handover of the former Portuguese enclave to China in 1999.

Sensevy, Quilio and Mercier analyse the principles and rationale of a curriculum for WNA teaching in the first grade in France. The ways in which this curriculum is grounded in results of the current research in WNA are discussed.

Kaur presents the primary school mathematics curriculum in Singapore, focusing on the model method, an innovation in the teaching and learning of primary school mathematics. The method, a tool for representing and visualizing relationships, is a key heuristic pupils' use for solving whole number arithmetic (WNA) word problems.

Textbooks

There is no doubt about the important influence of textbooks on the teachers' practices at all school levels. The importance of textbooks can be documented by the existence of conferences specialising in this focus (e.g. International Conference on Mathematics Textbook Research and Development ICMT 2014, Jones et al., 2014). Examples of research dealing with textbooks for WNA are presented in the following two contributions.

Zhang, Cheung and Cheung investigate four sets of primary mathematics textbooks used in Hong Kong using content analysis.

Alafaleq, Mailizar, Wang and Fan examine how equality and inequality of whole numbers are introduced in primary mathematics textbooks in China, Indonesia and Saudi Arabia.

Questions for discussion in the working group

Papers in Theme 4 address part of the questions raised in the Discussion document as the background questions. Some questions are not explicitly addressed in the papers but they are related to them and will serve as a basis for discussions in the sessions in Theme 4, namely:

(1) What are some desirable constituents of teacher education programmes that prepare competent and effective teachers to teach WNA?
(2) How can teachers build up on the knowledge children acquire outside school?
(3) How to prepare teachers for promoting the development of student's metacognitive strategies during the learning of WNA?

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Papers in Theme 4

Alafaleq, M., Mailizar, M., Wang, L., & Fan, L. How equality and inequality of whole numbers are introduced in China, Indonesia and Saudi Arabia primary mathematics textbooks.

Askew, M. Seeing through place value: An example of connectionist teaching.

Barry, A., Novotná, J., & Sarrazy, B. Experience and didactical knowledge – the case of didactical variability in solving problems.

Brombacher, A. National intervention research activity for early grade mathematics in Jordan.

Cao, Y., Li, X., & Zuo, H. Characteristics of multiplication teaching of whole numbers in China: The application of the nine times table.

Ekdahl, A-L., & Runesson, U. Teachers' responses to incorrect answers on missing number problems in South Africa.

Gervasoni, A., & Parish, L. Insights and implications about the whole number knowledge of grade 1 to grade 4 children.

Kaur, B. The model method – A tool for representing and visualising relationships.

Lin, P-J. Teaching the structure of standard algorithm of multiplication with 2-digit multipliers via conjecturing.

Pearn, C. Same year, same school, same curriculum: Different mathematics results.

Sensevy, G., Quilio, S., & Mercier, A. Arithmetic and comprehension at primary school.

Wong, I. N., Jiang, C., Cheung, K-C., & Sun, X. Primary mathematics education in Macau: Fifteen years of experiences after 1999 handover from Portugal to mainland China.

Zhang, Q-P., Cheung, K-C., & Cheung, K-F. An analysis of two-digit numbers subtraction in Hong Kong primary mathematics textbooks.

Zhao, X., Van den Heuvel-Panhuizen, & M., Veldhuis, M. Classroom assessment techniques to assess Chinese students' sense of division.

HOW EQUALITY AND INEQUALITY OF WHOLE NUMBERS ARE INTRODUCED IN CHINA, INDONESIA AND SAUDI ARABIA PRIMARY SCHOOL TEXTBOOKS

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Abstract

The comparison of whole numbers is an important concept that should be introduced to young learners in mathematics. However, studies on how this concept is presented in mathematics textbooks is rarely reported and discussed in research literature. This study is intended to examine, and document as well, how equality and inequality of whole numbers are introduced in primary mathematics textbooks in China, Indonesia and Saudi Arabia. Six textbooks were selected from these different countries, and then the textbooks were analysed with focus on the examples of comparison of whole numbers presented in the main text of these textbooks. Findings revealed a high level of consistency in the way of introducing the comparison of whole number in the textbooks across the three countries. However, differences were also found in term of contexts employed and grade levels of the introduction. Possible reasons for the similarities and differences were also discussed in the paper.

Key words: comparison of whole numbers, mathematics textbooks, primary mathematics education, textbook comparison, textbook research

Introduction

Whole numbers is one of basic sets of numbers that includes all counting numbers, or positive integers, plus zero (Frobisher et al., 1999). Mathematics educators and researchers have maintained that number is an important and complex concept in the primary grades for various reasons (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). One of these reasons is that children come to schools in the first year with a widely different mathematics skills and knowledge based on their everyday experiences. For instance, some children developed counting concept before getting to schools whereas, others come with deferent ideas about numbers.

By the primary grades, students begin to use whole numbers in order to count since the whole numbers is a fundamental factor in learning mathematics to establish clear connections between numbers and relations (Gilmore and Bryant, 2006). Griffin, Case and Siegler (1994) pointed out that every child could develop number sense when teachers succeed to develop this sense by using informal activities and then children will be able to succeed in early mathematics and beyond. According to Terezinha and Peter (2009) "It is considerably more difficult for children to use numbers to represent relations than to represent quantities. Understanding relations is crucial for their further development in mathematics" (p. 3). Moreover, the NCTM *Standards* (2000) consider "equality is an important algebraic concept that students must encounter and begin to understand in the lower grades" (p. 94).

The present study is intended to investigate how the equality and inequality of whole numbers are introduced in primary mathematics textbooks in China, Indonesia and Saudi Arabia. Researchers have argued that textbooks reflect mostly what would happen in the classrooms (e.g., Floden, 2002). In addition, studies about how comparison of whole numbers is presented in mathematics textbooks is rarely reported and discussed in research literature. By conducting this study, we hope it can in a sense contribute to filling the gap in research literature by documenting how the concept of equality and inequality about whole numbers in different educational systems is presented and exploring the possible reasons for the differences and similarities.

Method

Selection of Textbooks

To investigate how China, Indonesia and Saudi Arabia textbooks introduce the equality and inequality of whole numbers, we have selected six textbooks at the primary grade level from the three counties. For Chinese and Indonesian textbooks there are a variety of mathematics textbook series being used in primary schools and we selected the latest and most popular series in both countries. For Saudi Arabia, we chose the national textbook, as this is the only series being used in Primary schools and this series is published by the Ministry of education of Saudi Arabia.

Textbook Content Analysed

In this study, the focus of our analysis is all the examples presented in the main text of the textbook as we believe that examples, as a key component of the mathematics textbook, provide most important pedagogical orientation for teachers' classroom teaching.

Coding Procedure

According to the aim of the study, we first identified the units in the selected textbooks that contain the comparison of whole numbers, then we coded the examples according to the following five categories of methods introducing the concept of the equality and inequality of whole number: (1) Comparison by Counting (or simply counting), (2) Comparison by one-to-one correspondence, (3) Comparison using the number line, (4) Comparison by identifying the number of digits in the whole numbers, and (5) Comparison by place value. In addition, we also took into account the context of the examples and the grade level they are introduced.

The first coding process was conducted by the authors and checked by external coders, in order to measure the reliability. The Inter-rater agreement according to the Intra-class Correlation Coefficient (ICC) on China, Indonesia and Saudi

Arabia textbooks coding in general was 0.92, 0.99, 0.97 respectively, which indicates a high agreement in coding.

Results and Discussion

Tab. 1 reports the total numbers of the examples of using different methods provided in the mathematics textbooks examined in the study.

Method	China	Indonesia	Saudi Arabia
Counting	0	7	31
One-to-one Correspondence	8	3	20
Number Line	3	4	13
Number of Digit in Whole Numbers	3	0	12
Place Value	5	13	28
Total	19	27	92

Tab. 1: Numbers of the examples of using different methods

From Tab. 1, we can find that overall there exists a high level of consistency in the three countries' textbooks in terms of the types of methods used to compare whole numbers. However, Indonesian textbooks have not used the number of digit in whole numbers method, and it seems that Indonesian textbooks rely on the place value of whole numbers more than other methods. Furthermore, we have not found any counting method related to whole numbers comparison in Chinese textbooks. We can also easily see that the numbers of examples in Saudi textbook are the most among the three countries and Saudi textbooks have paid more attention to the counting method. In addition, we can find that the whole series of Saudi textbooks provides the largest number of examples, which might to a degree reflect the desire of the textbook authors to provide teachers and students with useful textbooks, since all teachers adhere to use this series of textbook.

An interesting result is that the three countries textbooks have started with the concept of equality and inequality by introducing One-to-one correspondence method. This result goes along with Piaget ideas about children' concept of number as he argued that children do not understand and realise numbers and the relations between them by verbal-based counting. However, children will agree that two sets are equal if they manage to match items in one set to the other and vice versa, and in the case of inequality (see Fig. 1 and 2), one-to-one correspondence is the basis for building the concept of number (Piaget, 1965).

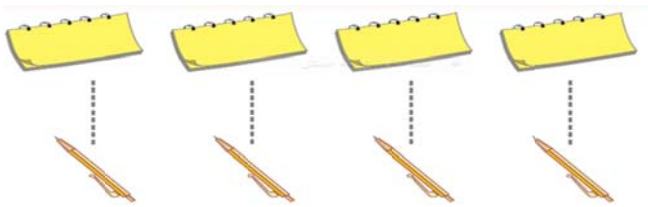

Fig. 1: One-to-one correspondence from Saudi textbooks

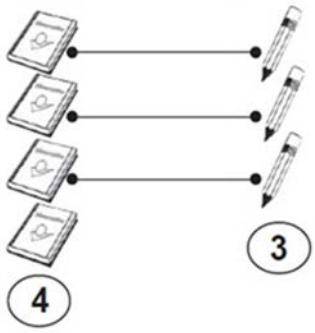

Fig. 2: One-to-one correspondence from Indonesian textbooks

Grade Level	China	Indonesia	Saudi Arabia
Grade 1	One to One Correspondence; Place value; Number line	One to One Correspondence; Counting	One to one Correspondence; Counting; Number Line
Grade 2	Number Line; Place Value; Digit	Counting; Place Value	Counting; Number Line; Place Value
Grade 3	None	Number Line	Number Line; Place Value; Digit
Grade 4	Digit	Place Value	Number Line; Place Value; Digit
Grade 5	None	None	None
Grade 6	None	None	None

Tab. 2: Methods used for comparison of whole numbers at different grade levels in different grade levels in different countries

Tab. 2 presents the distribution of different comparison methods according to grade levels in the three countries. We can notice that the concept of equality and inequality exist in grade 1, 2, 3 and 4, and we have not found any of these methods are used in grade 5 and 6. It is clear that the in grade 1 and 2 the three countries textbooks all present the comparison of whole numbers in a variety of methods.

Moreover, Chinese grade 1 textbook contains the largest number of comparison methods. Correspondingly, Chinese textbooks progress fastest in introducing the concept of equality and inequality among the three countries. In addition, in Chinese textbooks, comparison of whole numbers is only introduced in Grade 1, 2, and 4, but not in Grade 3.

It should also be noted that in the Indonesian textbooks, the number line is used to compare whole numbers in Grade 3, but this method has been used in Saudi and Chinese textbooks in earlier grade levels. Inhelder, Sinclair and Bovet (1974) suggested that children's concept of distance and length like how many steps have to be taken between numbers can develop their metric Euclidean conceptions, and they emphasised to start with non-number line task contexts then improve it to number line. It appears that the way Saudi textbooks introduce the comparison of whole numbers is consistent with the idea suggested by these researchers. For example: In Grade 1, the textbook started with non-number line examples in order to explain the meaning of "before, after" and "more than, less than" and from that point the textbook introduces the number line as a comparison method.

Tab. 3 shows the context regarding the items that have been used to introduce the concept of equality and inequality. Gregory et al. (1999) defined context "as any information that can be used to characterise the situation of an entity, where an entity can be a person, place, or physical or computational object" (p. 302). From Tab. 3 we can see that the textbooks in all the three countries provided a variety of contexts to introduce the concept, and a large consistency is evident, although Chinese textbook appeared to offer slightly richer contexts.

Country	Context used
China	Animals; Food; Household Items; Area and Population; Measurement; Transportation Modes
Indonesia	Animals; Food; Transportation Modes; Household Items; School Items; Toys
Saudi Arabia	Animals; Food; Transportation Modes; School Items; Toys; Household Items.

Tab. 3: Contexts used for introducing comparison of whole numbers in different countries

The importance of providing suitable contexts for developing children’s understanding has been recognised for a long time. For example Dewey (1951) once emphasised that learning would not accrue when it takes place with disconnected events, and learning should be integrated with life process. From this point, we think that some contexts provided in the textbooks we examined could be improved or changed so they are more related to children's daily activities and social environment.

Tab. 4 shows the grade levels in which the comparison of whole numbers up to different values is introduced in the textbooks in the three countries. From Tab. 4 we can find that Chinese textbooks introduce the comparison of whole numbers up to 100 in Grade 1, whereas Saudi and Indonesian textbooks introduce the whole number comparison up to 20 and put off the comparison of larger numbers to Grade 2. Moreover, Chinese textbooks introduced comparison symbols in Grade 1 but Saudi and Indonesian textbooks introduce them in Grade 2. Despite that, there is a gap in Chinese grade 3 textbooks as we have not found any comparison and the remaining content of comparison is only introduced in Grade 4. Whether this gap or discontinuity in introduction of the comparison of whole numbers might after students' learning is worth further attention. In addition, it is also noticeable that Indonesian textbooks introduced the comparison of whole numbers up to millions only, while in the other two countries, the comparison is up to billions.

Grade Level	China	Indonesia	Saudi Arabia
Grade 1	Numbers 0 to 100	Numbers 0 to 20	Numbers 0 to 20
Grade 2	Numbers up to thousands	Numbers up to 500	Numbers up to 999
Grade 3	None	Numbers up to 999	Numbers up to 9999
Grade 4	Numbers up to millions and billions	Numbers up to one million	Numbers up to millions and billions
Grade 5	None	None	None
Grade 6	None	None	None

Tab. 4: Grade levels where comparison of whole numbers are introduced

Summary and Conclusion

This study examined China, Indonesia, Saudi Arabia mathematics textbooks at the primary schools level in order to reveal and document how the three countries introduce the equality and inequality of whole numbers. The results of the textbook analysis showed that there are many similarities and some differences among the three countries' textbooks. The results revealed a high level of consistency in the way of introducing the comparison of whole numbers in the textbooks across the three countries, which in a sense reflects the unique nature of mathematics, in other words, the concept of mathematics is essentially the same everywhere.

On the other hand, Saudi textbooks offered the largest number of examples for introducing the comparison of whole numbers. Moreover, Chinese textbooks introduce the comparison of whole numbers in Grade 1 by using larger numbers than the two other countries. This result indicates that Chinese textbooks introduce large numbers to students at a younger age (6 to 7), while Indonesian textbooks only introduce the comparison of whole numbers up to one million, the other two countries it is up to billions. Further study is needed in order to understand better why there are such differences in the textbooks in the three countries.

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SEEING THROUGH PLACE VALUE: AN EXAMPLE OF CONNECTIONIST TEACHING

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Abstract

This paper reports on a five-year longitudinal study of developing teaching in South African primary schools. Initial video data from 30 grade 2 classes (mainly 7- or 8-year olds) across ten schools revealed teaching practices that were, largely, disjointed and lacking coherence. Three years on, video from the same schools show lessons are becoming more coherent. Data from one of these latter lessons, focused on place value, is analysed to illustrate how coherence is constituted, and that this teacher's actions, examples and talk display elements of a 'connectionist' orientation to teaching and learning. Thus it is argued that within a context focused around whole class teaching it is still possible to engage learners with mathematics in ways that go beyond merely re-producing procedures demonstrated by the teacher.

Key words: connectionist teaching, effective teaching, place value, South Africa

Introduction

In the context of national and international test results presenting a bleak portrait of mathematical performance in South Africa, the Wits Maths Connect–Primary (WMC–P) longitudinal research and development project, is developing and investigating interventions to improve mathematics teaching and learning in ten government primary schools. Analysis of baseline data from 2011 (observation and videos of lessons from each Grade 2 class in project schools) revealed teachers' selection and sequencing of tasks resulted in a lack of coherence in and across tasks, and in task enactment (Askew, Venkat and Mathews, 2012). This weak teaching coherence exhibited 'extreme localisation' and 'ahistoricity' (Venkat and Naidoo, 2012), and thus impairing possibilities for learners to understand number as a connected network of ideas.

Three years later, video data from Grade 2 classes (in the same ten schools) show improvements in coherence and pacing, and analysis is now examining nuances in how teachers bring coherence to their lessons. This paper illustrates this analysis by examining one lesson through the lens of a *connectionist orientation* to teaching (Askew, et al 1997). The content of the lesson is place value, identified in the discussion document framing ICMI study 23 as crucial for understanding whole number arithmetic (WNA). Hence this paper makes a contribution both to understanding teaching of WNA generally, and place value in particular.

Theoretical Background

A recent review of research into 'great' teaching identified six necessary components: deep pedagogical content knowledge (PCK), high quality instruction, a demanding yet supportive classroom climate, effective classroom management, teacher beliefs and professional actions (Coe, et al., 2014). Two of these components were noted as having strong research evidence of impact on learning: PCK and quality of instruction.

A study of effective teachers of numeracy (essentially defined as number and operations) in England, provided evidence supporting the importance of these components (Askew et al., 1997). Studying 100 teachers over a year, the teachers' practices (as observed) and beliefs (as expressed in interview or imputed from observations) were examined in relation to mean class gains in pre- and post-test assessments of skill and understanding in number. Two archetypes of teacher 'orientations' - *transmission* and *discovery* - towards teaching and learning were identified as associated with narrower learner gains.

In contrast, many teachers of classes showing the highest mean learning gains over the year displayed characteristics of what the researchers dubbed a *connectionist'* orientation, characterised by beliefs and practices that included:

- connecting different areas of mathematics and different concepts in the same area through a variety of words, symbols and diagrams
- using pupils' descriptions of their methods and their reasoning to help establish and emphasise connections and address misconceptions
- discussing concepts and images to exemplify the teacher's network of knowledge, thus supporting learning (adapted from Askew, et al 1997).

In this paper, mathematical connections are viewed core to a pedagogy working with highly inter-related example sequences (Watson and Mason, 2006a). In related writing, these authors point to the need for simultaneous working with horizontal relationships within examples and vertical patterns between examples (Watson and Mason, 2006b). Concomitant with this simultaneity is a shift to viewing mathematics teaching and learning as a network of connections rather than a sequential and hierarchical enterprise (Davis, 2009).

Materials and Methods

The lesson focused on here was selected as a 'telling case' (Mitchell, 1984) of connectionist teaching. It is typical of lessons in South Africa in having extended instances of whole class talk and the tasks the teacher introduced were also typical of the range of tasks observed more broadly. Unusual, however, is the extent of the teacher's careful connecting together of the various lesson elements.

Mrs S (pseudonym) is an experienced Foundation Phase teacher, teaching one of five Grade 2 classes in a suburban government school working with a historically disadvantaged pupil cohort. She teaches a class of 38 children, slightly below the provincial and national mean class sizes. The language of instruction is English, although this is not the home language of the majority of the pupils. Video data on lessons on the same topic (place value) were collected from the other two classes whose teachers were also videotaped in 2011: early analysis of those two lessons suggests fewer connectionist oriented moves in the teaching.

South Africa's national curriculum statement (DBE, 2011) prescribes that by the end of Grade 2 the expectation of working with numbers up to 99 and that learners can decompose two-digit numbers into multiples of tens and ones/units and identify and state the value of each digit (p. 253). Maintaining emphasis on the numerical 'value' of digits within teaching has been noted as important to developing pupils' understanding of place value (Thompson, 2000). Analysis of vignettes from the observed lesson, presented in italics below, points to the foregrounding of this emphasis.

Results: Vignette and Analysis

Prior to the lesson, Mrs S had listed in a column on the board the numerals 13, 19, 27, 45, 67, 93. After pupils read these out, she said she wanted learners to break the numbers down. A girl asked to break down thirteen replied 'ten plus three'. Alongside the '13' Mrs S wrote '= 10 + 3'. Other learners were asked to break each number down similarly until '93 = 90 + 3' was written on the board.

> T: Very interesting, eh?
>
> Class: (chorus) Yes.
>
> T: This number [pointing to '13'] is now ten plus three [moves her hand along, tracing under '= 10 + 3' written on the board]. And this? [pointing under '19']?
>
> Class: Ten plus nine [T moves hand under '= 10 + 9' along with the chorus].
>
> T: Now here? [sliding her hand down to under '27']
>
> Class: Twenty plus seven.
>
> Mrs S continued to run her hand down to the next numeral and along underneath the expanded notation in time with the class chorusing the expansion.
>
> T: Now there is something happening here. Look here [gestures down the column of tens]. Now we have two digits this side, now the remainder is one [gestures down the column of ones]. These are tens [points to column of tens] and here we have [points to the ones, rising, questioning intonation in voice]
>
> Class: Units
>
> T: Or ones

Of note here is the teacher's immediate making of both 'horizontal' and 'vertical' connections (Watson and Mason, 2006b) in that her talk and gestures draw attention both to the horizontal expansion, and to the vertical commonalities across the examples. This contrasts with the sequential working with individual examples highlighted by Venkat and Naidoo (2012) where the dominant practice focused only on the horizontal level.

> T: Now we can introduce our blocks. We did this in term one remember?
>
> Teacher picks up a stick of ten interlocking cubes, joined to make a 'ten-stick' attaches one ten to the board, close to the left of the '10' in '13 = 10 + 3'
>
> T: And here [pointing to the '3'] we need?
>
> Class: Three ones.
>
> T: Okay, three, am I okay? [Holding three **ten** sticks up next to the digit '3']
>
> Class. Noooo.

T: So what can I use?

Class: [Some say 'three ones', some 'three units']

T: So where are the ones? [Child comes to teachers' desk and hands over three single cubes.] I thought these [holding up the three ten-sticks] were the ones because this [holding up a single ten stick] is one. Okay, the small ones. Why? Because ten of them will make one ten. I must put how many?

Class: Three

T: Three of them [Attaches three single cubes to the board, to the right of and close to the digit 3 in 10 = 3']

Here the teacher explicitly addresses two foci. First, her actions and talk raise the issue of the possible confusion between referring to a ten-stick as 'one ten' and needing three 'ones': her playing at getting it wrong draws attention to the need to be clear about the different referents of 'three' in the talk. Through her 'error' she uses images to distinguish 'three tens' from 'three units'. Second, the careful positioning of the artefacts near the symbols, the literal proximity of the concrete and symbolic, reinforces the connection of signifier and signified. Although we have records of other teachers using base ten blocks, the physical is not usually so carefully coordinated to cohere with the symbolic.

A similar process was gone through with nineteen. First a ten stick was attached to the board, next to the '10' and then, nine single cubes attached to the right of the '9'. As these were being attached, the class count got ahead of the number of cubes attached. Mrs S stopped and said 'You are counting in the air because I did not put it [holds up a single cube] up.'

Here, Mrs S' attention is not only on her actions but also on the learners' involvement: the counting is not simply oral reciting, but counting to keep track of the number of cubes. Her attention encompasses both her production of examples and images and the sense that learners are making of this – a key attribute of a connectionist orientation.

T: Is it okay for me to put one other down here? [Gestures adding in another single cube to the nine attached to the board.]

Class: No.

T: Okay what would happen if I put an extra one in here? What do you [indicating a response from a particular pupil] think would happen?

P1: It would be ten plus ten.

T: It would be ten [indicating the collection of single cubes] plus this one [pointing to the original ten-stick]. Then what would we be having in the tens?

Class: Twenty

T: Because if we have ten we have more than nine and we make it another ten [Gesturing circles around the nine and the imaginary act of picking the collection of cubes up and moving them to the left, to join the ten].

This is another example of the teacher checking in on the learners' understanding, not taking for granted that they know what she knows. Once again, she goes beyond the 'immediate answer' for the 'immediate example' –

questioning what would happen in an imagined example where 'an extra one' is added. And her gesturing further attends to connecting representations.

T: Now I am coming to this one [puts her hand below '27']. It is changing now. There [points to the '10' in '10 + 3' and '10 + 9'] it was one, one, one ten [gesturing to underline the '10' in '10 + 3'], one ten [gesturing to underline the '10' in '10 + 9']. Now it is two [pointing to the '2' in '27'], so I have twenty [points to the '20']. So how many of these blocks [holds up a ten stick]?

P2: Two tens.

T: Two tens, thank you. Making what? Two tens making?

Class: Twenty.

Here is more horizontal and vertical connecting. The 2 in 27 entails 20, and the 2 in 27 also marked as a change from 'one ten' in the previous examples.

Seven units were attached to the board, then pupils invited to do the same for 45, 67 and 93 (with sticks and units reused, so finally only 93 had cubes next to it).

T: Right, now we know how to break these numbers into tens and?

Class: Units.

T: Now we are going to do a similar activity using the same numbers. I just want to see whether you have observed something. I will underline the number and then you will tell me the value, what does it stand for? Don't tell me that it's tens or units or ones, here I want the value, how many [makes a circular cupping motion with hands]. Okay?

Here the teacher draws attention to the fact that what is coming up is not completely new but connected to prior learning. Learners are 'let into' what interests her: whether they have observed something. Learners are expected to have agency – to note patterns they may have observed not simply remembered. Another connection is marked through the emphasis on saying the value designated not simply which place a digit is in, but.

T: What is the value of that one? [Underlining '1' in '13'.] The answer is there already. In breaking down we show the value in another way. Okay? Now I want you to tell me the value of that one [in '13']

P3: Ten.

T: It's a ten, that's (unclear) isn't it. So the value of that number is ten. [Writes '10' to the right of the equation.]

T: What is the value of nine in that number? [Underlines '9' in '19']. A?

A: Nineteen.

T: She is saying nineteen. Is she correct?

Class: No.

T: Can somebody come here and explain?

P4: Nine.

T: Nine. Why is it nine?

P4: 'cos it's in the unit.

T: Just as a reminder, remember A, it is like this, tens, units [Writes T U above each number.] So nine is under the units, under the ones [pointing to the position of '9' relative to the label 'T U'] so the value of this number [circling the '9' in '19'] is only nine [writes '9' to the right]. There it is A [underlines the '9' in '10 + 9'] Okay? There, okay? There it is. Nine, so the value of this number [Circling gesture around the '9' in '19'] is nine [writes over the '9' to the side again]

The teacher chains together the different representations here, whilst extending the representational repertoire. In her utterance 'the answer is there already. In breaking down we show the value in another way' she directs attention to another connection, then by reframing 'showing the value' as associated with 'breaking down' not only are two potentially discrete ideas connected, but learners are also encouraged to connect with what they already know. In focusing on the value of the digits, not simply the place they occupy, the introduction of T U labels is delayed, but subsequently brought in as another representation to help a particular learner. Given that this notation would have been used before this connects back again to previous lessons. Multiple links between place value features are again made explicit: that these are all various way of signifying the same underlying idea.

T: What is the value of this seven? [underlining '7' in '27']?

P5: Seven.

T: Why is it seven?

P5: Because it is under the units.

T: There it is [points to the '7' in '20 + 7']. What we did by breaking down, we are actually doing the value of the numbers, breaking them down. Okay, what is the value of the two in that same number?

P6: Twenty.

T: Why are we saying twenty?

P6: Because it is two tens.

T: It is two tens, so if I have two tens [picks up a ten in each hand] it means I have a ten [holds up one ten stick] and another ten [holds up the other ten stick] and if I put them together [moves hands together] I have [rising intonation]

Class: Twenty

T: So if I have two tens [holds the two ten sticks by the '2' in '27'] I have twenty. There it is [one hand keeps the two tens sticks on the board, the other gestures under the '20' in '20 + 7']. I have twenty [points to '20' in '20 + 7']. So the value of the two is twenty.

The teacher thus coordinates a dance between the different representations - there is a choreographing of the coordination of the digits, place, value, cubes, actions and language. The learners' attention is repeatedly drawn to what is similar, how things that look different have the same underlying meaning.

T: What is the value of the five? [Underlining the '5' in '45']

Class: Five.

T: It's there [gesturing below the '5' in '40 + 5']. And the value of the four?

Class: Forty.

T: Forty. There in front of you it is there. [Points to the '40' in '40 + 5']. So after breaking the numbers down [Runs her hand up and down the column of tens] it becomes easy for us.

The connection with breaking numbers down is reiterated with the teacher drawing attention to learners using what they already know - knowing the value of a digit is based in the knowledge of how to break numbers down, rather then deduced from the position of the digit. In addition to the directionality forward from past learning, there is multi-directionality in the present – breaking down is connected to all the elements introduced into the learning space: digits, place, value, cubes, actions and language. This contrasts with other lessons where the focus tends to have a uni-directional flow in its playing out, limited in time to the immediately present, and to the particular examples in isolation.

T: The value of this six [pointing to the '6' in '67']

Class: Sixty.

T: There it is [underlining gesture to the '60' in '60 + 7']

T: So How many tens do we have?

Class: [Some say six, some sixty]

T: How many? [Sense of confusion from the class.]

Teacher picks up ten paper tens strips [up to now the physical artefacts have been cubes - card units and card ten-strips are used for the first time].

T: How many tens do we have?

Teacher picks up a number of ten strips in one hand, and counts them, moving from one strip from one hand to the other as the class count, one, two, … , six.

T: If we have six tens it means we have ten [removes one ten strip from the hand holding all six, holds it up on the count of ten. As the strips are moved back to the first hand one at a time, the class picks up the count, but now in tens]

Class: Twenty, thirty, forty, fifty, sixty.

The double count here loops back to the idea of the strips or sticks referring either to one ten, or ten ones. And we see the teacher varying her question: from what is the value of the six to how many tens – thus learners are not allowed to fall into a routine of thinking the answer will always be the same.

The sequence ended with Pupil A identifying the '3' in '93' as three ones. The class were given ten strips and unit squares to model numbers independently.

Discussion

The commentary on this case study reveals a teacher demonstrating many of the practices Askew and colleagues identified as associated with a connectionist orientation to teaching and learning. The extent of horizontal and vertical connections reflects a view of mathematics teaching and learning as based in a network of connections, which, as Davis (2009) notes, points towards 'dwelling with highly connected ideas … and exploring local networks.' (p. 263)

In the original study, Askew et al. note that there was no association between particular forms of classroom organisation – pupils working individually, in groups or as a whole class – and orientations to teaching. That said, at the time of the study there was much more individual or group work in English classrooms, than whole class teaching. Given the long history of whole class teaching in the South African context and evidence of limited moves in sub-Saharan countries to overtly learner-centred pedagogies (Schweisfurth, 2011), it is encouraging to see that the tenets of connectionist teaching can be enacted in such an environment. Future work will include examining whether the evidence for such an orientation towards teaching is linked to learning outcomes.

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EXPERIENCE AND DIDACTICAL KNOWLEDGE

The case of didactical variability in solving problems

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Abstract

Experience is often seen as a source that allows teachers to construct didactical or pedagogical knowledge. The research presented here tries to explore this common belief. The authors worked with 30 elementary school teachers. They show that the knowledge of variables that determine the difficulty of an additive problem differs considerably from one teacher to another. They show that neither the teaching experience nor teacher education can account for these differences: thus it must be the differences in pedagogical beliefs that enable to explain these differences in the didactical knowledge.

Key words: arithmetical problem, construction of knowledge, didactical variability, elementary school

> "The more the student has been drilled in formal exercises, the more it is difficult for her, later, to restore a fruitful functioning of concepts so acquired. "Application" of learned, ready-made knowledge goes badly because the logic of the articulation of the acquisitions which compose it is exclusively that of the knowledge itself and because the role of situations has been excluded *a priori*."
>
> (Brousseau, 1997, 43)

Introduction

Learning mathematics is not restricted to learning algorithms only but it is manifested by identifying conditions for their use in new situations. This criterion allows us to state that the child has learned something. In Novotná and Sarrazy (2005), the effects of variability in the formulation of problem assignments on students' flexibility when using the taught algorithms in new situations were investigated. The research described in the above mentioned paper was developed in the framework of the Theory of didactical situations in mathematics.

The following question was investigated: How can one explain that some students are able to use the taught knowledge in new contexts, while others, although "they know" the taught algorithms, are not able to re-contextualise this knowledge? The research confirmed the following hypothesis: Inter-individual differences in the sensibility to didactical contract (which is measured by an index) are the effect of a teacher's didactical variability in the environment of arithmetical problems.

The model used in Novotná and Sarrazy (2005, p. 701) is based on the following idea: “The more the *same* form of didactical organisation presents the modalities of different realisations, the more uncertainty attached is added to this form. To satisfy the teacher’s expectations, such a student has to ‘examine’ the domain of validity of their knowledge much more than a student who is exposed to a strongly ritualised (repetitive) teaching and therefore a much reduced variability. In other words, a strongly ritualised teaching would allow the student to know in advance what they have to do and thus to adopt a behaviour *ad hoc* (adapted). On the other hand, by the interruptions of introduced routines, a strong variability makes the following strategies futile (controversial): the students cannot rely only on the indicators of introduced routines (semantic indicators, triggers …) and therefore cannot either anticipate or master the liaison of sequences which allows him to discover the behaviours expected by the teacher. This model was not refuted by our results.”

In this paper, the ideas from Novotná and Sarrazy (2005) are developed further, using the methodology first presented in Sarrazy (2002).

Background

The idea behind this research is a relatively simple one: under which conditions can teachers’ experience from their mathematical education contribute to an increase in their didactical knowledge? By “didactical knowledge” we understand the fact that the teacher knows (because he/she “does it” and not necessarily because “he/she tells it”) that an arithmetical problem with a classical structure “initial state – transformation – final state” is more difficult if the question concerns the initial state, and such a problem is also more difficult than a problem where transformations of states are combined or relative (see Vergnaud’s typology, 1990). The concept of didactical variability was introduced by Sarrazy (2002) in order to classify a teacher’s capacity to create arithmetical problems of very different difficulties.

Differences in variability:

1) have impact on the phenomena of sensibility to contract (the higher the variability is, the more easily pupils adapt their knowledge to new contexts) (Sarrazy, id.);
2) allow us to renounce the myth of creativity in mathematics (Novotná and Sarrazy, 2011).

The paper comes out of a very recent study (Barry, 2014) and attempts to get insight into the origins of differences in variability among teachers: Is it linked to their initial teacher education (humanist, scientific, …), to their education in didactics of mathematics, to their experience from teaching mathematics (short, long), to their pedagogical beliefs (active, traditional)?

To put this very simply, the more diverse the situations are, the more chance to get adapted and to verify permanency of their knowledge the pupils will have. The teacher will have to face potential interactions, questions, unanticipated digressions, will be forced to answer and thus develop their knowledge based on experience. On contrast, the more closed the situation will be, the lesser the reactions of the teacher and the pupils to the milieu will be. Therefore we will test the hypothesis that a teacher's pedagogical beliefs contribute to reveal the differences in variability among teachers. In other words we want to show how pedagogical ideologies can raise didactical phenomena, i.e. *connaissances* (isolated pieces of knowledge in the sense of the Theory of didactical situations) and what effect they have on *savoir* of the properties of milieus (in our case word problems) proposed to pupils.

Materials and Methods

Thirty elementary school teachers participated in the research. A questionnaire was used to collect two types of data: 12 questions focused on biographical data (length of teaching practice, education etc.) and 12 questions aimed at finding out respondents' pedagogical beliefs independently on their mathematical knowledge. Semi-structured interviews were conducted, recorded and transcribed. The respondents were asked to pose three additive problems of increasing difficulty without using a textbook; the problems were used to determine the index of didactical variability using the method introduced by Sarrazy (2002).

Description of the research sample

Biographical variables

- The sample consisted of 30 teachers, the majority of them were female (21 out of 30, 70%).
- 14 teachers taught 9-year-old pupils (46.67%), 8 taught 10-year-old pupils (36.67%) and 8 both 9- and 10-year-old pupils (26.67%).
- The overall length of teaching practice was between 1 and 36 years (m = 13). The teachers were divided into two groups: "novice" teachers with less than 10 years of teaching practice (16 out of 30, 53.33%) and "experienced" with 10 and more years of teaching practice (14 out of 30, 46.67%).
- The length of teaching practice in the same grade was between 1 and 36 years (m = 7). The teachers were divided into two groups: "novice" teachers with less than 5 years of teaching practice (18 out of 30, 60%) and "experienced" with 5 and more years of teaching practice (12 out of 30.40%).
- Bachelor level study: 11 teachers out of 30 (36.67%) have bachelor exam in arts, 19 out of 30 (63.33%) bachelor exam in science.
- Finally, as to their teacher training, 14 teachers out of 30 (46.67%) completed their qualification at IUFM.

Pedagogical variables

We focused only on the most discriminating pedagogical variables; these concern repetition, classroom management and relationships between understanding operations and their operational mechanism. For example, as far as the variable repetition is concerned ("repeat until the pupil has understood") the teachers fell into two groups of the same size (50% "more or less agree"). Slightly more than one half of the teachers (53.33%) emphasised the benefits of repetition. A majority of the teachers (73.33%) agreed that it was important "not to allow pupils to intervene at their will". A significant majority (86.67%) agreed on the idea that "a pupil can learn only through problems". 66.67% of the teachers believe the teacher must teach the way to solve problems and that the pupil must then rediscover it and apply it. Finally, 43.33% of the teachers think that it is possible to "learn without understanding".

As far as the idea of distinguishing the mechanism of an operation from its understanding is concerned, a small majority of the teachers (56.66%) think that "this difference is more needful for weak pupils"; 43.33% of the teachers answered that "pupils do not distinguish between the two because it is not necessary to understand a mechanism in order to apply it correctly"; and only 36.67% of the teachers think that "usually a pupil first learns the mechanism and only later starts to understand the operation".

Index of variability

90 problems (30 x 3) were analysed using the same model of calculation introduced by Sarrazy (2002). The model consists of 12 variables divided into three categories: numerical, rhetorical and semantic-conceptual. For each of these 12 variables (summarised in Tab. 1), the number of the observed variations (Vo) was determined. This number was then related to the number of possible variations (Vp) in the set of the three problems. The index of variety is the sum of the observed variations divided by the possible variations: $IV = \frac{\sum_{i=1}^{12} Vo_i}{\sum_{i=1}^{12} Vp_i}$.

Type of variables	*Code*	*Modalities assessed*
Numerical variables		
1 Type of numbers	*TN*	Authorises a procedure other than the classical one / does not authorise…
2 Irrelevant data	*ID*	Present / absent
Rhetorical variables		
3 Semantic indices	*SI*	Present / absent
4 Trigger in the question	*DE*	Present / absent
5 Syntagmatic organisation, temporal organisation of events	*SO, TO*	Correspond / do not correspond
6 Position of the question	*PQ*	Beginning / middle / end
7 Formulation	*FO*	Classical / narrative

Semantic-conceptual variables			
8	Type of problem	*TY*	6 modalities corresponding to the 6 additive structures
9	Nature of the unknown	*NU*	Initial state, final state, transformation
10	Order of numerical data in the explanation of the problem, operative order	*SO, OP*	Correspond / do not correspond
11	Trigger, mathematical operator	*DE M*	Correspond / do not correspond
12	Semantic index, mathematical operator	*SI M*	Correspond / do not correspond

Tab. 1: Variables used to evaluate index of variability

Let us show an example of how IV was determined for the teacher 22.

Easy	Mark has 8 marbles. He wins 8 marbles. How many marbles does he now have?
Medium	Mickael is 25 years old. Mark is 15 years older than Mickael. Jules is 3 years older than Mark. How old is Jules?
Hard	Mum is doing some shopping. She buys 3 apples for a total of 2.95€, a chicken for 3.75€ and a bunch of leeks which costs 4€. On her way back, she buys paper (1.70€) and 2 baguettes (1.80€). How much did she spend in total?

Tab. 2: Teacher 22's three additive problems

Variables	Vo	Vp
Type of numbers	1	1
Irrelevant data	0	1
Semantic indices	0	1
Trigger in the question	1	1
Syntagmatic / temporal organisation	0	1
Position of the question	0	2
Classical or narrative formulation	1	1
Type of problem	2	2
Nature of the unknown	2	2
Order of numerical data in the explanation of the problem,	1	1
operative order	0	0
Trigger, mathematical operator	0	1
Semantic index, mathematical operator		
∑	8	14
IV	0.57	

Tab. 3: Calculation of teacher 22's variety index (IV)

Legend: Vo = number of observed variations; Vp = number of possible variations; IV = index of variability

We grouped the 30 IV into 7 classes: We can observe a relatively homogeneous discrimination (the distribution is normal).

Lower bound	Upper bound	Centre	Size	Relative frequency
0.067	0.140	0.104	2	0.067
0.140	0.214	0.177	5	0.167
0.214	0.288	0.251	7	0.233
0.288	0.362	0.325	4	0.133
0.362	0.436	0.399	5	0.167
0.436	0.510	0.473	4	0.133
0.510	0.583	0.546	3	0.100

Tab. 4: Distribution of variety indices in 7 groups

Pedagogical style

Two modalities were defined: 1) predominantly traditional pedagogy (PT) and predominantly active pedagogy (AP); the first modality is knowledge centred, the other is learner centred. We cannot explain here the whole set of all variables, the scope of the paper does not allow that, we will focus on the four variables that proved to be significant for distinguishing among teachers in the perspective of their variability:

Q1.3- REPET: "Repeat until the pupil understands."
Q6.3- MECA: "The pupil learns the mechanism and understands later."
Q9- DIRIG3: "It is possible to learn without understanding."
Q14- REGL3: "The pupil can understand a rule without knowing how to apply it. This ability develops later."

Results

Neither teacher education, nor attitude towards mathematics (the type of bachelor exam), nor the length of teaching practice can be the ground for explaining the difference in IV; the only thing that can be seen as relevant for these differences are pedagogical beliefs as stated by the teachers: "active" teachers have a slightly greater mean (IV = 0.421) than "traditional" teachers (IV = 0.286; Student's t-test, s., p = 0.56). The observations confirm our expectations: the more open the situations proposed by the teacher are, the higher the probability of unexpected events is (pupils' answers, type of thinking etc.); the teacher will therefore have to react to these unexpected situations (will have to find a counterexample, analyse a pupil's mistake quickly, estimate the effect of a didactical variable etc.) These feedbacks seem to create forms of adaptation to these situations that are not necessarily conscious but are efficient. The differences in variability between the two types of teachers confirm these constructions of didactical knowledge. The interviews clearly show that with the same variability and length of practice, some teachers were not aware of variations they carried out, while others were able to formulate clearly a certain number of variables: both the more obvious like adding supplementary data, to the more subtle variables like the relationship between semantic indices and the mathematical operator or between syntagmatic organisation and operational order. None of the teachers was able to evoke variations of additive structures, let alone implement variations of this dimension.

Discussion and Conclusion

The question of experience is not reduced to a simple question of time – this finding is similar to the findings of Chopin (2011) for didactical visibility. Didactical variability does not relate to the length of experience but has the form of a piece of knowledge emerging from the teacher's action.

These results show very interesting points (the role of pedagogical beliefs about didactical knowledge) but also invalidate interesting beliefs: it is for example surprising that the type of education (both initial teacher education and continuing professional development), general length of practice or the length of practice in the same grade have no impact on a teacher's variability. Didactical knowledge (taught during their education) does not necessarily show as didactical knowledge in a situation (variability); here we find again one of the central questions of the theory of didactical situations: having an isolated item of knowledge (connaissance) may not be sufficient for development of knowing (savoir). Our research confirms the central idea of teacher education: one isolated piece of didactical knowledge is obviously insufficient for development of a teacher's knowledge.

This work allows us to start reflecting on the conditions of structuring such experience. The anthropo-didactical framework seems to be suitable for the study of this experimental knowledge, the piece of knowledge perceived as the product of intersection of pedagogical and didactical fields.

The findings presented in this paper have important consequences for teacher education and for teaching strategies.

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NATIONAL INTERVENTION RESEARCH ACTIVITY FOR EARLY GRADE MATHEMATICS IN JORDAN

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Abstract

This paper reports on a national sample based intervention research activity conducted in Jordan to improve performance in early grade mathematics as measured using the Early Grade Mathematics Assessment (EGMA). The endline study demonstrated that the proportion of children doing mathematics with understanding in the treatment group increased by 72,5% (from 13,7% to 24,2%) while the proportion in the control group increased by 8,7% (from 16,0% to 17,9%). The impact suggests that deliberate and structured daily focus on foundational whole number skills can support the development of children's ability to do mathematics with understanding.

Key words: EGMA, intervention, structured support, whole number

Introduction

To gain insight into student facility with foundational skills and to better understand characteristics among Jordanian schools that are associated with student performance, the Jordan Ministry of Education (MoE), with support from USAID/Jordan, administered a national sample based survey at the end of the 2011/2012 school year that included the Early Grade Mathematics Assessment (EGMA). This paper focuses on the EGMA results and the mathematics component of the intervention activity that followed.

The EGMA is an oral assessment designed to measure a student's foundational skills in mathematics in the early grades. EGMA was developed based on the work of Baroody et al. (2006), Chard et al. (2005), Clements and Samara, (2007), and Foegen et al. (2007).The subtasks assess number identification, quantity discrimination, missing-number identification, word problem solving, as well as addition and subtraction all with whole numbers. There are two addition and subtraction subtasks. In the Level 1 (L1) subtasks children are asked to solve addition/subtraction problems, with sums/differences below 20, without the aid of paper and pencil, the items range from problems with only single digits to problems that involve the bridging of the ten. In the L2 subtasks children are asked to solve addition/subtraction problems that involve the knowledge and application of the basic addition and subtraction facts assessed in the L1 subtask. Students are allowed to use any strategy that they want, including the use of paper and pencil supplied by the assessor. The problems extended to the addition and subtraction of two-digit numbers involving bridging.

In the 2012 National Survey Brombacher et al. (2012) found that although students answered the more procedural addition and subtraction L1 items correctly and with confidence—83.6% for addition and 79.4% for subtraction in grade 2, and 81.6% for addition and 75.9% for subtraction in grade 3—student performance dropped by 31% (in grade 2) and 27% (in grade 3) from L1 addition to L2 addition, and by more than 47% (in grade 2) and 41% (in grade 3) from L1 subtraction to L2 subtraction. The 2012 National Survey also suggested that memorisation plays a large role in the way that children know and learn mathematics.

Following on from the 2012 National Survey, an intervention focused on foundational skills in whole number arithmetic was implemented in 45 treatment schools. In this paper, I report the impact of that intervention activity as measured through an endline study involving the intervention schools and 110 control schools.

Materials and Methods

The intervention program consisted of the introduction of deliberate, structured, and developmentally appropriate daily practice in foundational skills for whole number mathematics in Grade 1 to Grade 3 mathematics lessons.

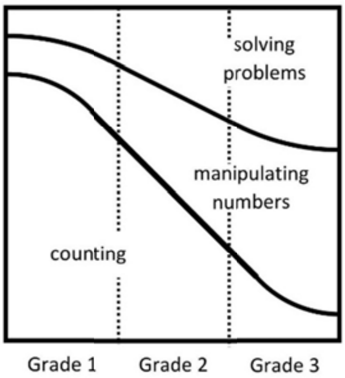
Teachers in treatment schools spent the first 15 minutes of every mathematics lesson revisiting and reinforcing foundational skills. They did so every day, so that the students experienced this activity as part of the classroom program—as a routine "warm-up" activity to the curriculum's lesson for the day. The 15-minute activity's key feature was for it to become part of the daily routine using the same structure every day, with the rationale that as students (and teachers) became familiar with the routine, it would go quickly and not require a large amount of explanation; it would provide both the needed exposure to and the practice with key foundational skills.

In addition to addressing the foundational skills that the 2012 National Survey had identified as being underdeveloped in grade 2 and grade 3 students, the different activities for each of the skills introduced teachers to more research-based pedagogical practices than those that the 2012 National Survey had seen in use in early grade classrooms. However, rather than introducing these practices through direct instruction, the program did so through immersion. By conducting the different activities as part of a daily routine, teachers were implementing more effective pedagogies. As the teachers gained confidence in conducting the activities, it was hoped that they would reflect on what they were doing and would recognise the value of the pedagogies.

The daily routine in mathematics was designed to support children in developing a more robust sense of number, defined in terms of:

- Being able to work fluently and flexibly with numbers and number concepts;
- Having a rich understanding of the meaning of number; and
- Having a wide range of effective strategies for solving a variety of number problems.

The daily mathematics routine in the Jordan pilot intervention involved three distinct but interrelated elements: counting, manipulating numbers and solving



problems with whole numbers. The amount of time allocated to each element of the routine varied according to the developmental state of the children. In Grade 1 the children did more counting and solving problems and less manipulating numbers while in Grade 3 the focus was more on manipulating numbers and problem solving and less so on counting.

The three components of the routine were introduced for the following reasons. Counting supports the development of a sense of number in at least three different ways: counting gives meaning to the number names, counting contributes to the development of a sense of the quantity/amount, and counting is an essential tool for early problem solving. Manipulating of numbers is important as it contributes to the development of both the meaning of number and the ability to estimate, calculate with and solve problems involving numbers. The solving of problems in the daily routine was regarded not only as a purpose for or application of mathematics, but even more so as a way of helping children to develop mathematics in general and mathematical skills in particular. By solving appropriately structured problems, research evidence has shown that children can add, subtract, multiply and divide without formal awareness of these terms or formal methods associated with these operations. The solving of problems also gives meaning and purpose to the doing of mathematics – they help children to experience mathematics as sensible activity.

For each of the three different components of the daily routine there were a series of different activities that the teacher could use. The activities varied in terms of complexity (including number range) depending on the developmental state of the children.

The counting component of the routine included the activities listed below. Taken together the activities create a developmental trajectory that supports the development of counting over the three years from Grade 1 to Grade 3.

- Rote counting activities:
  - Counting in ones, counting rhymes and songs, and counting in steps
- Rational counting activities:
  - Counting small sets of counters in ones
  - Counting out small groups of counters
  - Estimating and counting larger sets of counters in ones
  - Counting in groups
  - Counting large sets of counters in groups

The manipulating numbers component of the routine included the activities listed hereafter. Taken together these activities also create a developmental trajectory that supports the development of number manipulating skills over the three years from Grade 1 to Grade 3. All of the activities are used throughout the three years with the difference across the years being the number range within which the activities are conducted.

- Single digit arithmetic
- Arithmetic with multiples of ten, hundreds and thousands
- Completing tens (hundreds and thousands) including adding and subtracting to and from multiples of ten
- Bridging tens (hundreds and thousand)
- Doubling and halving – to develop efficient division and multiplication strategies
- Interrelated multiplication facts

The problem solving component drew on the literature on the different problem types. The kinds of problem used varied across the three years with the focus in the early months being on problems that develop an understanding of the basic operations and the problems towards the end of the three year cycle focusing on the development of concepts such as fractions as well as ratio, rate and proportion. The problem types used included:

- Problems that support the development of addition and subtraction including:
    - Change, combine, and compare problems
- Problems that support the development of the division concept:
    - Sharing, and grouping
- Problems that support the development of the multiplication concept:
    - Repeated addition, and situation with a grid or array type structure
- Problems that support the development of the following concepts:
    - Fractions, ratio, rate and proportion including sharing in a ratio.

The materials for the pilot intervention in Jordan consisted of three different publications: The teacher manual, the daily lesson notes and a workbook with a daily written activity for the students.

Each of the activities for each of the components of the routine were described in the teacher manual. The teacher manual was used to convey to teachers the purpose of the activities in terms of their role in developing the skills they seek to develop as well as to describe to the teacher how to conduct each activity as part of the daily routine in the classroom. The daily lesson notes indicated to the teacher which activity to do in each of the segments of the daily routine and provided some specific details for those activities for the day. For example, if the activity required students to count a large pile of counters using groups, the lesson notes would suggest to the teacher an appropriate number of counters to be counted and the size of the groups to be used in counting the counters. Finally, there was a set of workbooks for the students. In the workbooks there was a page for each day of the semester with activities that supported the whole number-related focus of the day. The written activities in the workbook matched the activities transacted by the teacher and the class during the fifteen minute routine of activities and, in so doing, reinforced the concepts being developed on the day.

The scope and sequence of activities that the lesson notes and workbook activities were based on represent a developmental trajectory. This trajectory, as well as the daily lesson notes and workbook materials, were developed by the Ministry of Education with the technical support of the author made possible through USAID/Jordan. The process involved a team of individuals representing the Ministry's curriculum development unit, the teacher supervisors and teachers from each of the grades being targeted by the programme. The development process for the materials for the first semester took a little over six months.

The research questions alongside the intervention sought to establish:

- If daily practice of foundational skills through deliberate, structured, and developmentally appropriate activities support children to do mathematics with understanding as measured by EGMA?
- What conditions support teachers in implement the daily routine and the associated activities with fidelity and confidence?

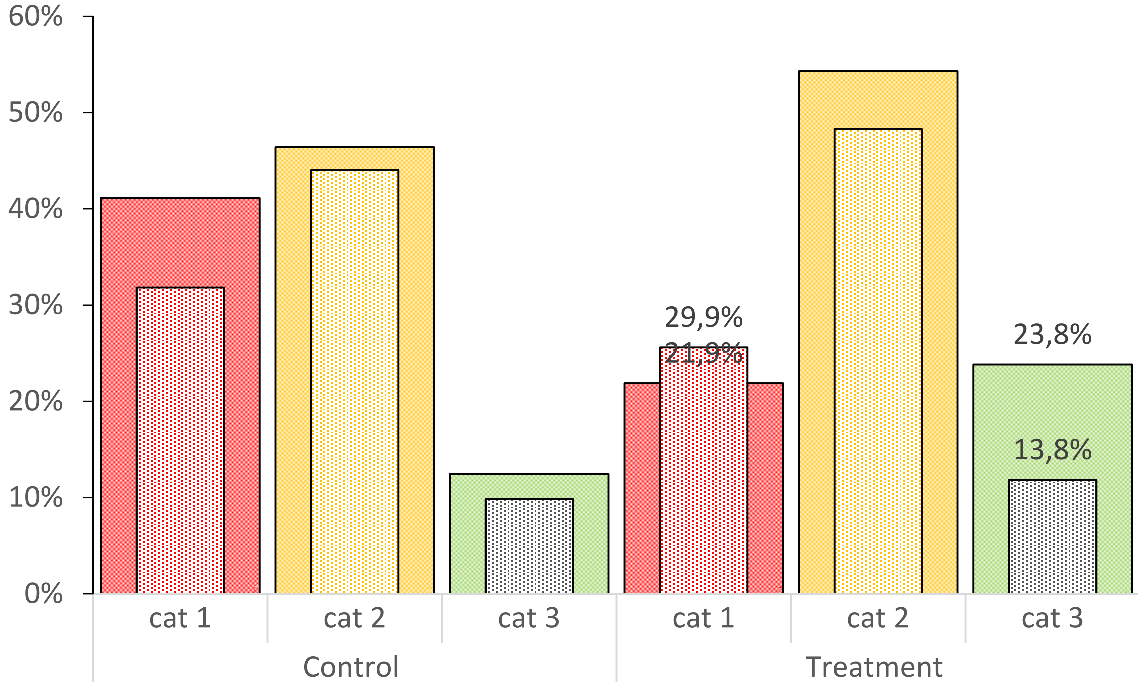
Schools for the intervention pilot were selected from the 2012 National Survey sample such that there were at least two, and preferably four or more, schools in a school district (field directorate) with at least one supervisor available to provide training and support to two schools. A total of 20 MoE supervisors were assigned to the intervention. They were responsible for training more than 300 teachers in the 43 schools across 12 education districts. Training was conducted in two stages: (1) training of trainers and (2) training of teachers. The training of trainers (MoE supervisors) was provided by the author, while the training of teachers was provided by the MoE supervisors. The intervention was implemented during the 2013/2014 school year by more than 400 teachers in 347 classrooms across 43 schools, reaching approximately 12,000 students.

Results

To measure the impact of the intervention pilot, an endline study was conducted in May 2014 once again using the EGMA instrument.

Brombacher et al. (2015) found that overall, the EGMA results indicate that the intervention was successful in raising mathematics achievement in treatment schools. To gain an overall impression, composite mathematics classifications were developed based on the more conceptual and cognitively demanding subtasks of the EGMA. Students were classified as belonging to one of three categories: Non-mathematician or Early mathematician (cat 1) (either missing number and/or addition and subtraction L2 below 30%); Emergent mathematician (cat 2) (missing number and addition and subtraction L2 both above 30%); and Mathematician (cat 3) (missing number and addition and subtraction L2 both above 80%).

Fig. 1 displays the changes in the proportion of students in each of the categories (described in the previous paragraph) in the treatment and control schools from 2012 to 2014 both for treatment and for control schools. This figure provides direct evidence of the overall effectiveness of the intervention. While the percentage of non-mathematicians or early mathematicians remains relatively consistent across years for the control group, there are large reductions in the proportion non-mathematicians or early mathematicians in treatment schools (from 30% to 22%). Additionally, while the proportion of mathematicians remains constant for control schools, the increase of the proportion in the treatment schools (14% to 24%) is statistically significant. In other words, the intervention did exactly what it was intended to do. While there were virtually no gains in control schools from 2012 to 2014, there were significant gains across treatment schools in terms of reducing the proportion of the lowest performers and increasing the proportion of the highest performers. These results are extremely promising, particularly because the intervention was implemented for only one school year.



	Control			Treatment		
	cat 1	cat 2	cat 3	cat 1	cat 2	cat 3
Endline	41,1%	46,4%	12,5%	21,9%	54,3%	23,8%
Baseline	37,1%	51,4%	11,5%	29,9%	56,3%	13,8%

Kev: narrow bar: baseline: wide bar: endline

Fig. 1: Overall treatment effect for Mathematics categories

These results suggest that it is possible to increase the number of mathematicians in early grade classrooms by providing deliberate, structured, and developmentally appropriate practice in foundational skills for reading and mathematics.

This intervention set out to research whether daily practice of foundational skills for could increase the number of students doing mathematics with understanding. From the wide range of evidence collected, including in particular the observations of the teacher coaches, it would appear as if, in general, the intervention was implemented with greater fidelity than not and that it had the desired impact. The implication may well be that there is much to be gained by an intervention that systematically addresses only those key elements of a teaching and learning program that have been shown to be deficient, instead of replacing the entire program.

A hypothesis of this intervention was that if teachers were introduced to more effective pedagogies through immersion, that is, by asking teachers to implement a limited number of carefully structured routines on a regular (daily) basis, teachers would recognise the benefits of the approach and more generally assimilate some of that approach into their teaching. At this stage of the intervention, it is not possible to know to what extent teachers have actually incorporated the intervention practices more generally into their teaching (although there are some anecdotal claims that they have). Nevertheless, it is clear from teachers' responses that they claim to have seen benefits from the intervention. Teachers claim that students enjoyed the intervention activities and that students benefited from the intervention because they appeared to perform mathematics as a result of the intervention activities. Teachers also claim that the intervention exposed them to new and more effective teaching approaches.

Discussion and conclusion

Encouraged by the positive results, it is nonetheless critical to examine the different components of the intervention to see what lessons can be learned—lessons that will inform future interventions and improve their chances of success.

Classroom Support Teachers in the intervention schools received both direct training and school-based support. It is clear from the data that school based support visits contributed to the impact of the intervention In particular each additional school-based support visit is associated with an increase of 0.8% in the proportion of mathematicians in the class. School- and classroom-based support to teachers who implemented this intervention enhanced the success. The more frequent the support, the more effective the implementation was.

Teacher Training Another variable that had a significant impact on the success of the implementation was the proportion of the training that the teachers had attended. Teachers who attended more of the training had a greater proportion of mathematicians in their classes than teachers who attended less training.

Fidelity of Implementation The feedback provided by supervisors about their classroom visits gives a range of different ways of evaluating the fidelity with which teachers implemented the intervention. In particular, supervisors reported about (1) the particular lesson (in the lesson notes) that the teacher was implementing; (2) the extent to which teachers were following the lesson notes as they should have been; (3) whether or not the teacher was actively monitoring student understanding during the lesson; (4) the type of student participation in the lesson; and (5) the extent to which students had worked in their workbooks and teachers had marked the workbooks. In the analysis of the data, all of these variables were positively associated with the intervention's impact. Being on the expected page of the lesson notes was associated with a 15% increase in the percentage of mathematicians in a classroom. 70% of the mathematics classrooms in which teachers followed the teacher guide and lesson notes were among the top performing classrooms. Classrooms, where teachers encouraged student participation, were more likely to be among the top performing classrooms for classrooms where students were not actively encouraged to participate in the lessons, not a single classroom was in the top performing districts.

The results of the National Intervention Pilot for Early Grade and Mathematics suggest that the results of the 2012 National Survey say more about the way that children are learning than about their ability to learn or not. The intervention results provide encouragement that through deliberately structured and focused attention to foundational skills in mathematics, children's performance can be dramatically improved. The Intervention Pilot also provides a wealth of information on the role of key variables and factors that need to be considered as the activities of the intervention are both incorporated into national policies and taken to scale.

Acknowledgements

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CHARACTERISTICS OF MULTIPLICATION TEACHING OF WHOLE NUMBERS IN CHINA: THE APPLICATION OF NINE TIMES TABLE

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Abstract

A tradition of Chinese mathematics teaching is stressing on the basic knowledge and basic skills. Most of the Chinese students are able to perform multiplication accurately in a short time. One of the key reasons is that "nine times table" is applied in multiplication successfully. In order to investigate how "nine times table" is taught in China, the primary mathematics textbook and mathematics teachers' manual published by People's Education Press, and a typical classroom instruction were analysed in this study. There are three main findings: Firstly, "nine times table" is divided into several teaching plots. Teachers taught it individually at first, than taught all of them together. Secondly, there is a high level of teaching objects of "nine times table". Students were asked to recite it proficiently. Thirdly, teachers usually take advantage of rhythmical image of the Table and help the students recite the Table based on having a good understanding of it in variety of ways.

Key words: characteristics, mathematics curriculum China, multiplication teaching, nine times table

Introduction

Chinese mathematics teaching stresses the learning of basic knowledge and basic skills. *Two Basic Teaching* is considered to be one of the properties of Chinese mathematics education (Fan, 2004). Chinese students have high-level calculation ability and they are able to calculate rapidly and accurately (Cai, 1998; Silver and Kenney, 2000). They perform very well on tasks requiring computation and applications of formulas (Cai, 2002). There is a crucial factor contributing to this phenomenon: "nine times table" which is central to multiplication teaching. In the early 1980's, there was an increasing public voice that the "times table" must continue to hold its place in a basic curriculum (Trivett, 1980). Kay (2014) pointed out that there is simply no way that a typical student can solve multiplication problems quickly without memorising the multiplication table. Schon, Ebner and Kothmeier (2012) mentioned that learning the multiplication table is a central subject of mathematics in primary school. To examine how "nine times table" is taught and whether teachers are able to help students to recite it thoroughly, the mathematics curriculum, mathematics textbooks and teachers' manuals for primary school are analysed in this study to answer these questions.

"Nine Times Table"

"Nine times table" begins with *one one beget one* and end with *nine nines beget eighty-one,* totally 45 sentences. It benefits from words of figures in Chinese characters which are monosyllabic words and there are 5 syllables per formula at most. Therefore, the times table is read with a catchy rhyme and is full of rhythmical image. This is one of the important reasons for Chinese primary school students to memorise it easily.

As shown in the Tab. 1, the first sentence of each column is always two same numbers multiply together. For example, the first sentence of the third column is *three threes beget nine*. In addition, from the very beginning of the second column, each column decreases one sentence than the previous column. At the same time, the first number of each column is the former number of the previous column while the second number stays the same. Last but not least, the first number remained the same while the second number increase to nine regularly in the same column formulas. For instance, the seventh column of the sentences is as follows: *seven sevens beget forty-nine, seven eights beget fifty-six, seven nines beget sixty-three*.

一一得一 (1×1=1)								
一二得二 (1×2=2)	二二得二 (2×2=4)							
一三得三 (1×3=3)	二三得六 (2×3=6)	三三得九 (3×3=9)						
一四得四 (1×4=4)	二四得八 (2×4=8)	三四十二 (3×4=12)	四四十六 (4×4=16)					
一五得五 (1×5=5)	二五一十 (2×5=10)	三五十五 (3×5=15)	四五二十 (4×5=20)	五五二十五 (5×5=25)				
一六得六 (1×6=6)	二六十二 (2×6=12)	三六十八 (3×6=18)	四六二十四 (4×6=24)	五六三十 (5×6=30)	六六三十六 (6×6=36)			
一七得七 (1×7=7)	二七十四 (2×7=14)	三七二十一 (3×7=21)	四七二十八 (4×7=28)	五七三十五 (5×7=35)	六七四十二 (6×7=42)	七七四十九 (7×7=49)		
一八得八 (1×8=8)	二八十六 (2×8=16)	三八二十四 (3×8=24)	四八三十二 (4×8=32)	五八四十 (5×8=40)	六八四十八 (6×8=48)	七八五十六 (7×8=56)	八八六十四 (8×8=64)	
一九得九 (1×9=9)	二九十八 (2×9=18)	三九二十七 (3×9=27)	四九三十六 (4×9=36)	五九四十五 (5×9=45)	六九五十四 (6×9=54)	七九六十三 (7×9=63)	八九七十二 (8×9=72)	九九八十一 (9×9=81)

Tab. 1: The "nine times table" in elementary school mathematics textbooks

The Division of Teaching Plots of "Nine Times Table"

"Nine times table" includes formulas of nine numbers, from one-times table to nine-times table. The writers of mathematics textbook for primary students divided it into six teaching plots, according the length of each formulas and the level of students' knowledge and experiences. A common way is to present these six teaching plots separately at first and afterwards put them all together. The six teaching plots and their sequence are as followed: pithy formulas of five, pithy formulas of two, three, and four, pithy formulas of six, pithy formulas of seven, pithy formulas of eight and pithy formulas of nine. Their contents and the number of each plot are listed in Tab. 2.

As it is shown in Tab. 2, the numbers of divided teaching segments are roughly equivalent, ranging from five sentences to nine, with differences in relative equilibrium. Pithy formulas of one—*one one beget one* haven't been compiled as an independent teaching plots in the textbook. The reason is that the content of pithy formulas of one is easy to understand and it is short, which has only one sentence. So, students do not need to study it specifically. At last, besides the pithy formulas of five, the pithy formulas of the rest numbers are taught according to the logic sequence from small too big.

Name	Content	Quantity
No.1 Pithy Formulas of Five	one five beget five two fives beget ten three fives beget fifty four fives beget twenty five fives beget twenty-five;	5
No.2 Pithy Formulas of Two, Three and Four	one two beget two two twos beget four; one three beget three two threes beget six three threes beget nine; one four beget four two fours beget eight three fours beget twelve four fours beget sixteen;	9
No.3 Pithy Formulas of Six	one times six beget six two sixes beget twelve three sixes beget eighteen four sixes beget twenty-four five sixes beget thirty six sixes beget thirty-six；	6
No.4 Pithy Formulas of Seven	one sevens beget seven two sevens beget fourteen three sevens beget twenty-one four sevens beget twenty-eighty five sevens beget thirty-five six sevens beget forty-two seven sevens beget forty-nine；	7
No.5 Pithy Formulas of Eight	one eight beget eight two eights beget sixteen three eights beget twenty-four four eights beget thirty-two five eights beget forty six eights beget forty-eighty seven eights beget fifty-six eight eights beget sixty-four；	8
No.6 Pithy Formulas of Nine	one times nine beget nine two nines beget eighteen three t nines beget twenty-seven four nines beget thirty-six five nines beget forty-four six nines beget fifty-four seven nines beget sixty-three eighty nines beget seventy-two nine nines beget eighty-one;	9

Tab. 2: The content and sequence of each teaching plots of the “nine times table”

The curriculum expects from experience of taking five as counting unit, that students can count five, ten, fifty,...easily, so teaching pithy formulas of five primarily is more suitable for student's life experience and prior knowledge base and is able to lay a solid foundation for further learning of the sentences of the rest of the numbers.

The number of sentences increase gradually, when the students have mastered pithy formulas of five, they will further study pithy formulas of two, pithy formulas of three, pithy formulas of four and pithy formulas of six. After a unit of learning, students will master the six times table which contains pithy formulas from one to six. At this time, teachers normally require students to summarise what has learned so far with the goal to form a six times table including numbers from one to six. Until students have finished learning pithy formulas of seven, pithy formulas of eight and pithy formulas of nine, teachers will ask students to summarise what they have learned to master the whole “nine times table”. Then, the learning of the “nine times table” is completed. In other words, at this time students will be required to master the whole table other than pithy formulas of a single number. Through these strategies, students finally master the “nine times table” which further provides students the foundation to learn multiplication and division of the whole numbers as well as decimal in later grades.

The Arrangement of Teaching Time

Generally speaking, the “nine times table” is taught in the first term of grade two in primary school. Usually, the textbook spares two units to write the content of the “nine times table”. Before the learning of the multiplication, students learn the addition and subtraction of whole numbers less than 100. Taking the textbook published by People's Education Press as an example, Tab. 3 summarises how the “nine times table” is organised and how much time is devoted for its teaching.

As shown in Tab. 3, students start to learn multiplication at an earlier grade and start to learn the “nine times table” immediately after they have some preliminary understanding of multiplication and the basic cognition of the meaning of multiplication. The learning of the “nine times table” lays the foundation of learning multiplication knowledge and obtaining multiplicative computation ability. Not only is the “nine times table” divided to two units to study, but also the two concerning units was separated by the content of graphics and geometry which is called observing objects to ease the students' memorisation pressure.

In general, the total number of teaching hours in a semester are approximately 60 hours, of which 20 hours, namely 33.3% of the total number are devoted for the learning of the “nine times table”. The aim of the last part of each “nine times table” learning unit is not only to learn new content but also to summarise all the contents in the unit, so it takes more time. The rest of the parts of those units take two or three teaching hours. The numbers of class hours are almost positively correlated with the amount and difficulty of the content. The teaching hours in Tab. 3 are only suggested by the textbooks writers and teachers may alter them according to the real situation of teaching progress and students' academic background and understanding.

Grade	Unit	Content	Recommended Lessons
The First Semester of Grade Two	The Fourth Unit Multiplication Table（1）	A Brief Introduction of Multiplication	3
		Pithy Formulas of Five	1
		Pithy Formulas of Two Pithy Formulas of Three Pithy Formulas of Four	3
		Pithy Formulas of Six	4
		Reflection and Review	1
	The Sixth Unit Multiplication Table（2）	Pithy Formulas of Seven	2
		Pithy Formulas of Eight	3
		Pithy Formulas of Nine	4
		Reflection and Review	2

Tab. 3: Content organisation and arrangement of teaching times
Note: Each lesson has 40 minutes.

Teaching Objects

As early as 1902, China had formulated a national mathematics curriculum standard which was used as the guideline for textbook compiling, teaching, and examination. The current practiced curriculum standard known as the *Mathematics Curriculum Standard for Compulsory Education (2011 Edition)* was formulated by the Ministry of Education and published in 2011. In this standard, for the "nine times table", it is described that students should be able to do oral calculation of multiplication fluently. This is a high level requirement, on one hand, taking the learning of the "nine times table" as the foundation and precondition of further learning of whole numbers multiplication and division, and even real numbers multiplication and division. On the other hand, students are required to achieve the level of fluency in oral calculation. That is to say, students need to compute intuitively without any assist, such as pen and paper, or calculator. They are capable of calculating results correctly after a short while of consideration, and they even reached the level of automatic extraction, that is to say, getting the answer immediately after reading the arithmetical calculating.

A Typical Classroom Instruction

Although teachers' classroom teaching methods are quite different, the basic steps of classroom teaching of the "nine times table" are quite similar, all of which include the process of the sentences formation, understanding the origin and significance of the sentences, memorising the sentences, and applying the sentences to calculate. The process of the classroom teaching of “nine times table” will be shown with Pithy Formulas of Seven (the first class) as an example below. The lesson was taught by a teacher from Chongqing and the mathematics textbook used by the teacher is published by People’s Education Press. There were five teaching steps in this lesson including reviewing pithy formulas of six, exploring, cognising, memorising, applying the pithy formulas of seven. The main activities and its teaching time used are summarised in Tab. 4.

In every teaching step, the teacher played an important role in activating students’ enthusiasm, and arousing students’ positivity, and he arranged many interesting activities. Taking the step of memorising the pithy formulas of seven as an example, in order to help students learn the sentences through diversified activities, and strengthen the memorising of the sentences, the following activities were designed by the teacher: reading the sentences, sampling some students reciting, reciting the sentences between desk mates, and guessing the sentences in disrupted order. At the stage of memorising the pithy formulas of seven, the following assignments were arranged: practicing an exercise in the textbook, answering questions from a popular cartoon character named Yangyang Xi. These activities were lively and vivid, which could deepen students’ understanding of the pithy formulas of seven, and also could make the students feel the accuracy and validity of the sentences in solving problems.

In the entire teaching activities, teachers paid attention to the understanding, exploration and comprehension of the sentences, and emphasised particularly on the firmly memorising and fluently using the sentences. At the end of this class, teachers demanded the students to recite the sentences to their families, which could make students review the sentences in time and memorise the sentences deeply. As for the time spent on each step, the focus of this lesson were exploring, understanding, and then preliminarily memorising, simply applying the pithy formulas of seven, which are the core of “nine times table” teaching stated above.

This lesson is a typical example of the “nine times table” teaching, and the teaching process fully reflects the characteristics of the “nine times table” teaching. The teacher tried to make his students actively involved through designing diversified teaching activities, improve the learning requirement gradually to a high level of difficulty. In addition, he created a positive atmosphere which made students memorise the sentences gradually, pay attention to the comprehensively memorising. He also made his students participant in the process of compiling the sentences and made his students realise the advantages of solving the calculation problem with the sentences, and avoided students’ rote memorisation.

Teaching Steps		Time Spent （Min）	Percent age
Content	**Main Activities**		
Reviewing the Pithy Formulas of Six	Practicing some exercises about Pithy Formulas of Six	2	5%
Exploring the Pithy Formulas of Seven	Playing a seven-piece puzzle and sharing with their patterns； Filling the table with the help of seven-piece puzzle; Writing multiplication equations of number seven according to the filled table; Compiling Pithy Formulas of Seven with the help of the written multiplication equations;	17	42.5%
Understanding the Pithy Formulas of Seven	Exploring the rules of Pithy Formulas of Seven; Understanding Pithy Formulas of Seven from different angles;	7	17.5%
Memorising the Pithy Formulas of Seven	Reading the sentences ; Reciting the sentences through various activities ;	8	20%
Applying the Pithy Formulas of Seven	Using the sentences to calculate; Using the sentences to solve problems in real life;	6	15%

Tab. 4: Time spent on each teaching steps in Pithy Formulas of Seven (the first class)

Conclusion

Chinese mathematics teaching emphasises the base and its training is the essence of Chinese mathematics education. After a long time of development, many educational traditions with Chinese characteristics have formed. The teaching of “nine times table” is one of the outstanding cases. Learning of the “nine times table” helps to improve students’ computational capability, which further lays a solid foundation for their future study and application in daily life.

The “nine times table” is not only the basis for students’ future learning, such as the learning of integer multiplication, but also the foundation of learning division.

The learning of “nine times table” not only happens in mathematics curriculum, but also happens in the social and cultural life frequently. For example, there is a famous Chinese saying, *regardless whether three sevens beget twenty-one*, to express the meaning of acting decisively and using *having the “nine times table” in mind* to describe a person who is astute. In the literary works of *Journey to the West*, there has the depicts like Tang Priest and his apprentices experienced *nine nines beget eighty-one* sufferings during the journey to the west and so on. What’s more, some parents require their children to memorise the “nine times table” as a part of family enlightenment education. Therefore, some students already know the “nine times table” before their formal education. The “nine times table" has been a part of Chinese mathematics culture, which is integrated into Chinese social life.

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TEACHERS' RESPONSES TO INCORRECT ANSWERS ON MISSING NUMBER PROBLEMS IN SOUTH AFRICA

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Abstract

This paper examines differences in how three Grade 3 South African teachers responded to students' incorrect answers in whole class teaching of the part-whole relationship in additive missing number problems. Nine video recorded lessons, taught by three teachers, were analysed, with attention paid to teaching episodes containing incorrect students' answers. The variation theoretical analyses indicated differences in the ways teachers responded to incorrect answers. We argue that different ways of responding to incorrect answers may provide different learning possibilities.

Key words: addition and subtraction, incorrect answers, part-whole relationship, primary mathematics, South Africa, variation theory

Background and Theoretical Framing of the Study

Classroom interaction has been the subject of study in several research projects. The common initiation-response-evaluation (IRE) interactions (Mehan, 1979) have been studied to analyse how teachers react to and evaluate students' responses. Different, and more productive ways of reacting to students' incorrect answers have also been demonstrated, involving evaluation of incorrect answers by opening up for class discussion, or explicitly pointing out the incorrect answer, or remaining neutral and asking follow up questions (Chin, 2006). In the current study, our interest in teachers' responses to incorrect answers was driven by wanting to understand the role of different responses in students' learning possibilities. Where interaction patterns delimit students' contribution to correct/incorrect answers only, the cognitive demand level tends to be low (Lampert, 1990). Furthermore, if interaction patterns do not allow for students' arguing and reflecting upon incorrect answers, students have less opportunity to think reasonably. While there are studies in South Africa that have looked at teachers' handling of errors (e.g. Brodie, 2010), the empirical sites for these studies have usually been within secondary or tertiary mathematics. At the same time though, several recent studies have noted problems within *Whole Number Arithmetic* (**WNA**) teaching in South Africa - with some notable absences in offers of criteria for deciding on correctness of learner offers (Hoadley, 2006), and others pointing to lack of support for progression in teachers' handling of learner offers (Venkat, 2013). Taken together, these studies point to the usefulness of attention to how teachers work with learner errors in the context of **WNA** in South Africa. To examine the teachers' responses to incorrect answers and relate that to learning possibilities in this study, our analysis of the data was framed within a variation theoretical perspective (Marton and Booth, 1997). One fundamental assumption within this framework is that the experience of difference is a necessary condition of learning to discern. We do not learn by experiencing similarities but by being aware of how instances differ. So, for example, when it comes to generalisation, the experience of differences must come before the experience of sameness (Marton, 2015). Thus, possibilities to experience contrasts and differences are important for learning. For instance, the effectiveness of instruction presenting examples simultaneously and contrasting them was found in a comparative study of teaching and learning the equal sign (Hattikudur and Alibali, 2010).

While this points to the need to plan in advance for learners' experience of contrasts, incorrect answers in the lesson provide teachers with 'on-the spot'-opportunities to contrast incorrect answers or answers that are not the expected, with alternative suggested answers. So, incorrect answers can be seen as a source that teachers can use to make differences discernible and thus, widening the space of learning. We assume that it is more likely that incorrect answers can become a contrast and hence, provide greater potential for learning, if the teacher responds to offers as 'possible' answers and juxtaposes and explicitly contrasts them. The aim of this paper is to describe the differences in teachers' ways of responding to students' incorrect answers and to discuss what that may imply for students' learning of part-whole-relationship in additive missing number problems in the context of South Africa.

Design and Data Analysis

This paper draws on data from a small scale intervention project carried out in a government suburban primary school in Johannesburg, with Grade 3 class sizes of 35+ and English as the language of teaching. Three Grade 3 teachers worked cooperatively with a research team in a professional development project similar to lesson study (Yoshida, 1999), i.e. planning, enacting, observing and evaluating lessons. The study lasted one year and included three cycles of three lessons, each over a three week period. For the purposes of this paper we sampled nine video recorded lessons (three from each teacher) out of a total number of 27 lessons. Since we were also interested in seeing shift over time, lessons were sampled from the beginning, the middle and the end of the project. The dataset included one lesson in February 2013, one lesson in October 2013 and another in February 2014. The lessons focused on a more 'structural' approach to additive relations teaching that contrasted with the more operational approach prevalent in South Africa (see Venkat, Ekdahl and Runesson, 2014 for more detailed discussion). The aim was to support learners in solving problems involving missing addends/ subtrahends/start numbers (e. g. $8 + \square = 11,\ 11 - \square = 3,\ 8 = \square - 3$). In these nine lessons teachers mainly worked with problems in number sentence format but also linked to a triad diagram - a representation which pushes towards a more structural approach to additive relations. In the jointly planned lessons different activities on this theme were handled, some designed by the research team, some in collaboration with the teachers. The lessons focused on part-whole relationship tasks, for example, identification of parts and whole and evaluating the missing number in given problems transformation of missing number problems in number sentence form into a triad diagram. Carefully contrasted additive relations and operations problems like: $11 - \square = 3 \ and \ 3 = 11 - \square$ were used within the lessons.

Lessons focused on these themes, including students' incorrect answers, were selected for analysis. Only teacher's reactions to incorrect answers in whole class teaching were analysed. To identify the different kinds of reactions to incorrect answers we looked at the task, the teachers' questions, the students' answers in the episode (could include both incorrect and correct answers) and the teacher's responses. In our analysis we used principles from variation theory (Marton and Booth, 1997) as analytical tools to examine learning possibilities. Variation theory states that learner's attention must be drawn to that which is intended to be learned by foregrounding this and opening it up as a dimension of variation, or by making a contrast. So, depending on what is compared to what, what is opened up as alternatives/variation, what is foregrounded and varied against a stable background, different things are made possible to learn. The analysis consisted of steps to identify what was foregrounded and backgrounded in teachers' reactions, the incorrect or the correct answer, or both of them. Additionally, how different answers were simultaneously possible for the learners to discern and how they were juxtaposed and contrasted by the teachers. Finally we analysed what was made possible to learn about part-whole relationships and missing number problems, depending on the different ways of responding to incorrect answers and the variation / contrast that was made possible to discern.

Results

From the analysis three different ways in which teachers responded to students' incorrect answers emerged. It is suggested that these offer different possibilities for learning about part-whole relationships used in missing number problems.

1. Producing the correct answer

The most frequent reaction from the teachers was to foreground the correct answer and background the incorrect. This could be seen when teachers reacted by ignoring the incorrect response or blamed students for not listening or thinking, e.g. '*Think before your raise up your hand*!' *(Teacher B, February, 2014)* or alternatively: '*Remember we are helping each other so we can come up with a correct answer*' *(Teacher A, Oct, 2013)*. Sometimes a student's correct answer was praised orally by the teacher and the class: '*Well done, well done!*' or with applause. Generally, students produced one or two incorrect answers before the correct answer was offered by another student. While the phrase '*Is she/ he correct?*' – was often utilized; only '*yes/no'* answers were taken in response. No contrast between the examples was made by the teachers and no explanation for the correct and expected answer was given or encouraged. Excerpt 1 illustrates an episode where the teacher reinforces the correct answer. The purpose was to find the missing start number in a missing number problem.

			On the board
1	Teacher:	What is the missing number? (written on the board)	
2	Student writes 12 as the missing number.		$\square - 2 = 7$
3	Class:	Oh! (reactions)	$12 - 2 = 7$
4	Teacher:	So, she is correct?	
5	Class:	No!	
6	Teacher:	(rubs out 12) Someone else?	$\square - 2 = 7$
7		Another student goes to the board and writes 9.	$9 - 2 = 7$
8	Teacher:	Did he get it correct?	
9	Class:	Yes!	
10	Teacher:	(Points to the number sentence) 9 take away 2 is equal to 7.	

Excerpt 1: Teacher A, February 2013

The excerpt shows that the teacher backgrounded the incorrect answer by initially asking the class to evaluate the first student's incorrect answer without asking for the reason why it was incorrect (line 4-5). By rubbing out the incorrect answer without explanations, the teacher closed the possibility for the students to understand why the answer was incorrect (line 6). The reaction to the correct answer produced by the second student (line 7) shows how the teacher just focused on the correct answer (9) and the incorrect (12) was backgrounded, therefore no contrast between these values was made. Since only a variation of correct and incorrect numbers were present, it was made possible to discern that 9 is the missing number in that specific number sentence (line 10) but not *why* 9 is the correct missing number. Furthermore, since no variation of rationales underlying the suggested answers was espoused this was not possible to learn. The only thing made possible to learn in these episodes was that there are correct or incorrect answers to missing number problems.

2. Focusing on either the incorrect or the correct answer

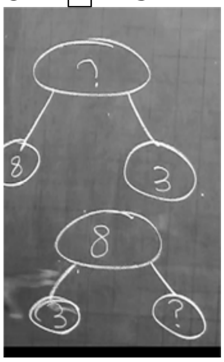
A different way to react to students' incorrect answers was to incorporate some focus on either the incorrect or the correct answer. Teachers responded to students' answers by using questions and explanations, providing openings for students to understand why one specific answer was incorrect/correct. Most frequently in this category, the correct answer was foregrounded with some rationale and the incorrect answer was left in the background. Teachers did not respond to the answers as two possible answers. Infrequently, an incorrect answer was foregrounded, with teachers following up to find out why a learner had given a wrong answer. Teacher reactions opened up reasoning about the part-whole relationship, operations and relations between numbers, related to the incorrect answer. Here the correct answer was backgrounded, not compared to the incorrect. The teacher seemingly took for granted that the students understood that the other produced answer was correct. Within these episodes multiple answers or values were not juxtaposed or explicitly contrasted and thus, no reasoning about differences between produced answers was possible. In episodes foregrounding incorrect *or* correct answers, in addition to making it possible to learn that there were correct and incorrect answers to missing part/ whole number problems, it was also possible to learn why an answer was incorrect/correct. However, there were limited possibilities to learn why other answers produced were incorrect/correct, because the answers were not explicitly contrasted to other answers.

3. Juxtaposing different answers

A variation of incorrect and correct answers was presented in these episodes. Here teachers reacted to more than one produced answer simultaneously. In Excerpt 2 the teacher asked the students to transform a missing number problem in number sentence form into a triad diagram.

On the board

$8 = \square - 3$



1	Teacher:	8equals mmm minus 3. (Draws a triad diagram below the number sentence). Someone put it here (points to the triad diagram) Don't put the answer, instead a question mark.
2	A student goes to the board and writes a correct answer.→	
3	Teacher:	Is it wrong?
4	Class:	Yes / No /mumble
5	Teacher:	Come and show us how you want it to be! (Draws a new triad) Let him do it, we want to see it.
6		Next student goes to the board, writes an incorrect answer. →
7	Teacher:	Is this correct (points to the triad)?
8	Class:	No
9	Teacher:	Why are you saying it is wrong? Explain! Don't say it is wrong because you don't want to tell me the reason. Why are you saying it is wrong?
10	Student:	Because it is not the whole, it is a part.
11	Teacher:	Yes, because it is not the whole it is a part. Why are you saying it is a part?
12	Student:	(inaudible)... get 11.
13	Teacher:	What are you getting 11 from? 11 is not here!
14	Student:	Because a whole (inaudible) of two parts (inaudible)
15	Teacher:	Explain, explain! This one (points to the first answer) puts the question mark here (points to the question mark). It´s supposed to be the ....?
16	Class:	Whole (just a few students answer)
17	Teacher:	Then he puts 8 here (points to the first example, left circle) and 3 here (points to the right) which means... we are looking for what?
18	Class:	Whole
19	Teacher:	This one (points to the correct example) is looking for the whole. And this one (points to the second, incorrect example) have 8 here and 3 here and a question mark here, which means this one is looking for a...?
20	Class:	Part
21	Teacher:	A part. But you say this one is wrong (points to the second example) and this one is correct (Points to the first) Why are you saying this one is correct?
22	Student:	Because.... (inaudible)
23	Teacher:	Why are you adding 8 and 3?
24	Student:	Because you say mmm minus 3.
25	Teacher:	We are subtracting 3 from ... (points to the number sentence), mmm minus 3, which means we are subtracting 3 from a whole.

Excerpt 2: Teacher B, October 2013

In this episode the teacher opened up for contrasting correct and incorrect answers, when asking the class if the correct given answer was *wrong* (line 3). Noticing the students' hesitation (line 4), the teacher encouraged another answer (line 5) and thereby opened up space for a variation of answers. Keeping both options visible and juxtaposed, both incorrect and correct answers were foregrounded, contrasted and compared. She then started a discussion based on the two simultaneously visible correct and incorrect triad diagrams (line 9) and invited the students to reason and argue for their answers (lines 9, 15, 21 and 23) and emphasised and compared different answers and utterances. Thereafter she revoiced offered explanations referring to relations between numbers and connections within and between representations (lines 17, 19, 21 and 25) – with explicit contrast and comparison in line19. Rationales for students' answers, made it more explicitly possible to discern part-whole relationship ideas.

In Excerpt 3 the response to incorrect answers and the task looked a bit different. The students were supposed to look for similarities and differences between two presented missing numbers problems written on a piece of paper.

1	Teacher:	Let's talk about the problems. There are two number sentences.	<u>On the board</u>
2	Students:	A whole is missing.	
3	Teacher:	A whole is missing? Are we missing a whole?	$11 - 6 = \square$ $\square = 11 - 6$
4	Class:	No / Part	
5	Teacher:	What are we missing?	
6	Class:	A part	
7	Teacher:	A part, okay we find out! Here on these two problems, what is the action?	
8	Student:	Take away	
9	Teacher:	We are taking away? What and what or from what? The operation is?	
10	Class:	11 minus 6	
11	Teacher:	And the operation here is what? (inaudible) What is happening, the action?	
12	Student:	6 take away 11	
13	Teacher:	Where is that? You are going to take 6 from 11, here (points) is that action. The action is 11 minus 6, understand? And here? (underlines the operation) No difference. So the answer you will get there is also the answer here! (Marks both number sentences). What is that answer?	
14	Student:	5	$11 - 6 = 5$
15	Teacher:	(writes 5) What is the whole?	$5 = 11 - 6$
16	Class:	11	
17	Teacher:	Yes 11. A bigger number, take away a part and left with another part. Here what is the answer? Look at the operation first.	

Excerpt 3: Teacher C, February 2014

When the students presented two different answers (line 2 and 4), the incorrect and correct answers were foregrounded simultaneously. The teacher got an opportunity to make un-planned differences discernible and started with a joint discussion (line 7) about the mathematical idea of operation as an action. In the reasoning offered, the answer itself (missing part or missing whole) is not in focus. Instead the differences and similarity between the two number sentences were made the focus (line 13). In comparing the number sentences the teacher underlined the operations drawing attention to the similarities. When the class had produced the missing number (line 14), the teacher returned to the question about the missing whole or missing part, connecting to the operation (line 15 and 17). Her response made it possible to learn what answer is incorrect and correct in terms of the relation between the three numbers, the meaning of the operation in the missing problem and identification of the whole and the parts.

These two teaching episodes involved contrast and provided simultaneity of discernment of incorrect and correct answers. In Excerpt 2 the teacher gave the learners possibility to discern and contrast both incorrect and correct answers, by leaving both on the board. In the other excerpt (Excerpt 3) while simultaneity was not as obvious in written form, in teacher's explanations, reasoning and pointing out, both incorrect and correct answers were compared and contrasted. In this way of responding to students' incorrect answers it is possible for learners to discriminate between correct and incorrect answers and also discern why one answer is incorrect and another correct. In the instances seen here, juxtaposing, and comparing and contrasting offers provided possibilities to learn about structural relations between numbers in missing number problems.

Overview comments

Based on the data gathered from the nine lessons on the different ways in which the teachers responded to the incorrect answers, the following categories were seen. Tab.1 includes the three lessons from Teacher A, B and C in February 2013, October 2013 and February 2014, the instances of each way of respond to incorrect answers (I, II, III) from the total instances of incorrect answers in each lesson (in brackets).

	February 2013 (n=14)			**October 2013 (n=19)**			**February 2014 (n=16)**		
	I	**II**	**III**	**I**	**II**	**III**	**I**	**II**	**III**
Teacher A	4(5)	1(5)	0(5)	5(8)	3(8)	0(8)	2(2)	0(2)	0(2)
Teacher B	1(2)	1(2)	0(2)	2(7)	4(7)	1(7)	5(8)	2(8)	1(8)
Teacher C	3(7)	3(7)	1(7)	1(4)	1(4)	2(4)	1(6)	2(6)	3(6)
Total	**8(14)**	**5(14)**	**1(14)**	**8(19)**	**8(19)**	**3(19)**	**8(16)**	**4(16)**	**4(16)**

Tab. 1: Summary of the instances of different ways of responding in the nine lessons

From this overview it would appear that there are some small shifts over time in teachers' way of responding to missing number problems, with increased instances of the more productive Type II and Type III responses. Further analyses of this data and the other 18 lessons in the three cycles are ongoing.

Conclusions

This study contributes to a discussion about the different ways in which teachers respond to students' incorrect answers in additive missing number problems and the implications these different ways may have on students' learning. Most frequently, teachers foregrounded the correct answer and left the incorrect answer unnoticed; a similar pattern seen in other studies (e.g. Mehan, 1979). We noticed that when the production of a correct answer was the focus, the students were just able to see if the answer was correct or incorrect. This might affect negatively on students' learning since they could not see the rationales behind the answers. However, even if a variation of incorrect and correct answers were present in a majority of these episodes, they were not contrasted by the teacher. Explicitly contrasting different answers, which is seen as a necessary condition for learning from a variation theory perspective (Marton, 2015), was seen in a few of the episodes only. However episodes, where either the incorrect or the correct answer was focused, must be considered as better options for learning than just affirming or negating an answer. Still, from a variation theory perspective, this gives limited possibility to discern the logic underlying the different answers. One can argue that the different ways of contrasting incorrect and correct answers effectively can be derived from the teachers' own mathematical knowledge base. When the teacher is more confident with the content, she might be more willing to handle learners' different offers. However in this study we do not specifically study how teachers' understanding of the topic impacts on the way of responding to incorrect answers. Finally, the results of this study raise questions about how teachers' responses to incorrect answers might affect students' learning of *Whole Number Arithmetic* (**WNA**) in South Africa. If teachers are unwilling to direct the students' attention to incorrect answers, and therefore background them, it might be a challenge to change teachers' attitudes towards how to respond to incorrect answers and help them to see these as a source for students' learning.

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INSIGHTS AND IMPLICATIONS ABOUT THE WHOLE NUMBER KNOWLEDGE OF GRADE 1 TO GRADE 4 CHILDREN

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Abstract

This paper provides a snapshot of the whole number knowledge of nearly 2000 Australian primary school children gained through a one-to-one assessment interview. The interview corresponds to a growth point framework that describes learning trajectories in counting, place value, addition and subtraction strategies, and multiplication and division strategies. The findings highlight the broad distribution of growth points in each domain for each grade level, and the wide distance between the lowest and highest growth points in each grade and domain. This demonstrates the complexity of classroom teaching and highlights the challenge of meeting each student's learning needs. Three issues related to children's Whole Number Arithmetic (WNA) emerged from the data. These suggest important themes for teacher professional learning and for refining mathematics curriculum and include: (1) interpreting 2-digit and 3-digit numbers; (2) using reasoning strategies as opposed to counting strategies in addition and subtraction; and (3) strategies for solving partially modelled and abstract problems in multiplication and division.

Key Words: arithmetic strategies, assessment interviews, teaching strategies

Introduction

Education outcomes for Australian children living in low Socio-Economic Status (SES) communities and Aboriginal and Torres Strait Islander communities are lower than for children not living in these communities (Commonwealth of Australia, 2008). An initiative launched by the previous Australian Government to address this issue in mathematics was a series of projects focused on how to close the numeracy gap for Australian children. This paper draws on the findings of one project, *Bridging the Numeracy Gap in low SES and Aboriginal Communities* (Gervasoni et al., 2011) that involved 44 schools in south-eastern Australia and Western Australia. Key approaches for improving mathematics learning in this study were: one-to-one interview-based mathematics assessments (Clarke et al., 2002; Gervasoni, Hadden and Turkenburg, 2007); using this data to guide instruction and curriculum development at individual, class and whole school levels (Gervasoni and Sullivan, 2007); and using the *Extending Mathematical Understanding* Program (Gervasoni, 2004) in the second year of school to provide intensive specialised instruction for those who were mathematically vulnerable. This paper reports on one aspect of this project; using the interview-based assessment and framework of growth points to gain insight about primary school children's whole number knowledge. Based on the insights gained, implications for WNA teaching and teacher professional learning will be discussed with the view to enhancing mathematics learning for all.

Using Frameworks and Interviews to Explore Whole Number Knowledge

Clinical assessment interviews are now widely used by teachers in Australia and New Zealand as a means of assessing children's mathematical knowledge. This is due to the experience of three large scale projects that informed assessment and curriculum policy formation in Victoria, NSW and New Zealand: *Count Me In Too* (Gould, 2000) in NSW, the Victorian *Early Numeracy Research Project* (Clarke, et al., 2002) and the *Numeracy Development Project* (Higgins, Parsons, and Hyland, 2003) in New Zealand. A common feature of each of these projects was the use of a one-to-one assessment interview and an associated research-based framework to describe progressions in mathematics learning (Bobis et al., 2005). A feature of assessment interviews is that they enable the teacher to observe children as they solve problems to determine the strategies they used and any misconceptions (Gervasoni and Sullivan, 2007). They also enable teachers to probe children's mathematical understanding through thoughtful questioning (Wright, Martland and Stafford, 2000) and observational listening (Mitchell and Horne, 2011). The insights gained through this type of assessment inform teachers about the particular instructional needs of each student more powerfully than scores from traditional pencil and paper tests, the disadvantages of which are well established (e.g., Clements and Ellerton, 1995). Because of the deep insight about children' mathematical knowledge gained through the use of one-to-one assessment interviews, the *Early Numeracy Interview* was chosen as the assessment tool for the *Bridging the Numeracy Gap* (BTNG) project. It was also anticipated that the data obtained would provide a rich snapshot of children's whole number knowledge.

The Early Numeracy Interview and Growth Points

The BTNG Project involved the assessment of primary school children's whole number knowledge at the beginning of each year using the *Early Numeracy Interview* (Department of Education Employment and Training, 2001) developed as part of the *Early Numeracy Research Project* (ENRP, Clarke et al., 2002). The data examined in this paper was drawn from this interview and the associated framework of growth points, so it is important that both are clearly understood. The principles underlying the construction and validation of the interview items and growth points, and the comparative achievement of children in the project and reference schools have been widely reported and are described in full in Clarke et al. (2002). A brief overview of the interview follows.

The *Early Numeracy Interview* is a clinical interview with a research-based framework of growth points that describe key stages of learning in nine mathematics domains, including the four whole number domains that are focused on in this paper: Counting, Place value, Addition and Subtraction, and Multiplication and Division. To illustrate the nature of the growth points, those for Addition and Subtraction follow. These describe strategies children use to solve problems. Each growth point represents substantial expansion in knowledge along paths to mathematical understanding (Clarke, 2001).

1. Counts all to find the total of two collections.
2. Counts on from one number to find the total of two collections.
3. Given subtraction situations, chooses appropriately from strategies including count back, count-down to and count up from.
4. Uses basic strategies for solving addition and subtraction problems (doubles, commutativity, adding 10, tens facts, other known facts).
5. Uses derived strategies for solving addition and subtraction problems (near doubles, adding 9, build to next ten, fact families, intuitive strategies).
6. Extending and applying. Given a range of tasks (including multi-digit numbers), can use basic, derived and intuitive strategies as appropriate.

The whole number tasks in the interview take between 15-25 minutes for each student and are administered by the classroom teacher. There are about 40 tasks in total. Given success with a task, the teacher continues with the next tasks in a domain (e.g., Place Value) for as long as the child is successful. The interview was refined during the BTNG project and renamed the *Mathematics Assessment Interview* (MAI) in 2010 (Gervasoni et al., 2011).

Gaining Insight about Children's Whole Number Knowledge

The data reported in this paper were collected in 44 Australian school communities in the States of Victoria and Western Australia. Participants included nearly 2000 Grade 1 to Grade 4 children (6-years to 9 years) who were assessed at the beginning of 2011 by their teachers using the MAI. Detailed interview record sheets were independently coded by research staff to determine the growth points children reached in each domain. This increased the validity and reliability of the data. The growth points for each student were entered into an SPSS database and analysed to determine the percentage of children on each growth point in each whole number domain and grade level.

Insights about Children's Whole Number Knowledge

Examination of the growth point distributions for nearly 2000 children gives a rich picture of whole number knowledge across the first five years of school. The following section explores the findings for Counting, Place Value, Addition and Subtraction Strategies, and Multiplication and Division Strategies.

Counting Knowledge

The Counting growth point distributions for Grade 1 to Grade 4 children are shown in Fig. 1. The data highlights a wide distribution of growth points in each grade. This demonstrates the complexity and challenge of classroom teaching, and the need for activities and instruction to be customised for individuals. Apparent also is the growth that occurs from one grade to the next; the median Counting growth point increased by one growth in each grade.

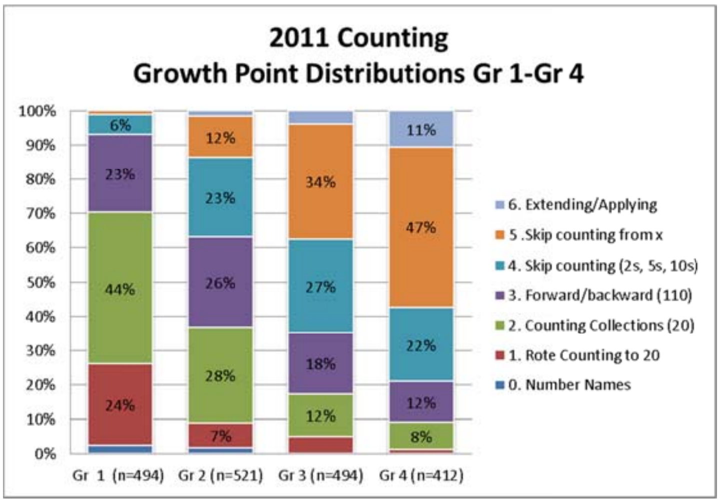

Fig. 1: Counting growth point distribution for Grade 1– 4 children

Examination of the growth point distribution for the Grade 4 children provides some insight about the growth of children's knowledge across the first four years of primary school. By the beginning of Grade 4 (fifth year at primary school), about 50% of children were working towards Growth Point 6 – extending and applying their counting knowledge. In contrast, about 10% of Grade 4 children could not yet count forwards and backwards by ones beyond 110, and another 20% could not yet skip count by 2s, 5s and 10s from zero. It is anticipated that these children (30%) would struggle with accessing some aspects of the Grade 4 curriculum and may benefit from customised instruction.

Place Value

Children's Place Value knowledge includes their abilities to read, write, order and interpret numbers. The growth point distributions are shown in Fig. 2.

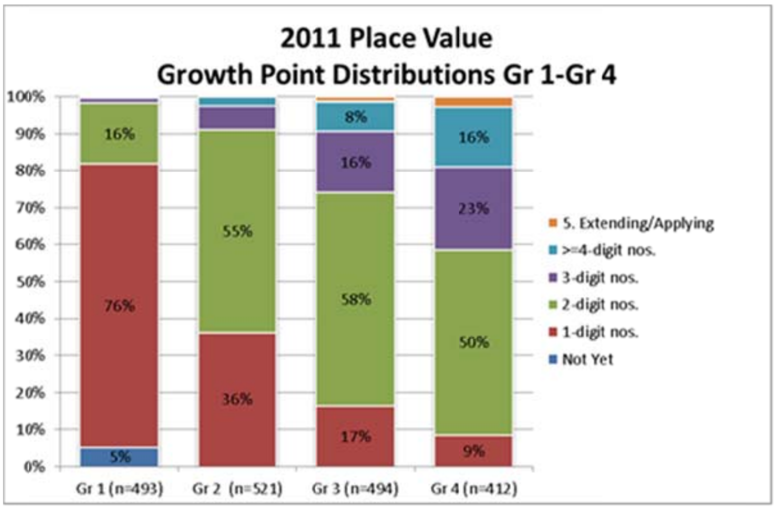

Fig. 2: Place Value growth point distribution for Grade 1-4 children

Again apparent is the wide distribution of growth points in each grade, but less growth is apparent from grade to grade compared with Counting. Indeed the median growth point remains GP2 (2-digit numbers) from Grade 2 to Grade 4. Reaching GP2 and GP3 (3-digit numbers) are significant challenges for young children. By Grade 4, 50% of children remained on GP2, and almost 10% on GP1. Thus the focus for 60% of these Grade 4 children was learning to interpret 2-digit and 3-digit numbers. This implied struggle with understanding 3-digit numbers well into Grade 4 is important for teachers to recognise. It was the interpretation of these quantities rather than the ability to read, write and order numerals that posed difficulty (Gervasoni et al., 2011). The complexity of teaching Place Value is highlighted further by the fact that 40% understand 3-digit numbers, while 15% reached GP4. This wide variation in the range of numbers children understand has implications for teaching arithmetic strategies.

Addition and Subtraction Strategies

A key learning focus in addition and subtraction is using reasoning strategies for calculating as opposed to counting strategies. The growth point distributions for Addition and Subtraction Strategies (Fig. 3) indicate that many children in all grades relied on counting strategies for calculating.

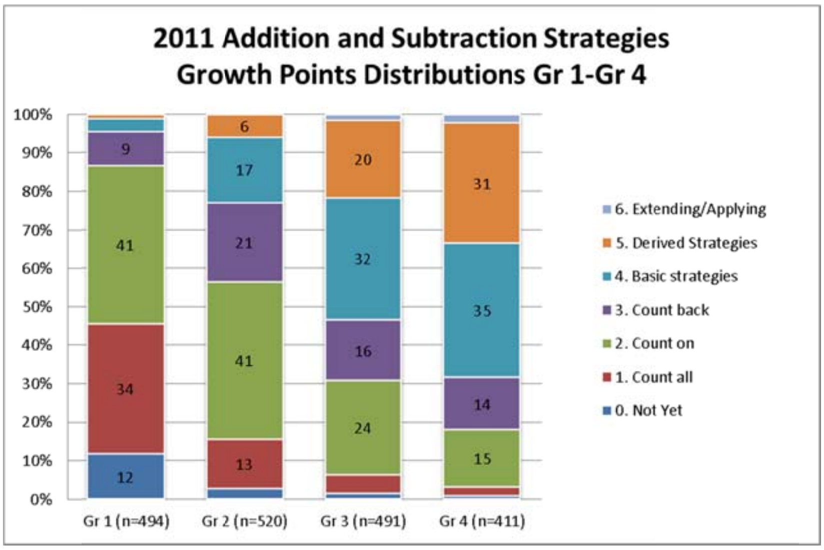

Fig. 3: Addition and Subtraction growth point distribution for Grade 1-4 children

The data in Fig. 3 indicate that 96% of Grade 1 children, 75 % of Grade 2 children, 46% of Grade 3 children and 30% of Grade 4 children used counting-based strategies for calculations, such as 4+4 and 10-3. The fact that so many Grade 4 children remain reliant on counting strategies for calculating, and that almost no Gr 4 students could solve mental calculations involving 2-digit and 3-digit numbers (GP6), is at odds with the tasks typically found in Grade 4 text books that involve calculations with much larger numbers.

Multiplication and Division Strategies

In Multiplication and Division, the key issue is children's ability to perform calculations without models being present, first in partial modelling situations and also when models are completely removed. Fig. 4 shows the growth point distributions for Multiplication and Division Strategies.

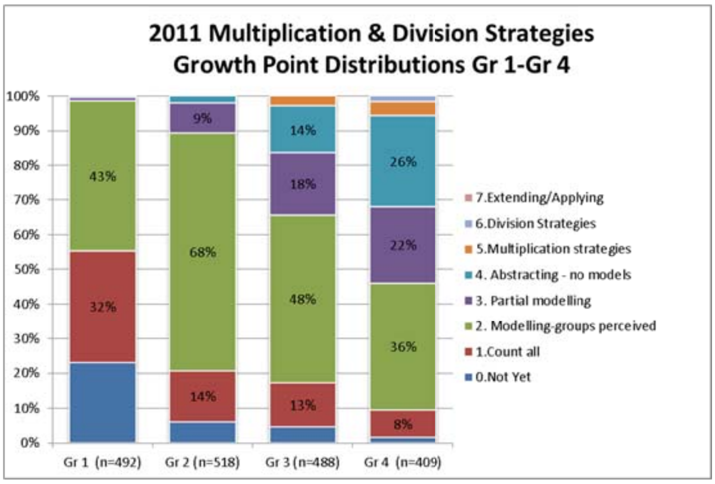

Fig. 4: Multiplication and Division growth point distribution for Grade 1-4 children

An outstanding feature of the growth point distributions shown in Fig. 4 is the large number of children in each grade on GP2 (models all objects to solve multiplicative and sharing situations). In Grade 3 only 35% of children were able to solve multiplicative problems in partial modelling or abstract situations. This increased to 50% of Grade 4 children. It is likely that children need more experience with an expanded range of multiplicative problems presented in partial modelling situations in order for their multiplicative reasoning to develop. Another feature of the data is that two-thirds of Grade 4 children were unable to solve multiplication problems without partial models present. This contrasts with the expectations implied in mathematics programs and text books.

Discussion and Conclusion

The findings presented in the previous section highlight several important issues related WNA learning. First is the wide distribution of growth points in each domain and each grade level, and the wide distance between the lowest and highest growth points in each grade level and domain. This highlights the complexity of teaching mathematics. The second issue is that progress through the growth points is more challenging for some growth points and in some domains. For example, progress from GP2 to GP3 in Place Value takes considerable time for many children. Knowledge about these challenging points and how to assist children to reach them is necessary to enable teachers to be most effective. This may be a useful focus for professional learning programs. The most challenging points identified in this study were: (1) interpreting 2-digit and 3-digit numbers; (2) using reasoning strategies as opposed to counting-based strategies in addition and subtraction; (3) and using strategies for solving partially modelled and abstract problems in multiplication and division. A third issue highlighted is the significant number of children on very low and very high growth points in most grade levels and domains. These children may be particularly vulnerable if teachers do not cater for their particular needs. These children are easily identified by referring to the growth point framework. It is also noteworthy that few children reached the highest growth points in each domain, even by Grade 4. This emphasises the importance of creating learning environments that enable all children to progress to the higher growth points.

In summary, the findings discussed in this paper suggest that there is no single ‘formula’ for describing children’s whole number knowledge or the instructional needs of children in a particular grade. Meeting the diverse learning needs of children requires teachers to be knowledgeable about how to identify each child’s current mathematical knowledge and customise instruction accordingly. This calls for rich assessment tools capable of revealing the extent of children’s knowledge in a range of domains, and an associated framework of growth points capable of guiding teachers’ curriculum and instructional decision-making. Assisting children to learn mathematics is complex, but teachers who are equipped with the pedagogical knowledge and actions necessary for responding to the diverse needs of individuals are able to provide children with the opportunities and experiences that will enable them to thrive mathematically.

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THE MODEL METHOD – A TOOL FOR REPRESENTING AND VISUALISING RELATIONSHIPS

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Abstract

The primary school mathematics curriculum in Singapore places emphasis on quantitative relationships when students learn the concepts of number and the four operations. The Model Method, an innovation in the teaching and learning of primary school mathematics, was developed by the primary school mathematics project team at the Curriculum Development Institute of Singapore in the 1980s. The method, a tool for representing and visualising relationships, is a key heuristic students' use for solving whole number arithmetic (WNA) word problems. When students make representations, using the Part-Whole and Comparison models, the problem structure emerges and students are able to visualise the relationship between the known and unknown and determine what operation to use and solve the problem. The model method has proved to be effective for making number sense and solving arithmetic word problems in Singapore schools.

Key words: four operations, heuristic, model method, number, Singapore

Introduction

In the history of education in Singapore, 1981 is a significant year as the New Education System (NES) (Ministry of Education, 1979) was implemented. The goal of the NES was to provide for every child in the system. In the late 1970s when low achievement in mathematics was a concern, it was decided that the primary mathematics curriculum (detailed syllabuses, textbooks, workbooks and teacher guides) would be developed by the Curriculum Development Institute of Singapore (CDIS). The CDIS was established in June 1980 and its main function was the development of curriculum and teaching materials. It was directly involved in the implementation of syllabuses and systematic collection of feedback at each stage of implementation so that subsequent revisions and refinements would be strategic (Ang and Yeoh, 1990).

At the CDIS the curriculum writers, who were experienced teachers from schools and the Ministry of Education, together with expertise of international consultants produced the first primary mathematics curriculum in 1981. The curriculum adopted the Concrete-Pictorial-Abstract approach for the teaching and learning of mathematics. This approach provides students with the necessary learning experiences and meaningful contexts, using concrete manipulatives and pictorial representations to construct abstract mathematical knowledge.

In 1983, the mathematics team writing the primary curriculum materials, led by Dr Kho, at CDIS made a breakthrough to address difficulties students were having with word problems. They introduced the 'Model Method' (Kho, 1987) in the curriculum for primary 5 and 6 students in the late 1980s. This method proven to be useful is now introduced to students in primary 1 (Ministry of Education, 2009). The method uses a structured process whereby students are taught to visualise abstract mathematical relationships and their varying problem structures through pictorial representations (Ferrucci et al., 2008).

The Model Method

The concrete-pictorial-abstract (CPA) approach of the primary school mathematics curriculum in Singapore is congruent with the concepts of the part-whole and comparison models. In the CPA approach students make use of concrete objects, while in the model approach they draw rectangular bars to represent the concrete objects. The rationale for the choice of rectangular bars is that they are relatively easy to partition into smaller units when necessary compared to other shapes.

Part-Whole Model

The part-whole model illustrates a situation when the whole is composed of a number of parts. The whole may have two or more parts. When the parts are given, the students can determine the whole. Sometimes the whole and some parts are given and other parts are unknown. Example 1 (Chan and Cole, 2013a, p. 27) shows how the CPA approach lends itself to the development of the part-whole model. Example 1 (shown in Fig. 1) is taken from a primary 1 textbook used in Singapore schools. It shows the introduction of the part-whole model concept.

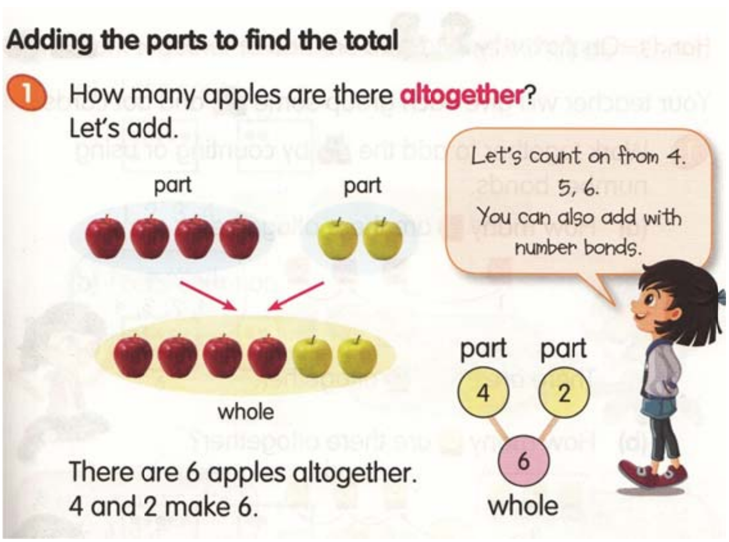


Fig. 1: Example 1 – The CPA and part-whole approach

In primary 2, students are introduced to the representation of the part-whole model which is shown in Fig. 2.

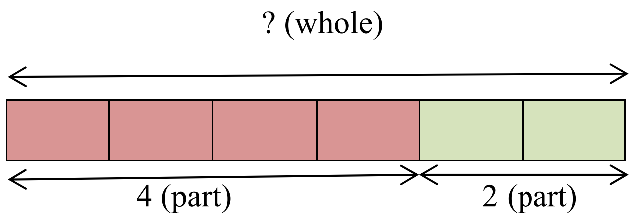


Fig. 2: The part-whole model

In the part-whole model (also known as the part-part-whole model) there is a quantitative relationship among the quantities: the whole and the parts. The sum of all the parts make the whole and can be easily found when the parts are known. Similarly any unknown part can be found when the sum of the parts and the other parts are known.

Comparison Model

The comparison model demonstrates the relationship between two or more quantities when they are compared, contrasted, or described in terms of differences. Example 2 (Chan and Cole, 2013a, p. 84) shows how the CPA approach lends itself to the development of the comparison model.

Example 2 (shown in Fig. 3) is taken from a primary 1 textbook used in Singapore schools. It shows the introduction of the comparison model concept.

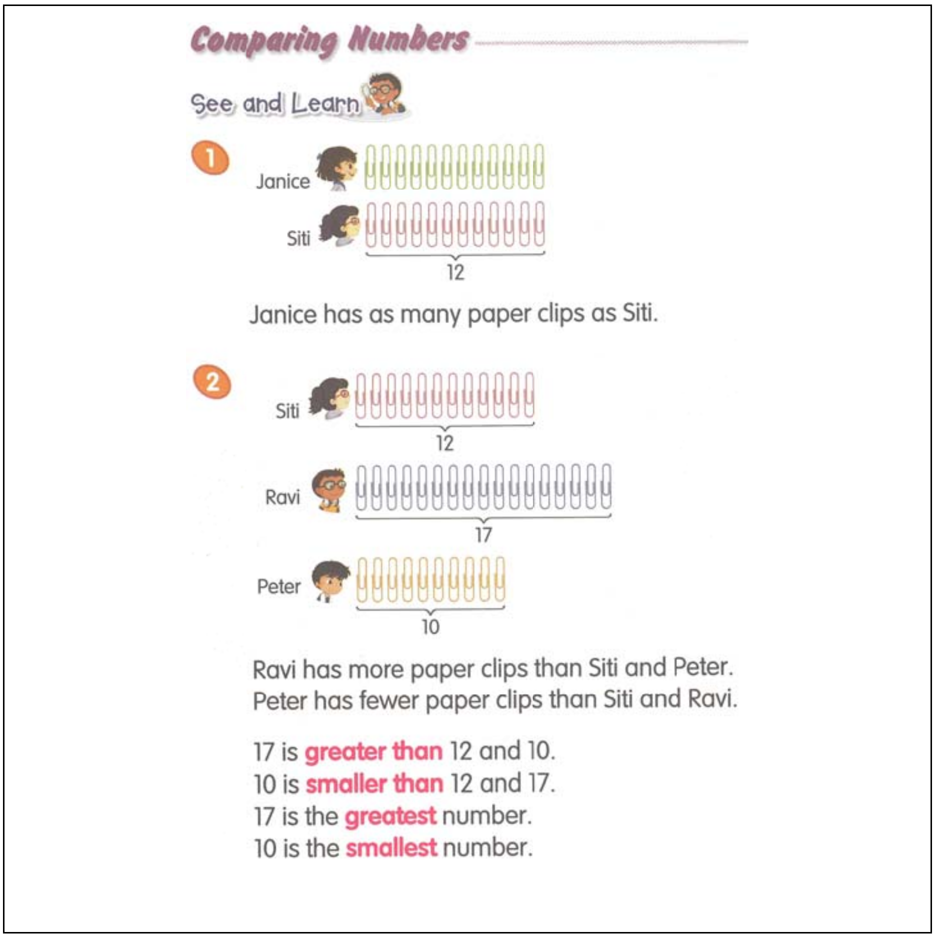


Fig. 3: Example 2 – The CPA and comparison approaches

In primary 2, students are introduced to the representation of a comparison model which is shown in Fig. 4.

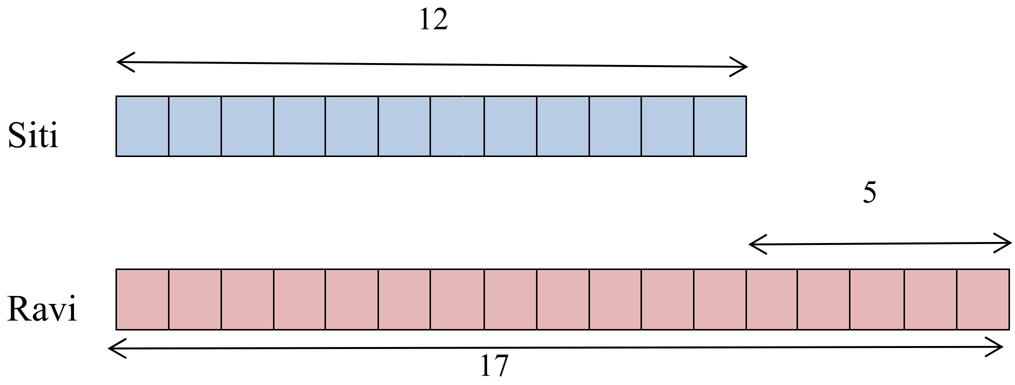


Fig. 4: The comparison model

Word Problems Involving the 4 Operations

The following examples, 3, 4 and 5, illustrate the use of models (part-whole and comparison) and the four operations to solve word problems. Example 3 (Chan and Cole, 2013b, p. 4) shows a typical word problem that is part of the curriculum for primary two. Example 3 (shown in Fig. 5) is taken from a primary 2 textbook used in Singapore schools. It illustrates how both the part-whole and comparison models are used to solve a 2-part word problem involving addition and subtraction respectively.

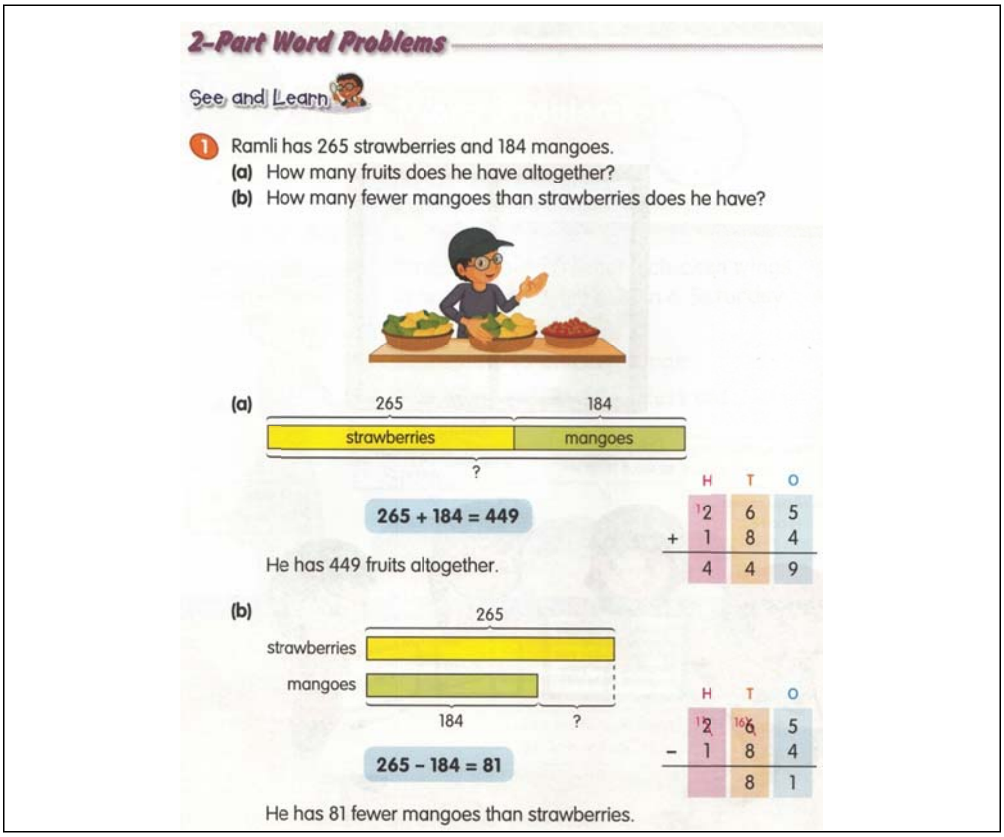
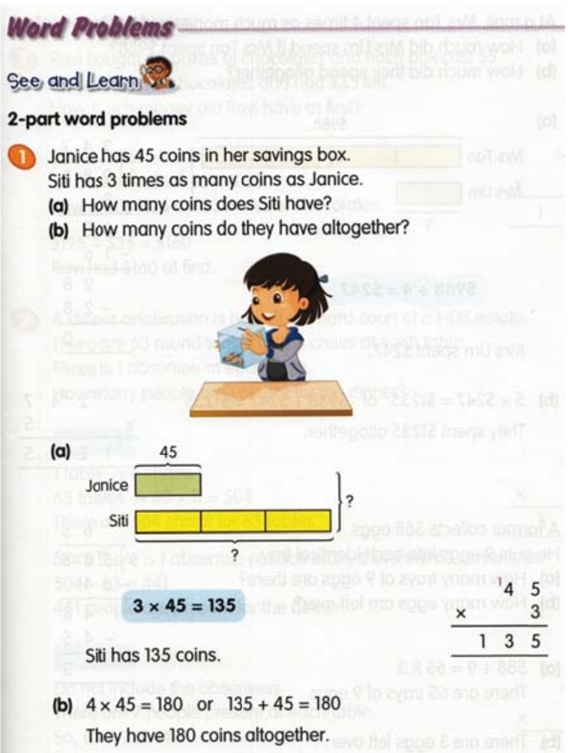


Fig. 5: Example 3 – Use of models to solve a two-part word problem

Examples 4 and 5 (Chan, 2014, pp. 119-120) show typical word problems that are part of the curriculum for primary three. Both examples 4 and 5, shown in Fig. 6 and 7 respectively, are taken from a primary 3 textbook used in Singapore schools. Example 4 illustrates how the comparison model is used to solve a 2-part word problem involving multiplication while example 5 shows the same but involving division.



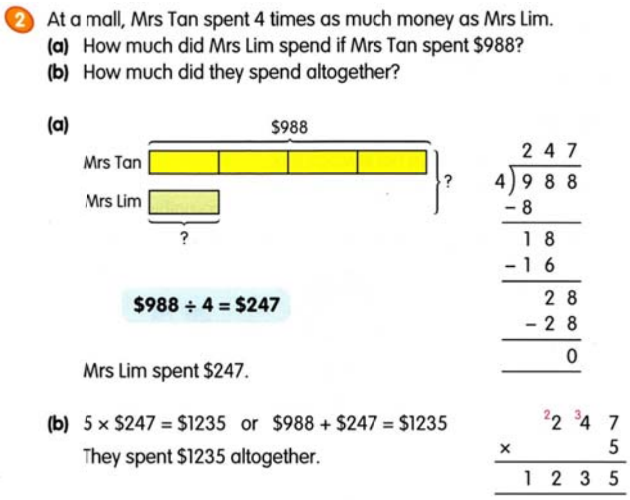


Fig. 6: Example 4 – Use of models to solve a two-part word problem

Fig. 7: Example 5 – Use of models to solve a two-part word problem

Problem Solving – Using Models and the Before – After Concept

The part-whole and comparison models are often used together with the before-after concept to solve word problems by students in primary five and six in Singapore schools. Example 6 illustrates how the part-whole model and the before-after concept may be used to solve a problem while example 7 illustrates how the comparison model and before-after concept may be used to solve a word problem which has a rather complex mathematical structure.

Example 6

In Mariam's aquarium, there are swordtails and guppies. 25% of the fish are swordtails. Mariam then buys more swordtails and puts them into the aquarium to double their number. What percentage of the fish are guppies after she doubles the number of swordtails?

Fig. 8 illustrates the solution process of Example 6. One way students may begin to solve this problem is by drawing the "before" part-whole model in which one of the rectangles represent the swordtails while the remaining three represent guppies. In the "after" part-whole model, since the number of swordtails has doubled, two rectangles represent the number of swordtails while three represent the number of guppies. The additional rectangle increases the number of rectangles to five and changes the percentage represented by each rectangle from 25% to 20%. From the "after" part-whole model, a student can conclude that now 60% of the fish are guppies

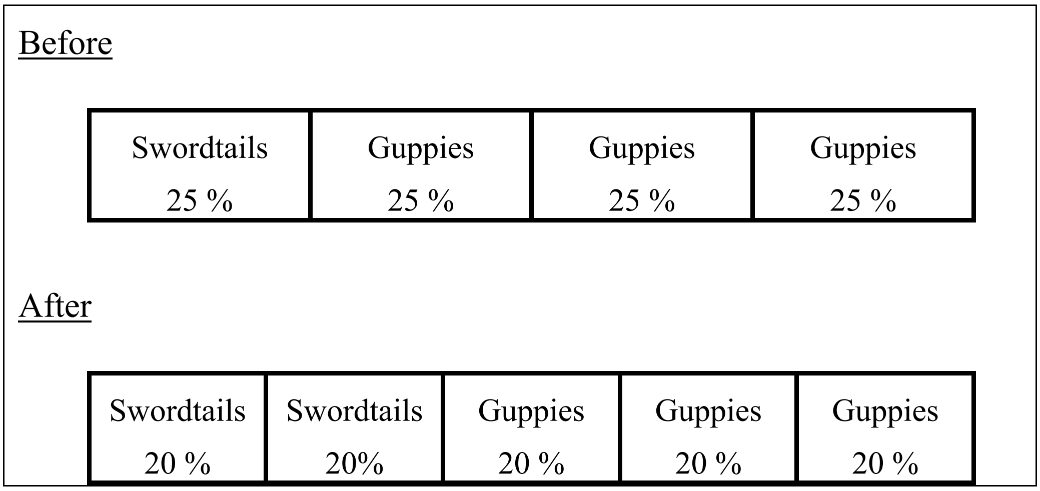


Fig. 8: Solution process of Example 6

Example 7

On Monday, the ratio of the number of beads Sarah had to the number of beads Emily had was 4:7. On Tuesday, after Emily gave 36 beads to Sarah, they both had the same number of beads. How many beads did Sarah have on Monday?

Fig. 9 illustrates the solution process, step by step, of Example 7. The comparison model is used together with the before-after concept. Steps 1, 2 and 3 belong to the before state while steps 4 and 5 belong to the after state. In step 1, a representation of the ratio 4:7 is made using the comparison model. The number of beads that Sarah and Emily had on Monday is represented by 4 and 7 blocks respectively. Next in step 2, the amount that needs to be shared is identified. As three blocks represent the amount to be shared, it is found that 1.5 is not easy to represent and so the number of beads that Sarah and Emily have is now represented by 8 and 14 blocks respectively. Step 4 shows the sharing taking place, on Tuesday, where Sarah is given 3 blocks by Emily. The three blocks are equal to 36 beads and therefore one block represents 12 beads. From Step 5, it apparent that Sarah had 12 x 8 = 96 beads on Monday.

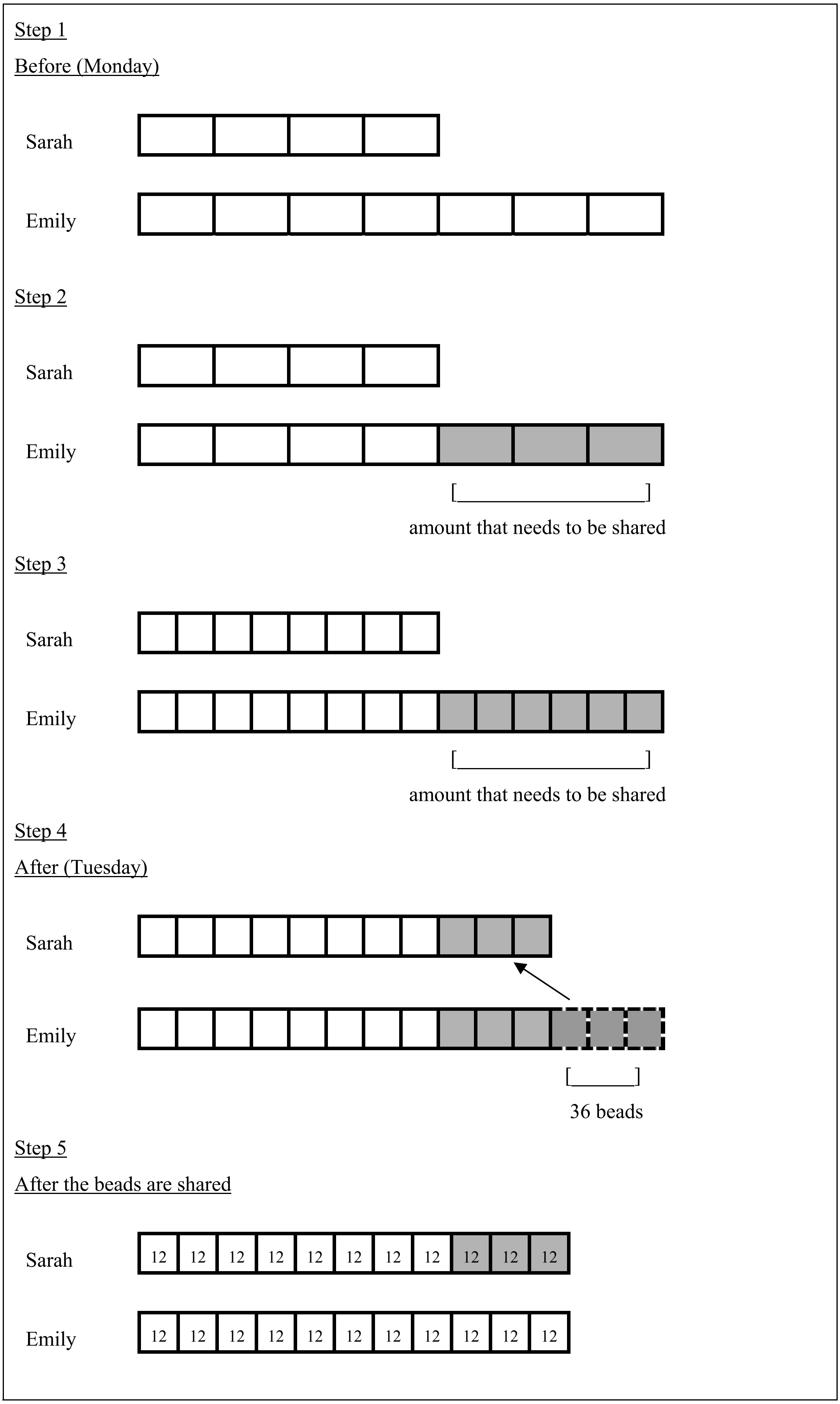


Fig. 9: Solution process of Example 7

What Research Says about the Model Method

A rigorous study by Ng and Lee (2009) of the model method clarified that the method engages students in capturing the inputs, the relationships between the inputs, and the output of the problem. Once students have constructed a model, they use it "to plan and develop a sequence of logical statements, which allows for the solution of the problem" (p. 291). Their study also noted that "average ability children's solution of word problems involving whole numbers could be improved if they learn to exercise more care in the construction of related models" (p. 311).

Conclusion

This paper illustrates a heuristic that has proven to be effective for whole number arithmetic (WNA) in primary school mathematics in Singapore. The heuristic known as the model method is a tool that students use to represent and visualise relationships, when solving number word problems involving the four operations. The concrete-pictorial-abstract approach of teaching mathematics appears to support the development of the models used in the method.

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TEACHING THE STRUCTURE OF STANDARD ALGORITHM OF MULTIPLICATION WITH 2-DIGIT MULTIPLIERS VIA CONJECTURING

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Abstract

This study was designed to teach the structure of multiplication algorithm via conjecturing. To help students consistently use multi-unit in each partial product in the algorithm, (A tens × B-ty[1] = (A×B) hundreds) and (A ones × B-ty = (A×B) tens) are two critical parts for 2-digit multipliers. 27 fourth-graders engaged in a conjecturing task. Pre-test, post-test, and students' worksheets were collected. The result indicates that conjecturing was an effective approach of teaching with understanding the structure of multiplication algorithm with 2-digit multiplier.

Key words: algorithm of multiplication, conjecturing, partial products, place value

Introduction

Students learn the multiplication algorithm more rapidly if they already know the structure and pattern of multiplication. Without this knowledge, progress is slow and difficult. The multiplication algorithm is usually taught sequentially, beginning with the simplest problems, those without requiring carrying (regrouping). The difficulty and complexity increase until multi-digit multiplier problems with carrying are presented. For example, 48×23 shows multiplying by a two-digit number with carrying. Thus, it is important that students understand what happens when carrying occurs. The major issue for multi-digit multiplication is what to multiply by what and how the place values of the digits in the multiplier affect the place values of the partial products. Research suggests that students' realisation of structure can be the factor of resulting into the differences between high and low achievers in mathematics (Mulligan, Prescott and Mitchelmore, 2004). Thus, helping students to understand the structure of the algorithm are essential of teaching multiplication.

A variety of multi-digit multiplication algorithms are used in different countries (Fuson and Li, 2009). There were variations in written methods. Some of the methods are a little longer because they include steps that help students make sense of and keep track of the underlying reasoning. The standard algorithm is especially powerful because they make essential use of the uniformity of the base-ten structure. This results in a set of iterative steps that allow the algorithm to be used for larger numbers. Unfortunately, in many classrooms, teachers teaching the standard algorithm often ask students memorise the steps rather

[1] 30 can be expanded as thirty (ones as counting unit) or 3 tens (tens as counting unit). B-ty means ones as counting unit, while B tens mean tens as counting unit. The standard algorithm of a two-digit multiplier 34 is always expanded as 34 = 30 + 4 instead of 34 = 3 tens + 4 ones.

than explain the meaning of the steps for them. For example, multiplying a whole number up to four digits by a two-digit number, fourth-graders do not use strategies based on place value and therefore cannot illustrate and explain the meaning of the calculation.

Fig. 1 shows various methods for recording standard algorithm with 1-digit multiplier. The algorithms of 327×4 are written in five methods. For the development of number concept, a student expanding 327 as 3 hundreds + 2 tens + 7 ones has higher level of cognition than those who expand it as 300+20+7. Method ID and IE have slightly higher level of cognition than Method 1A and 1 B. Each number in each row of Method 1D and 1E represents how many hundreds, tens, and ones rather than how many ones. 327×4 would rather be expanded as 12 hundreds+ 8 tens + 28 ones than as 1200+80+28, since 12 hundreds+ 8 tens + 28 ones is counted by multi-unit and 1200+80+28 is counted by ones.

For the distinction of Method 1D and 1E from Method 1A and 1B, Method 1D or 1E in the study is termed as standard algorithm with base-ten-units and Method 1A of 1B is termed as algorithm with ones-unit. The standard algorithm emphasises that computation is being approach on student sense-making. It is expected that students should be provided more opportunities to develop the standard algorithm of multiplication in Method 1D or IE.

Method 1C is an abbreviation algorithm recording the carries without showing each partial products, so that it is not a standard algorithm. A standard algorithm of multiplication is characterised as showing the partial products for recording the steps of the algorithm and it relies on decomposing numbers into base-ten units (place value) and then carrying out the computations with those units.

Method 1A and 1D proceed from left to right, while Method 1B and 1E from right to left. The method 1B and 1E proceeding from right to left are more convenient in dealing with carrying when it is needed.

Method 1A: Left to right with ones-unit 327 × 4 1200 300 ×4 80 20 ×4 28 7×4 1308	Method 1B: Right to left with ones-unit 327 × 4 28 7 ×4 80 20 ×4 1200 300 ×4 1308	Method 1C: Right to left recording the carries 327 × 4 21 1288 1308
Method 1D: Left to right with base-ten-units 327 × 4 12 3 hundreds ×4 8 2 tens ×4 28 7 ones ×4 1308	Method 1E: Right to left with base-ten-units 327 × 4 28 7 ones ×4 8 2 tens ×4 12 3 hundreds ×4 1308	

Fig. 1: Methods for the standard algorithm of 327×4

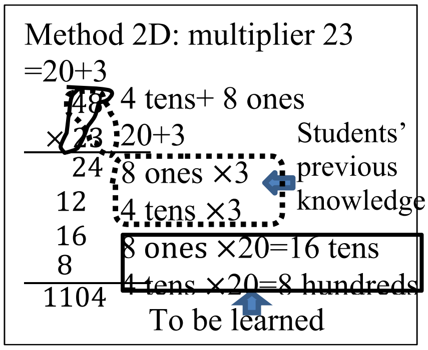
Textbooks or classroom teaching in Taiwan are expected that multiplication with two-digit multiplier moving from third grade where the approach of the standard algorithm with 1-digit multiplier is developed and explained via using visual models (diagrams) to the fourth grade where the approach of the SAM continues to be deepened and then is used fluently. Fig. 2 shows various methods of recording the algorithm of multiplication with 2-digit multipliers.

All methods in Fig. 2 have been utilised from past to present textbooks in Taiwan. The methods include: (1) the expanded form of the multiplier: 23=20+3, in Method 2A, 2B, 2C, 2D, and 2G, or 23=2 tens +3 ones in Method 2E and 2F; (2) the partial products showing in the algorithm are either base-ten-units or ones-unit: Method 2A, 2B, and 2C shows the partial products with ones-unit, while the Method 2D, 2E, and 2F, and 2G are base-ten-unit.

Within the methods by using base-ten-unit, the standard algorithm Method 2E used in a textbook of Taiwan is not consistent with the standard algorithm with one-digit multiplier, since 48 is expanded as 40+8 rather than 4 tens +8 ones. Furthermore, due to the multiplied and multiplier in Method 2E are simultaneously decomposed into tens and ones, it is difficult for fourth graders to transform 8 ones ×2 tens into 16 tens and 4 tens × 2 tens into 8 hundreds. It refers too complicated to communicating with students for the teachers in their teaching.

Method 2A: multiplier 23 =20+3 48 × 23 23=20+3 144 48 ×3 960 48×20 1104	Method 2B: multiplier 23 =20+3 48 × 23 20+3 24 8 ×3 120 40×3 960 48×20 1104	Method 2C: multiplier 23 =20+3 48 × 23 23=20+3 24 8 ×3 120 40 ×3 160 8×20 800 40 ×20 1104
Method 2D: multiplier 23 =20+3 48 4 tens+ 8 ones × 23 20 +3 24 8 ones ×3=24 ones 12 4 tens ×3=12 tens 16 8 ones ×20=16 tens 8 4 tens ×20=8 hundreds 1104	Method 2E: multiplier 23 =2 tens+3 ones 48 40 +8 × 23 2 tens +3 ones 24 8×3 ones 120 40 × 3 ones 16 8× 2 tens 8 40 × 2 tens 1104	Method 2F: multiplier 23 =2 tens+3 ones 48 4 tens+ 8 ones × 23 2 tens+ 3 ones 24 8 ones ×3 ones 120 4 tens ×3 ones 16 8 ones ×2 tens 8 4 tens ×2 tens 1104
Method 2G: Recording the carries 48 × 23 2 124 1 86 1104		

Fig. 2: Methods for the standard algorithm of 36×24



The Method 2D is performed with following steps of the partial products algorithm by multiplying the multiplier with 2-digit. The correct place value position in the partial products are maintained. To fulfil successfully, 48×23 is decomposed into (4 tens +8 ones) ×3 and (4 tens +8 ones) ×20 as shown on the top of the Method 2D, marked in dash and bold line respectively.

Step 1: 48= 4 tens + 8 ones , 23 = 20+3.
Step 2: 8 ones × 3= 24, is placed as the first partial product beneath the problem.
Step 3: 4 tens × 3=12 tens, is placed beneath the 24.
Step 4: 8 ones ×20=16 tens, is placed beneath the 12.
Step 5: 4 tens × 20 =8 hundreds, is placed beneath the 16.
Step 6: the partial products are added to arrive at the answer.

The calculation in the first three steps has been learned in grade 3. For fourth graders, step 4 and step 5 are new experience. Two critical concepts in the two steps for meaningful understanding the algorithm of multiplication with 2-digit multipliers are 8 ones ×20=16 tens and 4 tens ×20 =8 hundreds.

Given the analysis of difficulty in various methods and meaning-making, the step 4 and step 5 in Method 2D should be emphasised in teaching multiplication for fourth-grade students acquiring the meaning of partial products and correctly place the number in place value position. Thus, this study was designed to help students realise the structure of the standard algorithms via conjecturing. A research question to be asked is: How did students engage in conjecturing task for constructing the critical concepts of structure of the standard algorithm, A ones × B-ty = (A×B) tens, (e.g. 8 × 20 =16 tens) and A tens × B-ty = (A×B) hundreds, (e.g. 4 tens × 20=8 hundreds)?

Method

Participants and Context

Six teachers participated in the study to create conjecturing tasks for engaging students in the activity of proving. They were mutually supported in a professional program. We observed altogether the lessons when the tasks were carried out into classrooms. One of the teachers and her 27 students participating in the study have been engaging in the conjecturing for a year. They have learned the standard algorithm of multiplication with 1-digit in the third grade.

Tasks involved in the study were designed through ADDIER cycle. Analysis materials of textbook was the first step, moving forward Developing and Designing the tasks, Implementing the tasks into classrooms, and then Evaluating and Revising the tasks. One of the conjecturing tasks related to multiplication algorithm was introduced in Tab. 1.

The task was designed based on the stage of conjecturing suggested by Cañadas and Castro (2005): observing and organising cases, searching for and formulating a conjecture, validating the conjecture, generalising the conjecture, and justifying the generalisation. The task with a set of problems was fulfilled in three consecutive lessons.

<table>
<tr><th colspan="2">Objective of the task:
To know A ones × B-ty = (A×B) tens, and A tens × B-ty = (A×B) hundreds</th><th>Stages of Conjecturing</th></tr>
<tr><td>Task</td><td>A collection activity in 7-Eleven: A card has 10 columns. If you buy foods, then you get a column. The price can be deduced $3 in each column. The card can be used as long as you collect the 10 columns.
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①</td></tr></table>After collecting, you can further get a prize that the number appearing in a dice you rolled is the amount of cards you will get.</td><td></td></tr>
<tr><td>A</td><td>(1) Each of you are rolling a dice once. Write down the number of cards you got and attached those cards on the worksheet.
(2) How many $ 1 are deduced from the columns? Write it down with a number sentence.
(3) How many $10 are deduced from each card? How many $10 you can deduced from in total? Write it down with a number sentence.
(4) Check if your work is correct with your group.
(5) Gathering each of the sheets in your group altogether.</td><td>Establish & organise cases</td></tr>
<tr><td>B</td><td>(1) Do the two equations you generated are the same deduced prices?
(2) Write down the pattern that you find based on the two number sentences.
(3) Sharing the patterns you found and explain to your group.
(4) Write down the conjectures in your group.</td><td>Observing pattern
Formulate conjecture</td></tr>
<tr><td>C</td><td>(5) Comparing and categorising the conjectures made from each group.
(6) Explain the conjectures from your group to the whole class.</td><td>Validating Conjectures</td></tr>
<tr><td>D</td><td>(1) Can the conjectures be generalised into general cases?
(2) How do you state the conjecture in general?</td><td>Generalising</td></tr>
<tr><td>E</td><td>(1) How do you justify the generalisation?
(2) Are the conjectures still correct if you change the unit, 1, into 10?</td><td>Justifying</td></tr>
</table>

Tab. 1: The task for engaging in the structure of the algorithm via conjecturing

Students were grouped heterogeneously in groups of 4 or 5. After given the task, the students first worked independently and jotted down their responses on B4 paper; then they came together in groups to compare their solutions, and finally they shared their conjectures to the whole class. The lessons were videotaped and students' written work was collected.

To realise how the effect of teaching the structure of the algorithm of multiplication with 2-digit via conjecturing on students' understanding, five items were conducted on the pre-test and post-test, as seen in Tab. 2.

Judging the following items if they are correct and give the reasons.	
1. 60 of 3 ones is equal to 6 of 3tens.	2. 60 of 2 tens is equal to 6 of 2 hundreds.
3. 40 of 6 ones is equal to 4 of 6 tens.	4. 70 of 4 tens is equal to 7 of 4 hundreds.
5. 50 of 4 tens is equal to 5 of 4 hundreds.	

Tab. 2: Five items in pre-and post-test

Results

The data in Tab. 3 indicates that students had low percentage of correctness performing in the structure of multiplication before learning the 2-digit multiplier, even though they have learned the standard algorithm of multiplication with 1-digit multiplier in the third grade. Most of the reasons given in the pre-test were based on the same answer they computed, such as the reason for the item 2 is "Each answer is 1200". The reason of "0 in 60 moving to in 20 becomes 200" was given by students without understandings the structure underlying the change.

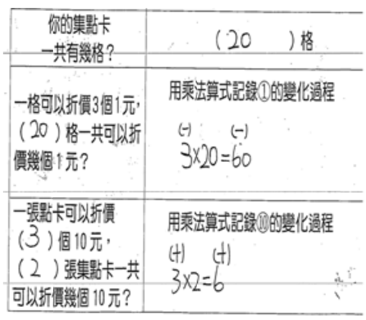
Item	1	2	3	4	5
Pre-test	48%	22%	52%	22%	22%
Post-test	100%	93%	100%	96%	93%

Tab. 3: Students' performance in pre- and post-test

The development of multiplication structure was preceded with the following.

Constructing cases as data for looking for patterns

After rolling a dice, the cases were collected from individual, to group, and then the whole class. Fig. 3 is an example collected from an individual student. Constructing cases was for helping students in observing a pattern and then to make a conjecture. We found that the activity of rolling a dice made them more likely to write down the number sentences with different counting units (ones and tens), such as $\underset{3}{(-)}\times 20=\underset{60}{(-)}$ and $\underset{3}{(+)}\times 2=\underset{6}{(+)}$ [2], as in Fig. 3.



ow many blocks do you ıve in the card?	(20) blocks
ach block can reduce 3 1. How many dollars can reduced by (20) blocks total?	Record the change of ○,1 by using number . $\underset{3}{(-)}\times 20=\underset{60}{(-)}$
card can reduced (3) ns. How many $10 can be duced by (2) cards?	Record the change of ○,10 by using number sentence. $\underset{3}{(+)}\times 2=\underset{6}{(+)}$

Fig. 3: Students' worksheet (Chinese version and English version)

[2] $2\underset{3}{(+)}\text{x}2=\underset{6}{(+)}$, here (+) means (tens), "+" is Chinese word of "number 10". it also represents tens in place value.

Formulating a conjecture

Task B were for providing students with the opportunity of engaging in formulating conjectures. In this stage, the conjecture each student formulated was based on his/her own case, so it has not been validated. Task B "*Write down the pattern that you find based on the two number sentences.*" was to ask students make a conjecture. For instance, student #13 make a conjecture is "*0 in 60 locating in* ${}^{(-)}_{1}\times 60={}^{(-)}_{60}$ *was cancelled and then got* ${}^{(+)}_{1}\times 6={}^{(+)}_{6}$" based on the two numbers ${}^{(-)}_{1}\times 60={}^{(-)}_{60}$ and ${}^{(+)}_{1}\times 6={}^{(+)}_{60}$.

Validating the truth of a conjecture

Validating the truth means to verify the conjecture for a new specific case but not in general. Task C gives students opportunity to improve their conjectures to be true based on new cases. For instance, the relationship of ${}^{(-)}_{1}\times 60={}^{(-)}_{60}$ and ${}^{(+)}_{1}\times 6={}^{(+)}_{6}$ stated from group 1 was "*complex changing to simple*". The relationship of ${}^{(-)}_{3}\times 20={}^{(-)}_{60}$ and ${}^{(+)}_{3}\times 2={}^{(+)}_{6}$ from group 4 displayed in Fig. 3c was *the unit "ones changed into the unit "tens"*. The students in group 4 stated with meaningful understand to explain how ${}^{(-)}_{3}\times 20={}^{(-)}_{60}$ is changed into ${}^{(+)}_{3}\times 2={}^{(+)}_{6}$ as "*in the collection card,* ${}^{(-)}_{3}\times 20$ *means $3 in each of 20 columns, but it can also be looked by rows, 3 tens in each card, 2 cards in total, the answer is 3 tens ×2=6 tens, represented as* ${}^{(+)}_{3}\times 2={}^{(+)}_{6}$. " Thus, the conjecture made by group 4 based on the two number sentences was the use of different unit "ones" and "tens", ${}^{(-)}_{3}\times 20={}^{(+)}_{3}\times 2$.

Generalising the conjectures and justifying the generalisation

Justifying the generalised conjecture involves giving reasons that explain the conjecture. Due to the limited knowledge of verification of the fourth graders, the interaction that took place between the students and the teacher was to sustain the engagement of generalising the conjectures and justifying the generalisation through the whole class discussion. The conjecture to be generalised was ${}^{(-)}_{3}\times 20={}^{(+)}_{3}\times 2$ to generalise in general cases, e.g. ${}^{(-)}_{8}\times 60={}^{(+)}_{8}\times 6$. ${}^{(-)}_{A}\times B\text{–}ty={}^{(+)}_{A}\times B$. To help students generalising the structure of multiplication with 2-digit multiplier into 3-digit multiplier, the teacher asked students the questions: ${}^{(-)}_{3}\times 200=(\ )$ or ${}^{(+)}_{A}\times B\text{–}hundreds=(\ ?\ )$ It is readily for students to generalise the structure to 3-multiplier and got the answer ${}^{(-)}_{3}\times 200={}^{(hund.)}_{3}\times 2$, and the generalisation is ${}^{(-)}_{A}\times B\text{–}hundreds={}^{(hund.)}_{A}\times B$. The reason students explain why ${}^{(-)}_{3}\times 200$ is equal to ${}^{(hund.)}_{3}\times 2$ was that "*in this problem, we can think in the same way as previous one. If a collection card has 100 columns, each column consists of $3 for a deduced price. The collection cards can also be seen as ten dollars in each row, 6 tens dollars in total*."

Conclusion

The result from the percentage students performed correctly in the pre-and post-test indicates that conjecturing is an effective approach for teaching students the meaning of the structure of standard algorithm of multiplication with multi-digit.

The task with a set of problems provided the fourth-graders to construct cases for observing and searching for pattern or relationships via collection card. Students formulated the conjectures based on their prior experience and knowledge. It is found that the cases to be constructed as the data for searching for the patterns are highly closed to the conjecture to be formulated and helped students to validate or justify the conjectures they made. For instance, students giving the reasons for explaining the truth $\frac{(-)}{3}\times 20=\frac{(+)}{3}\times 2$ and $\frac{(-)}{3}\times 200=\frac{(hund.)}{3}\times 2$ were based on the cases they constructed from the collection cards. There are two critical concepts of computing the standard algorithm of multiplication with multi-digit multipliers, such as 48×23 is decomposed into 48×3 and 48×20. 48×3 is not new for fourth grade students in Taiwan, but 48×20 is to be learned. 48×20 referred to this study is decomposed into (4 tens 8 ones) × 20 rather than (4 tens 8 ones) × 2 tens. Moreover, (4 tens 8 ones) × 20 is decomposed into 4 tens ×20 and 8 ones ×20. Helping students to understand 4 tens ×20=8 hundreds, 8 ones ×20=16 tens corresponding to consistent using multi-unit in each partial product algorithm of multiplication became the aim of the study, as seen in Method 2D in Fig. 2.

The conjectures lead to the conclusions that were the objectives of the lessons. The results also indicate that the nature of conjecturing is powerful, since generalising the conjectures to general cases involving in the study was the structure of 3-digit multiplier generalised by 2-digit multiplier. However, it is noted that the designing of task for conjecturing is a challenge for individual teacher if she/he is a novice of taking the conjecturing as an approach. Designing the task for conjecturing for students engaging in meaningful learning should be a focus of the teacher professional development.

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SAME YEAR, SAME SCHOOL, SAME CURRICULUM: DIFFERENT MATHEMATICS RESULTS

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Abstract

Year 4 students from a large metropolitan school in Melbourne were tested using the One Minute Tests of Basic Number Facts and a paper and pencil Number Screening Test developed by the author and colleagues. Observation of classes during the assessment procedures highlighted the vast difference in the students' speed and accuracy when recalling basic facts and the types of strategies they used when solving whole number arithmetic tasks. When the results were analysed there were differences noted in the class results and when the results were presented to the teachers their reactions to these results varied considerably. This paper will focus on the comparison of the Year 4 class data.

Key words: assessment, automaticity, basic number facts, word problems

Introduction

Many primary age students struggle to remember and reproduce basic number facts instantly. Hiebert and Lefevre (1986) noted that:

> manipulation rules. In general, the rules are more sensitive to syntactic constraints that … by the time students are in third and fourth grade, they have acquired a large array of symbol n to conceptual underpinnings. (pp. 20-21)

In previous research the author found that Year 3 and 4 students struggling with whole number arithmetic relied on rules and procedures even when these were inefficient and unreliable (Pearn, 2009). In Victoria Year 3 students range in age from 7.7 to 8.7 years while Year 4 students range in age from 8.7 to 9.7 years. Westwood (2000) states:

> Without easy recall of basic number facts, students have difficulty with even simple mental addition and subtraction problems. (p.45)

In March 1997, Australian state and territory education ministers agreed to a national goal that "every child leaving primary school should be numerate, and be able to read, write and spell at an appropriate level" (Masters and Forster, 1997, p. 1). A national plan to support this goal requires education authorities to provide support for teachers in their task of identifying students who are not achieving adequate literacy and numeracy skills. National numeracy benchmarks, articulating minimum standards, were developed for Years 3, 5 and 7. For example, Year 3 students are expected to remember, or work out, basic addition facts to 10 + 10, the matching subtraction facts and extensions of those facts. Year 5 students are expected to have achieved the Year 3 benchmark standard and also know or work out multiplication facts to 10 x 10 and use these to work out extensions of those facts.

Considerable research has been conducted about students' understanding of mathematical concepts and the strategies they use to solve mathematical tasks.

Gray and Tall (1994) have shown that students successful with mathematics use different types of strategies from those struggling with mathematics. Gray and Tall (1994) defined procedural thinking as being demonstrated when students are dependent on the procedure of counting and limited to the "count-all" and "count-back" procedures. While some students are dependent on rules and procedures other students give instantaneous answers. According to Gray and Tall (1994), the use of known facts and procedures to solve problems, along with the demonstration of a combination of conceptual thinking and procedural thinking, indicate that these students are proceptual thinkers. Gray and Tall (1994) defined proceptual thinking as:

> the flexible facility to ... enable(s) a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mental process. It includes both concepts to know and processes to do. (pp. 124-125)

Results from state-wide testing in Victoria revealed that Year 3 and Year 5 students from a large metropolitan primary school in the outer northern suburbs of Melbourne were not achieving at the level that the Principal expected and that there were many students not achieving the appropriate numeracy benchmarks. The author was employed by the Principal to provide ongoing professional development for the team of Year 4 teachers with teaching experience that ranged from two to twenty years.

In Victoria, Six Principles of Learning and Teaching P-12 (PoLT) (DEECD, 2007) have been designed to be used to reflect on practice and support professional dialogue to strengthen pedagogical practices. For example three of the principles state that students learn best when:

- The learning environment is supportive and productive
- Students' needs, backgrounds, perspectives and interests are reflected in the learning program
- Assessment practices are an integral part of teaching and learning

The Australian Association of Mathematics Teachers (AAMT) has Standards for Excellence in Teaching Mathematics in Australian Schools which provide targets "to which all teachers of mathematics can aspire and work towards in their professional development" (2006, p.2). The standards include three domains: professional knowledge, professional attributes and professional practice. The domain of professional knowledge includes: knowledge of students, mathematics and of students' learning of mathematics. For example:

> Excellent teachers of mathematics have a thorough knowledge of the students they teach. This includes knowledge of students' social and cultural contexts, the mathematics they know and use, their preferred ways of learning, and how confident they feel about learning mathematics (AAMT, 2006, p.2).

To provide further information about the mathematical skills and understandings of Year 4 students, two assessment instruments were used: The *One Minute*

Basic Number Facts Tests (Westwood, 1995 in Westwood, 2000) and a paper and pencil Number Screening Test (Pearn, Doig and Hunting, in press).

> Excellent teachers of mathematics regularly assess and report student learning outcomes, both cognitive and affective, with respect to skills, content, processes and attitudes. They use a range of assessment strategies that are fair, inclusive and appropriate to both the students and the learning context (AAMT, 2006, p.4).

Initial analysis of the Year 4 data highlighted the vast difference in the students' speed and accuracy when recalling basic number facts and the types of strategies they used when solving whole number tasks (Pearn, 2009). In a large scale research study of early years' teachers, Sullivan and McDonough (2002) reported that they attributed marked differences in achievement between classes to the teachers. This paper will also focus on the class analysis which also reveals distinct differences between Year 4 classes. These differences appear to reflect attitudes exhibited by teachers at the professional development sessions presented by the author.

Materials and Methods

There were 122 students from five Year 4 classes that ranged in size from 19 to 27 students. Students had been randomly assigned to classes. The students' ages ranged from 8.92 (8 years and 11 months) to 10.25 years (10 years 3 months). The average age of the students was approximately 9 years 7 months.

The Assessment Instruments

There are four *One Minute Basic Number Facts Tests* (Westwood, 1995 in Westwood, 2000). Each test has 33 items that focus on one of the four whole number processes: addition, subtraction, multiplication and division. These items are ordered randomly. The addition test items include one-digit addends with either one- or two-digit sums. The subtraction test items include one-digit minuends and subtrahends with a positive one-digit difference and some two-digit minuends and one-digit subtrahends with a one-digit difference. The multiplication test contains items with one-digit multipliers and one-digit multiplicands while the division test has six one-digit dividends and 27 two-digit dividends divided by a one-digit divisor with one-digit quotients. The first three items from each of the *One Minute Basic Facts Tests* (Westwood, 2000) are shown in Tab. 1.

Addition	Subtraction	Multiplication	Division
2 + 1 =	2 - 1 =	1 x 2 =	2 ÷ 1 =
1 + 4 =	5 - 1 =	2 x 3 =	4 ÷ 2 =
2 + 2 =	3 - 2 =	2 x 5 =	3 ÷ 1 =

Tab. 1: The first three items from the *One Minute Basic Number Facts Tests*

The Number Screening Tests 2A and 2B (Pearn, Doig, and Hunting, unpublished manuscript) were designed to identify students mathematically 'at risk' in Years 3 and 4. The tests included whole number tasks that had previously been found by the author to be difficult for students 'at risk' (see for example, Pearn, 2009). Both versions of the test were deemed to be of similar difficulty.

The Number Screening Tests contain 34 items that focus on whole number arithmetic. Eight items focus on counting, six on place value, six on addition, six on subtraction and one item had multiplication as the focus. There are seven word problems. The counting tasks include items that required students to complete the sequence of counting forwards by ones from a two-digit number (including bridging across 100), counting backwards by ones from two-and three-digit numbers, counting forwards by tens from multiples and non-multiples of ten, counting forwards by fives from a multiple of five and counting forwards by twos from a non-multiple of two. The place value tasks include items that require students to write the number that is one more or less, or ten more or less than a given number and order two and three digit numbers. The addition and subtraction tasks include one-digit and two-digit addends and subtrahends and one task requires students to find the missing addend. The subtraction whole number word problem:

> Tom's cat is 31 cm long.
>
> Mary's kitten is 17 cm shorter than Tom's cat.
>
> How long is Mary's kitten?

The author administered both assessments to ensure consistency with the administration. Observation of classes during the assessment procedures highlighted the vast difference in the students' speed and accuracy when recalling basic facts and the types of strategies they used when solving mathematical tasks (Pearn, 2009).

Results

This paper will discuss the analysis of the data from the *One Minute Basic Number Facts Tests* and from the Word Problems of the Number Screening Tests (2A and 2B). Both these assessment instruments are testing whole number arithmetic knowledge. In particular, the analysis will focus on the similarities and differences in the class data. For the *One Minute Basic Number Facts Tests* a student's score indicates the number correct for each of the tests. A student score of seven (7) indicates only that a student has correctly answered seven of the items correctly. This can be achieved in three different ways:

- attempt the first seven items and answer the seven correctly
- attempt any seven items from the test and answer the seven correctly
- attempt more than seven items correctly and answer seven correctly

In Tab. 2 students' scores (to 2 decimal places) are compared for the four processes: addition, subtraction, multiplication and division. Normal range was defined by Westwood (2000) as the range of scores for 50% of the students and the critically low score as one standard deviation below the mean for the age group. Students achieved higher scores in addition than subtraction. Subtraction scores are generally higher than multiplication scores which are generally better than scores for division. While some students were unable to correctly answer any items there were some students who completed the 33 items for each process in less than one minute.

	Addition	Subtraction	Multiplication	Division
Range of scores	2 – 33	0 - 33	0 - 33	0 – 33
Median	23	14	12	6
Mean	22.34	15.93	13.21	8.56
Standard Deviation	6.49	7.48	7.73	7.72
Normal range	17.92 – 26.75	10.84 – 21.01	7.96 – 18.47	3.27 – 13.77
Critical low score	16	8	5	1

Tab. 2: Scores for *One Minute Basic Number Facts* (n = 122)

Ten students correctly answered 33 addition facts within one minute. The mean for addition facts was 22.34, the median was 23, and the critically low score was 16. Five students answered 33 subtraction facts correctly in less than one minute while 13 answered 8 facts or less. The mean for subtraction was 16 and the median was 14. The critically low score for subtraction was 8. Four students answered all 33 multiplication facts correctly in less than one minute. The mean for multiplication was 13, the median was 12 and the critically low score was five. Three students correctly answered 33 division facts in less than one minute while 12 students correctly answered one or less facts in one minute. The mean was nine, the median was six and the critically low score was one for division.

All students completed a Number Screening Test. This paper will just focus on the results for the whole number word problems. Many students from all classes struggled with the word problems. There were 12 students (8%) who did not attempt any whole number word problem. There were 30 students (25%) who were only successful with one task but nine students (6%) correctly answered all six word problems.

Tab. 3 shows the percentage of students who were successful with each of the word problems.

Addition	Subtraction		Multiplication		Division
Task 1	Task 2	Task 5	Task 3	Task 4	Task 6
78	34	25	49	57	23

Tab. 3: Success with word problems (percentages)

While 78% of students successfully answered the addition word problem, only 34% of the students were successful with one of the subtraction problems, while only 25% were successful with the second subtraction task. Nearly half the students were successful with the multiplication problem and more than half succeeded with one of the division problems with only 23% successful with the second division problem.

Tab. 4 shows the means (to two decimal places) of the four *One Minute Basic Number Facts* (Westwood, 2000) by class. Class 2 had the lowest means for addition, multiplication and division and the second lowest for subtraction. Class 4 had the highest means for all four processes.

Class	Addition	Subtraction	Multiplication	Division
1 (n = 25)	23.08	16.16	12.04	7.56
2 (n = 19)	18.63	13.32	8.05	5.63
3 (n = 27)	21.42	13.04	12.23	6.5
4 (n = 26)	24.54	21.46	17.54	12.96
5 (n = 25)	22.79	14.33	14.54	8.83
Total	22.34	15.93	13.21	8.85

Tab. 4: Comparison of means by class

When the results for the addition facts were compared, the medians ranged from 20 (Class 2) to 24 (Class 4) with Classes 1, 3 and 5 having a median of 23 (see Tab. 5). The critically low scores for addition varied between 12 and 18. The median scores for subtraction ranged from 13 (Classes 2 & 3) to 20 (Class 4) with an overall median of 14. The critically low scores for subtraction varied from 9 to 14.

Tab. 5 highlights the differences between the scores for the multiplication facts for each class. The medians for multiplication facts ranged from 9 for Class 2 to 17.5 for Class 4 and the means varied from 8.05 for Class 2 to 17.81 for Class 4.

	Class 1	Class 2	Class 3	Class 4	Class 5
Range of scores	2 – 32	0 – 18	0 – 33	2 – 33	2 – 24
Median	11	9	13	17.5	13
Mean	12.04	8.05	12.41	17.81	14.4
Standard Deviation	6.69	4.13	7.86	9.8	5.39

Tab. 5: Scores for multiplication facts (by class)

Results for the division facts also highlighted the differences between classes. The range of total scores for Classes 1 and 2 were 0 – 6 and 0 – 5 respectively, 0 – 33 (Classes 3 and 4) and 0 – 21 (Class 5). Median scores for division facts ranged from one to 11 and the mean scores fell between 5.63 and 13.08.

When the scores for the word problems were analysed the median scores varied between 1 (Class 2), 3 (Classes 1, 3 and 5) and a score of 4 (Class 4) while the mean scores varied between 1.74 for Class 2 and 3.42 for Class 4.

Discussion and Conclusion

The seven principles of highly effective professional learning articulated in the Professional Learning in Effective Schools (Department of Education and Training, 2005) suggest that professional learning needs to be:

> … collaborative, embedded in teacher practice and bridging the gap between what students are capable of doing and actual student performance (p.4).

These assessment instruments were administered to determine the students' skills with whole number basic facts and solution to word problems and to determine whether students met the Year 3 numeracy benchmarks. The results shown in Tab. 3 highlighted the large range of whole number knowledge, and skills of all Year 4 students at this school (see also Pearn, 2009). The students had been randomly selected into their classes but the results varied considerably across classes as shown in Tab. 5.

These results were also being used to focus teachers on their students' whole number arithmetic knowledge and skills so that:

> … students initial conceptions then provide the foundation on which more formal understandings of the subject matter are built" (DE&T, 2005, p.5).

However, when presented with their results the teachers varied in their approach to analysing the results. The teacher of Class 4 carefully studied his students' tests and commented on individual students' results. Some students that he thought to be good at mathematics had good recall of learnt facts but were unable to solve simple word problems. Some students who were unable to recall number facts instantly were able to solve the word problems. At professional development sessions, usually conducted by the author after school, the teacher from Class 4 would ask insightful questions and expressed concern as to whether he was doing enough for his students. This teacher exemplified many of the attributes described by the AAMT Standards as the knowledge, skills and attributes required for good teaching of mathematics (AAMT, 2006). He appeared to have an excellent knowledge of the students he taught and was actively developing a coherent knowledge of mathematics appropriate for the level he was teaching and a rich knowledge of how students learn mathematics.

The Class 2 teacher flicked quickly through the tests and said that the results did not tell her anything she did not already know. At professional development sessions the Class 2 teacher would arrive late, leave early or make comments such as: "I already do that." but was unable to give examples of how this was implemented in her classroom.

These classes were not based on mathematical ability of the students. However the results from the *One Minute Basic Facts Tests* highlight the large diversity of the ability to recall basic number facts within each class and between classes. *The One Minute Basic Number Fact Tests* (Westwood, 2000) identified students struggling to recall basic number facts and those who had instant recall. The Number Screening Tests were designed to identify students mathematically 'at risk' but in this case were also able to identify Year 4 students who were already successful with whole number arithmetic.

A large number of the Year 4 students assessed using these two assessment protocols used inefficient counting strategies for both types of assessment. These strategies were demonstrated when the students tapped their fingers, blinked or rolled their eyes, and used tally marks on both tests. Teachers need to ensure that students develop more flexible strategies that allow them to develop fluency with number facts and know when and how to use them.

The author had been employed by the Principal to provide ongoing professional development for the Year 4 teachers to ensure that their students developed more efficient strategies when solving whole number arithmetic tasks. However, quality professional development is difficult to deliver when even one teacher is resistant and states: “I already do that”.

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ARITHMETIC AND COMPREHENSION AT PRIMARY SCHOOL

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Abstract

This paper presents a design-based research study for which a curriculum for first grade arithmetic has been built and implemented for two consecutive years in 60 and 120 experimental classes. We present the principles and rationale for this curriculum, which are based on the results of the current research in whole number arithmetic. Then we analyse some examples of the students' written work, offering comments on the basis of the principles and rationale previously delineated.

Key words*:* arithmetic, cooperative engineering, curriculum, first grade

Introduction

The paper presents a design-based research study for which a curriculum for first grade arithmetic has been built. This curriculum, given the name Arithmetic and Comprehension at Elementary School (ACE), has been developed within a team composed of teachers and researchers, and implemented in classrooms. Numerous researchers have studied elementary curricula intended for young schoolchildren (e.g., Brousseau, 1997; Fuson, 1990; Fyfe et al., 2014; McNeil and Fyfe, 2012; Bartolini-Bussi et al., 2011; Ma, 2010; Ding and Li, 2014), but it is difficult to find curricula especially designed to provide a *practical synthesis* of these works specifically for first grade. ACE is an attempt to build such a curriculum. In this paper we set two main objectives; first, to give readers a general understanding of the ACE project principles and rationale and second, to present some examples of students' work within ACE that can provide a window on the kind of whole number arithmetic comprehension that ACE may foster. For this perspective, the first part of this paper shall be dedicated to the description of the main features of the ACE curriculum. In the second part, we shall focus on materials and methods and explain how the student work that we present and analyse in the third part has been obtained. We conclude by discussing them in light of the issues concerning whole number arithmetic.

The ACE Curriculum

We focus first on the design aspect then on the mathematical aspect.

The ACE curriculum: A brief description of the research design

The ACE project experiments with the construction of number concepts in 6-year-old students (first grade) whom we are investigating. It gathers together five French research teams, each of which has designed a part of the whole curriculum. In this paper, we focus only on the Brittany-Marseille team, which is in of charge a domain called "Situations." The making of the curriculum is based on the implementation of a *cooperative engineering*, a specific form of design-based research (Cobb et al., 2003) that develops particular relationships between teachers and researchers (Sensevy et al., 2013).

The first year (2011-2012) of the experiment consisted of designing a curriculum for building of the concept of number in first grade. This design process was carried out in a specific way; the experimental situations were first carried out in four classes then redesigned online. The curriculum was implemented in 60 classes the second year of the experiment (2012-2013) and in 120 classes the third year of the experiment (2013-2014). This research was a quasi-experimental design. In effect, student learning in the experimental classes has been compared with student learning in control classes (pre-test/post-test assessment). Although the pre-test showed no significant differences between control and experimental classes, it is worthy to note the two main results we obtained by comparing performances of the post-test assessment: (1) For each year of the investigation (2012-2013 and 2013-2104), the students in the experimental classes outperformed the students in the control classes, particularly for the most conceptually demanding items (e.g., being able to decompose a given number using an additive method); and (2) For each year of the study, the gap between students from underserved communities (priority education zones, per the French system) and students from middle-class communities largely widened throughout the school year in the control classes but stayed at the same level in the experimental classes. That leads us to think that the ACE curriculum is a more equitable program than the traditional one. The ACE curriculum has been developed through a specific kind of evidence-based research.

8viewpoint, the ACE curriculum is based on the following principles: (1) Familiarising the students with numbers and relations within numbers by focusing first on "small numbers" for a long amount of time (Ma, 2011). (2) Giving prominent importance to the study of equivalence so that students become able to think of the equality sign not as a hint to produce an operation but as a relational sign (Brousseau, 1997; McNeil, 2014). (3) Using the arithmetic operations first as means to explore numbers and build significant relations between them; for example, in the core situation of this curriculum, the students are guided to refer to a number in an additive form (a sum) and to compare it, in particular, with other additive forms by using seminal conceptual strategies of relevant composition/decomposition ($3 + 4 = 3 + 3 + 1= 6 + 1$; $8 + 4 = 8 + 2 + 2 = 10 + 2$), decimal understanding ($24 = 20 + 4 = 10 + 10 + 4$), a topological approach to numbers. (4) Using manipulatives and representations in a systematic way by satisfying two criteria. The first one refers to the necessity of enabling the students to rely first on manipulative and concrete "objects" then to study iconic (analogical) representations of numbers then to write down equations in canonical form. This process seems very close to the tradition in Chinese textbooks (Bartolini-Bussi et al., 2011; Sun, 2011; Ding and Li, 2014) and can be thought of as "concreteness fading" (McNeil et al., 2012;

Fyfe et al., 2014). The second criterion lies in a "translational principle." To understand various properties of numbers, students had to compare different representations of the same mathematical reality to become progressively able to recognise the differences and the similarities between these representations. (5) The last principle of the ACE rationale holds as follows: To acquaint the pupil with the historical-cultural sense of mathematics (Bartolini-Bussi and Mariotti, 2008; Radford, 2014) and to apprehend the deep conceptual structure of mathematics (Richland, Stiegler, and Holyoak, 2012), students had to *write* mathematics and develop a first-hand relationship to mathematical writing. In a nutshell, the curriculum that we propose is characterised by a connected series of situations, all centred on the founding principles that we have presented below. The initial situation of this curriculum is the "Statements Game," which has been designed by the research team on the basis of the principles we stated above and which can be described as follows:

1. *One die is about to be thrown. Beforehand, the students use their fingers to make a "statement" (for example, a student shows two fingers on her right hand, and three fingers on her left hand).*

2. *The die is thrown. The students compare their statement with what is indicated by the die. If the sums are equal, the pupils have won.*

The continued complexification of the situation brings the students to make increasingly rich comparisons: the number of hands (students) is increased, as are the number and the nature of dice (1 to 10 dice are played with), etc. The students first play the game orally; then they write down the situations by utilising then conceiving the writing process as a way of designing the different games that they play. When students advance in their mathematical inquiry, the Statements Game situation gives them a concrete and basic reference, which they may always refer to give controlled meaning to a mathematical equation.

Materials and Methods

To give some examples of the students' production, we shall focus on a specific tool that was implemented in the ACE classroom, the "Journal of Number." At the end of a particular period of time (a learning unit of 2 to 6 sessions), students were asked to write down in this individual journal "something they know about mathematics," by following a general prompt given by the teacher, often on the basis of a previous student's production. The Journal of Number productions thus may be seen as a window on ACE's mathematical practice. Among the students' productions we have recorded, we have chosen a few that are representative of the average productions in most of the experimental classes. These are shown in Fig. 1.

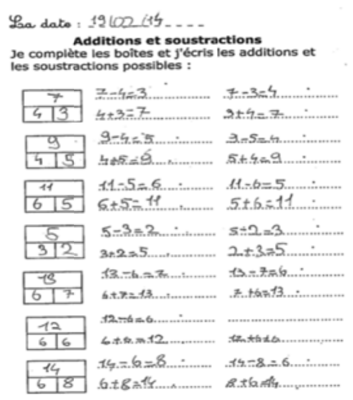
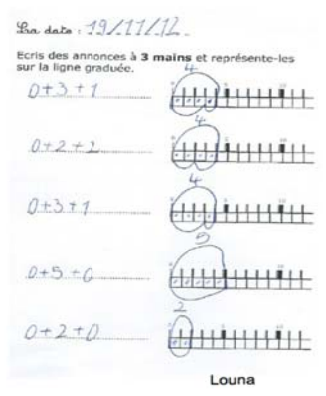
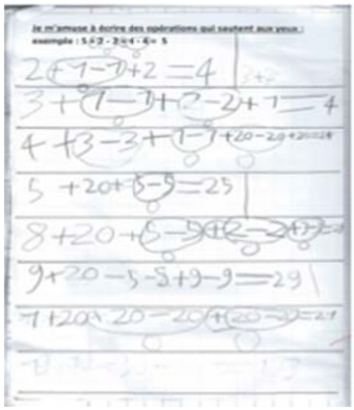
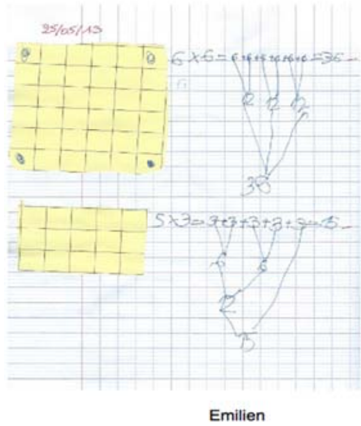
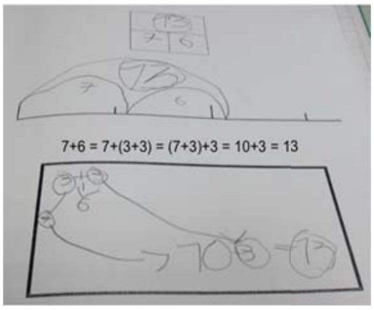
<table>
<tr><td></td><td></td><td></td></tr>
<tr><td>1. Using the number box</td><td>2. Translating representations</td><td>3. Playing a classroom game</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>4. Writing pieces sums and the pieces and coins necessary to pay</td><td>5. On the decimal system</td><td>6. Multiplicative structure</td></tr>
<tr><td>Yasser's method
15 – 7 : …
I compute 7 + … = 15
I know that 7 + 7 + 14
Thus 7+ 8 = 15
Thus 15 – 7 = 8</td><td>Vincent, Zina and Manel's Method
14 – 5 = …
I know that 14 – 7 = 7
I add 2, 7 + 2 = 9
14 – 5
14 -7 + 2 since 7 = 5 + 2</td><td></td></tr>
<tr><td>7. A method for computing</td><td>8. A method for computing</td><td>9. Mixing different strategies</td></tr>
</table>

Fig. 1: ACE – Sketching the Journal of Numbers Production

Example 1: Using the number box. I complete the boxes and I write down the possible additions and subtractions. This example refers to the study of the relationship between addition and subtraction through a tool, “the number box,” which enables the students to “invent” the numbers in the box and different relations and calculations of their own. Each calculation is based on the Statements Game.

Example 2: Translating representations. I write down three-term statements and represent them on the number line. In this example, students have to first refer to the concrete reference embodied in their fingers as it occurs in the Statement Game we previously described then represent the obtained statement on a line in an analogical way and in the usual symbolic way through an equation. The focus is on “small numbers.”

Example 3: Playing a classroom game. I play writing calculations which “jump out.” Example: 5 + 2 – 2 + 4 – 4 = 5. In this example, students have to invent different writings to play an “inverse relation” (Verschaffel et al., 2012) game in which they improve their systemic understanding of an equation.

Example 4: Writing pieces sums and the pieces and coins necessary to pay. In this example, the work in the Journal of Number enables students to build a strong relationship between the decimal system they are beginning to explore and understand and monetary “artifacts.”

Example 5: Elaborating on the decimal system and place value. Write down an additive writing with 6 or 7 terms. Write down this number in tenths and units. This example refers to the way decimal system and place value are worked out in ACE classes. The ten is introduced through composing-decomposing techniques and first appears as a way of easily comparing “long” additive writings. In this example, the students invent a given writing, which needs to represent some well-known decompositions (for example decompositions of ten) to be “converted” into a “tenths and units” number.

Example 6: A first approach on multiplicative structures. Students are simply engaged to build a paper rectangle and to write down a designation of this rectangle both in a multiplicative and an additive form. In this example, students elaborate the way multiplicative structures are presented in the ACE project. In the Statements Game, students face some “repeated dice throws” (i.e., 3 + 3 + 3 + 3 or 4 + 4 + 4, etc.). These throws are represented with rectangles (i.e., 4 x 3 or 3 x 4, etc.). Then these rectangles are referred to on the basis of two kinds of writing, additive and multiplicative. The study example shows how students are asked to produce concrete paper rectangles before referring to them with symbolic writings.

Examples 7 & 8: These two examples do not come directly from the Journal of Number, but they illustrate how the ACE curriculum (including the Journal of Number) makes the students able to elaborate some relevant mathematical strategies to compute some calculations. Each example represents a photo of the classroom blackboard on which the teacher has written the strategy used as dictated by some of the students. This strategy was aimed to be disseminated to the classroom collective. For Example 7 ("Yasser's method"): I compute 7 + … = 15; I know that 7 + 7 = 14; thus 7 + 8 = 15; thus 15 – 7 = 8. For Example 8 (14 – 5, "Vincent, Zina, and Manel's method"): I know that 14 – 7 = 7; I add 2, 7 + 2 = 9; 14 – 5, 14 – 7 + 2 since 7 = 5 + 2).

Example 9: Mixing different strategies and representations. In this example, the researchers have typed the long equation to represent in a canonical form the student's composing-decomposing strategy. After the box, the number line, which is used in an "approximation" manner, make understand the numbers both as "positions" and "movements" on the line. The composing-decomposing techniques are linked with the two systems of representation (the box and the number line) and enable the students to use their knowledge of doubles and decomposition of ten in a relevant way. It is worthy to note the inventive use of the sheet space necessary to display the calculation strategies.

Discussion and Conclusion

In this discussion, we would like to stress the most important points of this ACE project that we have presented in this paper. As we argue below, ACE is grounded in five principles (1) working out small numbers as long as necessary; (2) focusing on a deep understanding of mathematical equivalence; (3) engaging students in related arithmetical operations as a way of studying the decimal system through "conceptual techniques"; (4) making a systematic use of representations from the more concrete to the more abstract by linking them, by having the students able to compare them and to translate one to another; and (5) inciting students to write mathematics and to build a first-hand relationship to mathematics through this practice). Beyond these principles and their illustration in the Journal of Number, it could be worth stressing the following points.

Writing mathematics

Many scholars and researchers (e.g. Brousseau, 1997; Richland, Stiegler, and Holyoak, 2012; Chevallard and Sensevy, 2014) have contended that school mathematics need to achieve a conceptual density through an inquiry process that give them a kind of authenticity, a kinship with the cultural activity of professional mathematicians. We argue that one of the main criteria to appraise this kinship lies in the way mathematical writing is used in the classroom. In ACE, a strong hypothesis is that while learning arithmetic in first grade, students have to learn to write arithmetic at the same time and to invent some problems grounded in the concepts and techniques they are currently acquiring. In that way, the writing process is not confined to *the sole response to usual mathematical questions* of this grade, which may prevent students from understanding mathematics. Indeed, this kind of "response writing" often engages students to activate some routines built in the narrow arithmetical experience (McNeil, 2014) of the classical didactic contract (Brousseau, 1997;

Sensevy, 2012). On the contrary, in ACE, the "inventing process" which unfolds in the writing of mathematics is a way of strengthening the capabilities that students are building, while exploring their own new numerical relationships. Moreover, as we have seen in the study examples, it is very important to note that the mathematical writing is not only a means to foster a personal relationship to mathematics. In particular, through the Journal of Number practice, students can take other students' work as a point of departure and so relate their production to the collective endeavour. In the classroom, trying to build a kind of kinship, as with the cultural activity of mathematicians, means being able to organise a thought collective (Fleck, 1979; Sensevy et al., 2008), such as a research community.

Modeling

Another point we would like to emphasise lies in the kind of mathematics the ACE project attempts to promote. Our approach is grounded in Brousseau's epistemology of mathematical knowledge (Brousseau, 1997) and meets Freudhenthal's contention about the necessity of having children understand that mathematics can be a way of mastering phenomena in reality (Gravemeijer, 1994). In this respect, ACE can be viewed as a way of elaborating on concrete "systems" of the everyday life (for example, playing dice and trying to represent quantities on fingers) to "model" these systems through semiotic tools (Bartolini-Bussi and Mariotti, 2008) and the symbolic forms of the mathematical equations. This interest in the modelling process has brought us to develop a conceptual interplay between the concrete and the abstract within the classroom, which privileges a kind of dialogue and systematic relationship between the (more) concrete and the (more) abstract. Over the preference that many mathematics educators give to one or another of the two poles, we contend that this dialogue between the concrete and the abstract needs to foster a continual process of mutual reference (Fyfe et al., 2014), between various forms of representations, between systems and models. These orientations will constitute a main direction for future work, along with the dissemination of this new curriculum.

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PRIMARY MATHEMATICS EDUCATION IN MACAO: FIFTEEN YEARS OF EXPERIENCES AFTER THE 1999 HANDOVER FROM PORTUGAL TO MAINLAND CHINA

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Abstract

In the 21st century, around the world, many countries set a high priority seeking worthwhile curriculum practices and educational policies in order to establish an effective school system for the K-12 school-age children. However, no education system can provide Macao a ready-made curriculum model. If there is one such model available serving as an exemplar, adoption admittedly is generally difficult. This paper aims at introducing Macao's 15 years of experiences of primary mathematics education, after the official handover of the former Portuguese enclave to China in 1999. This study describes a model of educational practice on the topic of teaching of numbers, taking Macao's reality of an amalgamation of the cultures of the East and West into account in the discussion.

Key Words: amalgamation of eastern and western cultures, K-12 mathematics education, Macao

Background of Study

Macao has a population of 631,000, residing in a small land area of 30.3 km^2. For more than 400 years it had been governed by Portugal until 1999. With such a long history of 400+ years of cultural exchanges between the East and the West, Macao is unique in many respects regarding the fifteen years of free K-12 education provided to the school children. Today, Macao is a Special Administrative Region (SAR) of the People's Republic of China. Because of the "One country, Two systems" policy, Macao exhibits many fascinating features and qualities exemplary of an amalgamation of the Chinese and Portuguese cultures.

In academic year 2013/14, there are 77 schools, most of which (87%) are private. The 2013-2014 statistics show that there are 71,048 K-12 students, taught by 6,147 teachers in total. As at 2014, Macao is considered a high-income economy by the World Bank. The GDP is 697,502 MOP per capita, with an annual growth rate of 11.9%. The unemployment rate is very low (<2%), and the literacy rate exceeds 99%.

In the first decade of the new century, a series of initiatives promote the rapid development of Macao's basic education. For instance, in 2006, the Non-tertiary Education Law was promulgated which seeks to protect the right of K-12 education by law. Within ten years, Macao government's investment on non-tertiary education has been increased from MOP10.07 billion in 2002 to MOP32.92 billion in 2012 (Xinhua News, 2014). Education development fund is set up to promote various kinds of educational programs and activities.

Though the majority of K-12 schools in Macao are private, all students enjoy fifteen years of free obligatory basic education. Generally speaking, quality education for all is achieved admirably at the start of the new century.

Great achievements in education have been made in Macao in the past 15 years, particularly in mathematics education. For instance, two championships and three second prizes were won in five years in a row in American Regions Math League (ARML: International Group) since Macao first participated in 2009. The mathematics achievement in the latest PISA 2012 mathematics literacy survey has shown great progress as the ranking rose from 15th in 2009 to 6th in 2012 (Cheung et al., 2013). Macao's basic education is neither mediocre nor bad by international standard, and its experiences may be of interest to other economies to learn from.

As Macao is too small a place to have established its own curriculum standards for adoption in the new century, Macao schools and teachers are at liberty to adopt school mathematics curricula from Mainland China, Taiwan, Hong Kong, Portugal, and abroad. In order to provide the readers an impression regarding the adoption of mathematics curricula for basic education in Macao, this paper will briefly summarise Macao's fifteen years experiences of enhancing the quality of mathematics teaching and learning, as well as the measures adopted from initiatives learned from Mainland China, Macao's major Chinese-speaking basic education counterparts.

The Basic Mathematics Competence Requirements for Primary Schools in Macao: Local Curriculum Standards

As an international city, Macao government has made great effort to improve its education system in order to increase its manpower capacity even before it was formally returned to China in 1999. The first curriculum standard was issued in 1988, entitled "Macao Mathematics Syllabus (Trial)". It was revised in 1999. Six years later, the Education and Youth Affairs Bureau of Macao SAR Government (also called DSEJ) initiated the development of "Basic Competence Requirements in Mathematics (BCRM) for Primary Schools in Macao" (小学数学基本学力要求). BCRM was released in 2011 (DSEJ, 2011a) and was piloted at the lower primary level in eight schools in the 2012-2013 academic year (DSEJ, 2013a), and at the upper primary level in 2013-2014 academic year. The implementation of BCRM will have significant influence on mathematics curriculum and teaching practice. Note-worthy is the fact that BCRM stipulated the 'bottom lines' at various stages instead of the 'ceilings'. Therefore, the development of BCRM was not intended to unify all the courses and the teaching materials in Macao. Instead, schools have the freedom to develop school-based curriculum based on their educational visions and students' abilities. Like NCTM (2000), BCRM specified the learning objectives in five content areas: (A) Numbers and their operations; (B) Shapes and space; (C)

Measurement and its applications; (D) Data analysis and probability; and (E) Basic knowledge in algebra. The numbers of items under each area are 32, 24, 23, 13, and 6, respectively. The teaching of numbers and their operations takes up one-third of the whole content covered. Below are 27 of 32 items under "Numbers and their operations":

A-1-1 Being able to represent objects in daily life using numbers;

A-1-2 Being able to use arithmetic operations to solve related problems from daily life;

A-1-3 Be excited to be engaged in mathematical activities;

A-1-4 Being able to understand the meaning of cardinal and ordinal numbers;

A-1-5 Being able to recognize, read, and write numbers within 10,000 and to compare the magnitude of numbers;

A-1-6 Being able to understand the differences and relationships between digits and number of digits, the differences and relationships between numbers and digits, to understand the structure of numbers based on the understanding of counting units for different digits;

A-1-7 Being able to understand the meaning of addition and subtraction, to do addition and subtraction of numbers with answers less than 10,000;

A-1-8 Being able to understand the meaning of multiplication, and to multiply 3-digit by 1-digit numbers, and 2-digit by 2-digit numbers;

A-1-9 Being able to do addition and multiplication in a faster way using commutative and associative properties of addition and multiplication;

A-1-10 Being able to understand the meaning of division, and to do divisions up to 3-digit by 1-digit numbers;

A-1-11 Being able to understand the concepts of divisibility, indivisibility, quotient, and remainder;

A-1-12 Being able to conduct simple 2-step calculations.

A-1-13 Being able to understand the relationships among speed, distance, and time.

A-1-14 Being able to understand the process of approximation, and to do it in real contexts.

A-1-15 Being able to master mental addition and subtraction of numbers within 100.

A-1-16 Being able to master mental multiplication and division of numbers in tens and hundreds.

A-2-1 Being able to express one's thinking using mathematical language;

A-2-2 Being able to respect and understand other's approaches to solve mathematical problems;

A-2-3 Being able to develop a positive attitude towards mathematical explorations;

A-2-4 Being able to recognize numbers within one billion and to compare their magnitudes.

A-2-5 Being able to understand odd and even numbers, prime and composite numbers;

A-2-6 Being able to understand the meaning of factors and multiples of a number and the relationship between them;

A-2-7 Being able to understand the meaning of common factors, common multiples, greatest common factor, and least common multiples, and to be able to find them;

A-2-9 Being able to understand the features of numbers divisible by 2, 3 or 5;

A-2-10 Being able to do the four operations of numbers as whole numbers, decimals, fractions, and percent;

A-2-11 Being able to solve simple speed problems from daily life.

Elementary Mathematics Textbook Series in Macao: Some Salient Characteristics

DSEJ has mandated the basic literacy skills that children need to develop year by year and left it for Macao schools to decide what textbooks to use, because around ninety percent of the schools are private with no standard curricula and textbooks. If a school prefers to follow the curriculum system in Mainland China, it will select textbooks used in Mainland China, especially those widely used textbook series like those published by People's Education Press and Beijing Normal University Press. Similarly, if a school would like to implement the mathematics curriculum used in Hong Kong, it will select textbooks published in Hong Kong. For example, *Modern Mathematics for the 21st Century*, published by Modern Educational Research Society Limited. A small number of international schools also select textbooks used in Singapore, Canada, or United Kingdom. Wong (2005) edited the first elementary mathematics textbook series for the 1-6 grades of the local schools. However, it is note-worthy that it is the old version textbooks, not the revised versions of textbooks in Mainland China, Hong Kong, and Taiwan, was adopted in Macao schools. Regarding the teaching of numbers and their operations, we shall briefly introduce the two main features of the first textbook series for local schools (Wong , 2005).

(1) Developing algebraic ideas in early grades: Number puzzles

In the past two decades, it has been widely accepted in mathematics education that developing algebraic thinking in the early grades is necessary (Cai and Knuth, 2005). Number puzzles with some missing numbers are often used to develop children's concept of equations. The puzzles can be presented either in pictures or in numbers. Below is an example taken from the first-grade mathematics textbook. In these figures, 2-3 numbers need to be found and filled in the circles and spaces of the pictures so that the numbers satisfy the same number pattern embedded.

Fig.1: Number puzzles

(2) Localising ancient mathematics in Macao's mathematics curriculum

Chicken-rabbit problem is famous in Ancient China not only because it is a typical equation problem in two variables, but also because it has multiple solutions. Though it may be difficult for the second-grade students, we encourage them to use pictorial and tabular methods to help find the various solutions. Hopefully, they can eventually find all the solutions by themselves. In the process of searching for solutions, they develop their listing skills systematically. In the textbook, solution of ancient problem by the equation method is replaced by filling tables so as to find the answer.

Multiple Ways to Develop In-service Teachers' Teaching Abilities

In a recent OECD study many countries reported shortfalls in teaching skills and the difficulties in updating them (OECD, 2005; 2009). In 2007, the European Commission noted that incentives for teachers to carry on updating their skills throughout their professional lives were weak. It is interesting to note that Macao's teaching culture is largely shaped by western traditions in which teaching is regarded as a private practice with norms and structures that favour individualism and autonomy. The Macao education system is decentralised and fragmented without a united curriculum. Under different curricula, schools, and grade systems, teachers often have to work alone. Each new generation of teachers must start from zero in building their teaching experiences in their own way. Their weekly heavy teaching loads (normally more than twenty classes) make it difficult for them to redesign lessons for improvement. Under such a system, experienced teachers have less opportunity to share their experiences with the beginning teachers.

In short, it is reasonable to conclude that, from a system perspective, there is relatively little room for teacher professional development in Macao. Since 2006, Curriculum Reform and Development Council of Macao were established to implement assessment of in-service teachers' abilities and to reflect and give suggestions for further improvement. As a result, it has developed three initiatives learned from Mainland China for the professional development of their in-service teachers, namely: (1) Design Award Scheme for Teaching and Learning; (2) Pilot Scheme of the Elementary Curricula; and (3) Study Plan of the Leading Teachers. These initiatives may prove useful for the teacher education development in Macao.

A. Award Scheme on Instructional Design (ASID)

Award Scheme for Instructional Design (ASID) was initiated by DSEJ in 1996. It organises schools' assessment procedures every academic year. The instructional products include lesson plans (whether designed as a teaching unit, or a course spanning a semester/whole academic year), action research, and open demonstrations. It is hoped that teachers can be involved in selecting the teaching materials, and their abilities will be enhanced in due course. Those winning design will be uploaded to the DSEJ website for other teachers to use (DSEJ, 2011b).

B. Pilot Scheme of the Elementary Curricula

To implement BCRM, the Pilot Scheme of the Elementary Curricula was proposed and carried out in several schools. The core subjects are mathematics and Chinese. In the academic year of 2012-2013, it was conducted at the lower primary levels (Grades 1-3). And then it was extended to the upper primary levels (Grades 4-6). Specifically, the following activities were carried out:

(1) A group of four experts was formed to provide academic support and teacher training programs. They visited two schools four times a year. During each visit, they observed teachers' teaching activities and gave suggestions for further improvement. Meanwhile, they worked together with the schools' mathematics teachers, and discussed how to implement BCRM and adjusted their teaching plans.

(2) Selected mathematics teachers from an exchange program between Mainland China and Macao visited the participating schools every week so as to support the implementation of the scheme;

(3) Sharing sessions and workshops of in-service training were organised, with the local mathematics teachers informed beforehand.

After the scheme was completed, the products were collected and shared among peers on the Internet (DSEJ, 2011b).

C. Study plan of the leading teachers

Leading teachers play important roles in the professional development of in-service teachers, as well as in the implementation of intended programmes of study at school. A study plan has been developed for the 40 leading teachers in Macao, including heads of mathematics department, school leaders, and excellent mathematics teachers (DSEJ, 2011b). The study plan includes a variety of units, such as topic studies, curriculum studies, workshops, school visits, and sharing sessions. Subject-based experts from the Greater China region were selected as the instructors. This study plan could have policy implications for solving the problem regarding the lack of coherence and continuity as in the European in-service teacher education (OECD, 2005; 2009).

Making Phone Calls on Homework: A Help-Seeking System to Meet Individual Student Needs

According to the PISA 2012 main survey, about 11% of 15-year-old students with proficiency below level 2 are low-performing students who cannot function productively in society, and this percentage is small by international standard (Cheung et al., 2013). Immediate help, especially to slow learners or students with learning difficulties, is important for students' day-to-day learning. But how to make it come true is difficult for nowadays education systems. DSEJ initiated a programme called "Making Phone Calls on Homework" in 1997, which was actually undertaken by the Macao Association of Mathematics Education Research. It is now well accepted by local school children and their parents since they can just make a phone call, and the trained helpers will give some hints or guidance to help them finish their homework on time. However, because of the improvement of classroom teaching in schools, the number of calls phoned has been decreasing in recent years.

Implications for Mathematics Teaching and Learning in Non-Chinese Speaking Educational Systems

In the new century, Mainland China, Hong Kong, and Taiwan governments attempted to promote their own revised school curricula, and positioned the curriculum decision making policies in the Chinese historical, educational, social and economic contexts. While the initiatives have inspired innovative reforms in a number of schools, and in principle met with considerable support, on a wider scale implementation of the concerned reforms was actually hampered by insufficient resources, conceptual ambiguity and conservative resistance (Zheng, 2014). Different from other Asian educational systems in terms of freedom and democracy, and from the western systems in terms of amalgamation of the Chinese and Portuguese cultures, Macao offers a unique perspective for the educational practitioners to have a new look at Chinese mathematics education, as Macao is the only Chinese-speaking community so far without any curriculum reform. Two issues deserving attention of the readers are elaborated below:

Freedom and democracy

Macao schools are endowed with undoubting right of curriculum autonomy by Macao Law (see 38/94/M, 39/94/M, 46/97/M for details). These laws were promulgated in 1994 and 1997, aiming to regulate curriculum design for basic education from the preschool to the high school levels. Traditionally, teaching materials are selected and determined by schools at the stage of basic education in Macao. In other words, teaching materials, teaching contents, teaching language, and even curriculum design and course structure are quite free. The course management system should have given schools enough freedom to develop school-based curriculum, which cater for the individual needs of the students.

Internationalisation versus localisation

The publication of NCTM's 2000 Standards has affected the revision of mathematics curriculum standards in many countries in certain ways (Ma, 2013), which may understood as the process of internationalisation. For this, Ma (2013) has kindly reminded mathematics curriculum developers in Mainland China of maintaining their own traditions. In the process of adapting certain kind of practice working effectively in one culture, which may be called "localisation“, attention should always be paid to the local schooling contexts. In the past 15 years, DSEJ has left the schools to decide what mathematics curriculum they would like to follow, but focused on the professional development of teachers and provision of immediate help to the students. It is widely accepted that there is no such a curriculum that fits all the students. Therefore, DSEJ just set up BCRM as the bottom-line standards to be met by all students.

As a place of cultural exchange between the West and the East, Macao is unique in that it does allow some schools to maintain the tradition of Chinese mathematics education, but also allow other schools to develop their own school-based projects. The great improvement in student achievement in PISA 2012 indicated that Macao is successful in integrating the western and eastern cultures, and in maintaining a good balance between freedom and authorities. Macao's 15-year experience of mathematics education can provide a mirror for the policy-makers and researchers from OECD countries and the Great China regions, and beyond (OECD, 2005; 2009). For western countries, more efforts should be made to the professional development of mathematics teachers, particularly in improving their teaching skills. For eastern countries, more freedom needs to be delegated to the schools so that they can provide school curricula that really cater for the needs of their students.

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AN ANALYSIS OF TWO-DIGIT NUMBERS SUBTRACTION IN HONG KONG PRIMARY MATHEMATICS TEXTBOOKS

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Abstract

Addition and subtraction of whole numbers are the basic skills needed by students to solve more complex calculations. Two-digit numbers subtraction is a relatively challenging topic for primary school students as it involves problems concerning decomposition. Textbooks are the dominant tool for the teaching of mathematics in Hong Kong elementary schools. It is worthwhile to investigate how different textbooks organise their contents and what the aims and rationale are for this arrangement. Four sets of primary mathematics textbooks used in Hong Kong are analysed using content analysis. Most of these textbooks follow the arrangement suggested by the Curriculum Development Council. However, subtle differences exist in the teaching strategies they present and the design of the contents.

Key words: primary mathematics, teaching strategies, textbook analysis, two-digit numbers subtraction

Introduction

Addition and subtraction concepts have been developed using hands-on experiences with countable objects or the placing of value blocks. Fuson (1992) has suggested that a counting method is the most basic way to help children to learn simple addition. In brief, addition is the "putting together" of two groups of objects and finding how many there are in all. Subtraction means "how many are left" or "how many more or less there are." In school mathematics, addition and subtraction of whole numbers are the first stage of the curriculum (from Grade 1 to Grade 3). Although they are the basic skills that students need to solve other complex calculations, students often have difficulties in learning how to make these calculations, particularly in multi-digit operations (Young and O'Shea, 1981). For instance, students simply take the difference of two digits in each column, irrespective of which is the larger. Doing this produces an error like 63-24 = 21, in which students need to borrow but do not. Or in some cases, students do not need to borrow but nevertheless do so, eg.96 – 42 = 34 (Young and O'Shea, 1981, p. 155). Underlying these calculations, there is a complex thinking process. Subtraction with single digit numbers is relatively simple, while multi-digit subtraction is relatively difficult to learn because it involves decomposition. Many researchers have identified effective strategies for developing the multi-digit addition and subtraction abilities of children (Fuson, 1992; Fuson et al., 1997; Thornton, 1990; Torbeyns et al., 2009). However, there has been little focus on the influence of the curriculum material, in particular the textbooks, on the students' learning of the addition and subtraction of whole numbers.

It is accepted worldwide that mathematics textbooks have a major influence on classroom practice (Schram, Feiman-Nemser and Ball, 1989; Valverde et al., 2002). According to the results of the Trend in International Mathematics and Science Study (TIMSS), the textbook is a primary information source for teachers in deciding how to present the course contents (Schmidt, Mcknight and Raizen, 1997). 65% of grade 4 mathematics teachers in TIMSS 2007 used textbooks as the primary study material, and 30% used them as a supplement. In Hong Kong, schools may use textbooks and learning materials on the recommended textbook list (RTL) of the Education Bureau (EDB). These textbooks should be in line with the curriculum guides and assessment guides issued by the Curriculum Development Council (CDC). They should reflect the learning objectives and expected learning outcomes and also cover the core elements of the curriculum. Although the EDB suggests that it is not a compulsory requirement to use the textbooks and learning materials on the RTL, most teachers rely on the recommended textbooks. In TIMSS 2007, the number of grade 4 mathematics teachers using textbooks as the primary study material and as supplements are noticeably higher than the global results: 84% and 15% respectively (Mullis, Martin, and Foy, 2008, p. 290). In TIMSS 2011, the percentage of Hong Kong grades 4 and 8 mathematics teachers who used textbooks as a basis for instruction reached 88%, and the percentage of teachers who used textbooks as a supplementary resource was 11% (Mullis et al., 2012, p. 392-394).

Subtraction of multi-digit numbers is a relatively challenging topic for primary students as it involves problems concerning decomposition. According to CDC, the content of the textbooks should be closely in line with the curriculum guide for the subject (EDB, 2009). However, publishers or designers can arrange the topics and the formats of textbooks in different ways. Therefore, in this study we focus on the topic of two-digit subtraction (including decomposition) and analyse the structures and contents used in different textbooks. Using content analysis, we try to identify the differences and common aspects, and explore the reasons why students and teachers have difficulties in this respect.

Methodology

In the Hong Kong textbook market, there are eight packages of mathematics textbooks for the lower primary level (Grades 1-3) and nine for the upper primary level (Grades 4-6). In our study, four sets of textbooks are selected for analysis, all of which are widely used in Hong Kong primary schools. They are *21st Century Modern Mathematics* (Lam and Chan, 2006), *Primary Mathematics* (Lo, 2005), *Longman Primary Mathematics* (Leung and Lau, 2007), and *Primary Mathematics in Focus* (Hung, 2012), henceforth referred to as T1, T2, T3 and T4. In order to identify the structure and content of two-digit numbers subtraction the textbooks, we carried out an analysis of the content of the books in a systematic manner, i.e. we did content analysis (Krippendorff, 1980). We first noted the chapters in the books that matched the mathematics curriculum guide (CDC, 2002) for the topic two-digit subtraction taught in primary mathematics. Next we studied the arrangements of the teaching sequence and the teaching strategies that were demonstrated in the books.

Results and Discussions

According to the mathematics curriculum guide (CDC, 2002), three units related to two-digit numbers subtraction are to be taught in primary mathematics. These are *Basic addition and subtraction* (*within 18*) (**1N3**), *Addition and subtraction* (I) (*addition within 2 places; subtraction within 2 places, excluding decomposition*) (**1N5**), and *Addition and subtraction* (II) (*addition within 3 places; subtraction within 2 places*) (**2N2**). These are to be covered in Primary 1 and Primary 2. Details are shown in Tab. 1.

It is found that most textbooks follow the arrangement of the teaching sequences suggested by the curriculum guide. For the teaching unit **1N3**, all of the textbooks arrange it to be taught in the first semester of Primary 1. Three sets of textbooks (T1, T3 and T4) devote one chapter to the teaching of subtraction within 18. Only T2 covers this teaching unit in different chapters with the unit **1N2** (*Numbers to 20*) rather than devoting a unique chapter to teaching subtraction within 18. For **1N5**, all of the textbooks arrange for it to be taught in the second semester of Primary 1. T1 and T2 use two chapters to teach subtraction without decomposition. These were devoted to “2-digit number minus 1-digit number” and “2-digit number minus 2-digit number” respectively. T3 and T4 combine them and presented them in one chapter.

Furthermore, of all the textbooks, only T4 deals with “Successive subtraction (excluding decomposition)” and devotes an entire chapter to introducing this type of questions. As for **2N2**, all of the textbooks arrange this to be taught in the first semester of Primary 2 and use one chapter to introduce subtraction within 2 places (including decomposition). Before that, T1 and T2 use one chapter for revision of subtraction within 2 places (excluding decomposition) whereas T3 and T4 do not have such a revision chapter. The topic ‘Successive subtraction’ is also introduced in the same chapter in T1, T2 and T4, whereas T3 introduces it in an entirely separate chapter.

Textbook	Grade		
	Primary 1		Primary 2
	Semester 1	Semester 2	Semester 1
T1	Chapter 21: Subtraction within 18	Chapter 14: Subtraction within 2 places (2-digit number minus 1-digit number, excluding decomposition)	Chapter 9: 1. The relation between addition and subtraction 2. Subtraction within

		Chapter 15: Subtraction within 2 places (2-digit number minus 2-digit number, excluding decomposition)	2 places (excluding decomposition) Chapter 10: 1. Subtraction within 2 places (including decomposition) 2. Successive subtraction
T2	No unique chapter on Subtraction within 18	Chapter 8: Subtraction within 2 places (2-digit number minus 1-digit number, excluding decomposition) Chapter 9: Subtraction within 2 places (2-digit number minus 2-digit number, excluding decomposition)	Chapter 8: 1. The relation between addition and subtraction 2. Subtraction within 2 places (excluding decomposition) Chapter 9: 1. Subtraction within 2 places (including decomposition) 2. Successive subtraction
T3	Chapter 22: Subtraction within 18	Chapter 16: Subtraction within 2 places (excluding decomposition)	Chapter 5: Subtraction within 2 places (including decomposition) Chapter 6: Subtraction within 2 places (Successive subtraction)
T4	Chapter 16: Subtraction within 18	Chapter 7: Subtraction (1) (2-digit number minus 1-digit number, 2-digit number minus 2-digit number, excluding decomposition) Chapter 8: Subtraction (2) (Successive subtraction)	Chapter 5: 1. Subtraction (within 2 places, including decomposition) 2. Successive subtraction

Tab. 1: Arrangement of two-digit numbers subtraction content in the textbooks

Textbook	Example	Teaching sequence	Number of examples	Question Types
T1	- Use of Abacus - Number line - Picture for counting	Step 1: Revision on subtraction within 18 & 2-1a Step 2: 2-2a Step 3: 2-1b Step 4: 2-2b	7 21 3 15	Word problems Direct computation (horizontal and column form)
T2	- Use of Abacus - Number line - Picture for counting	Step 1: Revision on subtraction within 18 & 2-1a Step 2: 2-2a Step 3: 2-1b Step 4: 2-2b	10 10 8 13	Word problems Direct computation (horizontal and column form)
T3	- Use of Abacus - Number line	Step 1: 2-2a Step 2: 2-1b Step 3: 2-2b	5 5 17	Word problems Direct computation (horizontal and column form)
T4	- Use of Abacus	Step 1: 2-1a Step 2: 2-1b Step 3: 2-2b	2 4 13	Word problems Direct computation (horizontal and column form)

Tab. 2: Strategies for teaching two-digit numbers subtraction in the textbooks

Notes: 2-1a: Subtract a 1-digit number from a 2-digit number (excluding decomposition); 2-1b: Subtract a 1-digit number from a 2-digit number (including decomposition); 2-2a: Subtract a 2-digit number from a 2-digit number (excluding decomposition); 2-2b: Subtract a 2-digit number from a 2-digit number (including decomposition).

From Tab. 2, we can see the arrangement for teaching strategies dealing with two-digit numbers subtraction with decomposition. In particular, when we take a closer look at the use of the abacus in teaching 2-digit number minus 2-digit number, we find some interesting results. T2 and T4 were very similar. Both had more explanations on the process of subtraction by using an abacus compared to T1 and T3 (See Fig. 1). They decomposed all the steps for use with the abacus and interpreted each step correspondingly. T1 decomposed one tens-place bead into ten ones-place beads and at the same time circled eight beads to represent the operation of subtracting eight. It also provided an estimation result. Only one picture of a column form shows the calculating process in T1, while in the other three textbooks (T2, T3, T4), every step of the process is presented in a column form. T3 also shows the operation of decomposing 1 ten into 10 ones. This was a little different from the others. Firstly, it showed the transformation of one tens-place bead into ten ones-place beads; secondly it showed the process of subtracting eight. Both steps used the same symbol (crossing out). Students may not completely understand that the first step represents subtracting ten. This misunderstanding could also arise in T4. At the first step, one tens-place bead (orange) was circled and transformed into ten ones-place beads (green) and a dotted arrow was used to demonstrate the process. At the same step, nine ones-place beads were circled to demonstrate the process of subtracting nine. Students could conclude only that one tens-place bead is transformed into nine ones-place beads.

Conclusion

Through the content analysis of these four textbooks, it was found that although all of them followed the mathematics curriculum guide to design the teaching units, there still exist subtle differences in the structure and content of arranging subtraction within 2 places. All the textbooks use abacus to present the calculation of numbers together with horizontal and column form. To some extent, it is accord with Bruner (1960)'s stages of representation of learning theory. However, some textbooks give more explanations than others and provide detailed interpretation of the calculating process. Some symbolic representations of the calculating process may also lead students to misunderstand important elements in this topic.

As far as the teachers are concerned, it is crucial that they clearly and holistically understand the algorithm of subtraction. They should also carefully interpret the contents (including the pictures and symbols) in the textbooks. On the one hand, they need to consider the risk that students could misunderstand the presentation. On the other hand, the teachers should not follow the arrangement of the textbooks blindly. They could introduce more real life examples and encourage students to carry our more manipulations and have hands-on experience which may strengthen their numeracy. Since the presentation of the textbooks may lead to confusion or misunderstanding among both students and teachers, the ways in which they use the textbooks may also be important and this could be a direction for future research.

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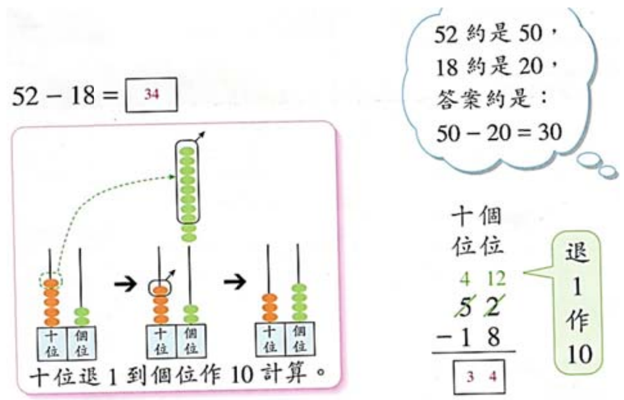
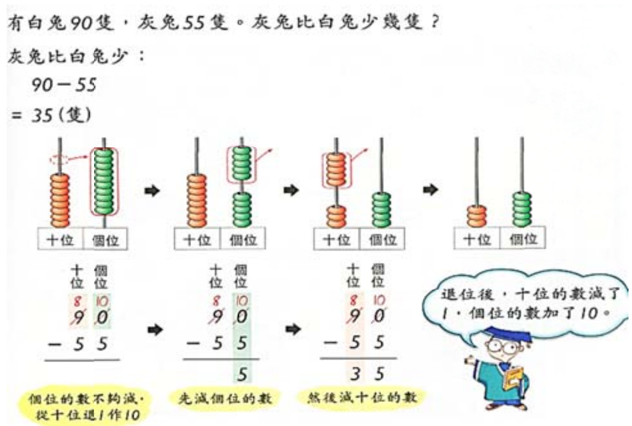
T1



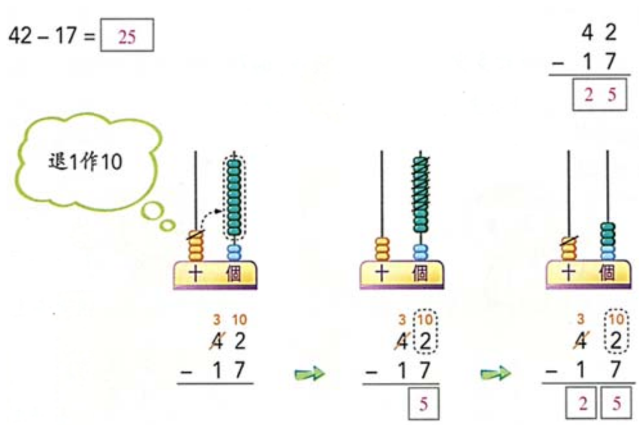
52 – 18 =

T2



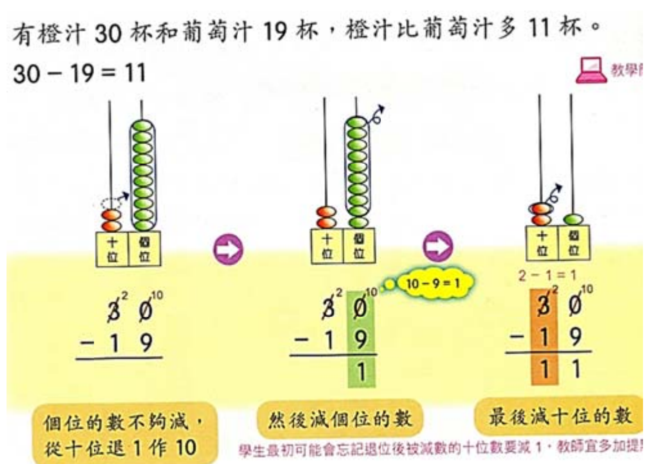
90– 55 =

T3



42 – 17 =

T4



30 – 19 =

Fig. 1: Use of abacus in teaching two-digit numbers subtraction

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CLASSROOM ASSESSMENT TECHNIQUES TO ASSESS CHINESE STUDENTS' SENSE OF DIVISION

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Abstract

This paper is about an explorative study of the use of classroom assessment techniques (CATs) by primary school mathematics teachers in China. Six female teachers and 216 third-grade students from two schools in Nanjing were involved. The focus was on assessing whole number arithmetic. Teachers' use of the CATs was investigated through lesson observations, feedback forms, interviews, and reports. In this paper we zoom in on one CAT in which students had to solve division problems without making use of the standard division algorithm, being the only procedure they had been taught. From the solutions teachers can infer whether their students really had deep understanding of the division operation. Only a few students could apply a solution strategy without using the standard algorithm. All teachers were initially unsure about what information they were supposed to find with this CAT and did not know how to deal with the results afterwards.

Key words: China, classroom assessment techniques, division, student work, textbook, whole number arithmetic

Introduction

Knowledge about students' learning is a sine qua non for educational decision making. Therefore, assessment – understood as the process in which students' responses to specially created or spontaneously occurring stimuli are used to draw inferences about their knowledge and skills (Popham, 2000) – plays a crucial role in education. Of the many different types of assessment that can be distinguished, formative assessment (Cizek, 2010) has the strongest link to teaching. It informs teachers directly about their students' learning processes so they can tailor their instruction to their students' needs. Formative assessment carried out by the teacher is often called "classroom assessment" (e.g., Mavrommatis, 1997). In fact, classroom assessment includes all teacher activities meant to collect information about their students' understanding of a particular topic.

To emphasize assessment's supporting role for teaching and learning, the Assessment Reform Group (1999) introduced the term "assessment for learning". This term was an eye-opener for many involved in assessing students' learning. In this new approach to assessment the focus shifts from a mainly test-based approach to one where assessment is more integrated with instruction and contains all kinds of informal assessment activities carried out by the teacher (Torrance, 2012).

From the moment that Black and Wiliam (1998) brought the power of classroom assessment to raise students' achievement to a larger audience, more and more research has been conducted on its practical applications. An example is the two-year project carried out with teachers in the United States (Leahy et al., 2005), in which eventually 50 “techniques” to improve teachers’ classroom assessment practice were developed. Characteristic of these techniques is that they blur the divide between instruction and assessment and are low-tech, low-cost, and usually feasible for individual teachers to implement. Another characteristic of these “classroom assessment techniques” (CATs), as we will call them hereafter, is that these are often well-known activities done by teachers that are now deployed in a new way with a specific assessment focus.

Inspired by the work of Wiliam and colleagues (Leahy et al., 2005; Wiliam 2011), a project aiming to improve classroom assessment was started in the Netherlands. Recently, two consecutive small-scale studies were conducted in Dutch primary schools to investigate the feasibility and effectiveness of the CATs in mathematics education (Veldhuis and Van den Heuvel-Panhuizen, 2014). In these studies, CATs were used to help teachers quickly find information about students’ abilities in whole number arithmetic and provide indications for further instruction. Two different formats were used: whole-classroom response systems directly informing the teacher and worksheets that the teacher has to check after the lesson. Results from the studies showed that using CATs had a positive effect on students’ learning. The mathematics achievement of students who were in classes where CATs were used improved considerably more than that of students from a national norm sample. Moreover, teachers and students reported enjoying the CATs and finding them useful. The present paper reports on an explorative study similar to the Dutch studies, but carried out in China.

In China, which has a long history of examination-oriented education, an assessment reform in basic education was kicked off by the Ministry of Education as part of the New Curriculum Reform in 2001. Therefore, during the last decade much attention has been given to putting assessment into the hands of teachers and help them perceive and practice the idea of assessment supporting teaching and learning (Zhang, 2009). However, despite this effort it was found that such assessment is only weakly relevant to teachers (Brown et al., 2011). Also, it seems that mathematics teachers in primary school in China tend to equate classroom assessment with their reactions to students’ different responses, and they do not include questions to reveal students’ thinking (Zhao, Van den Heuvel-Panhuizen and Veldhuis, in preparation).

The present study explored whether Chinese primary school teachers’ classroom assessment practice in mathematics education can be improved by applying CATs. Similar to the Dutch project the focus was on CATs to be used in teaching whole number arithmetic in the second semester of Grade 3; in particular the focus was on the standard algorithm of division. For this mathematical topic a package of CATs meant for two and a half weeks of teaching was designed by the authors of this paper. The package was tried out in February–March 2014. In this paper only part of the study is discussed.

Materials and Methods

Six female third-grade mathematics teachers (age $M = 32$; $SD = 7.23$ years) and 216 students from two primary schools in Nanjing tried out the CATs. Both schools are located in an urban district. School I, of which two teachers (Teachers A and B) and 60 students participated, has an average reputation. School II, of which four teachers (Teachers C, D, E and F) and 156 students took part, has a good reputation for its quality of education and the facilities in this school are better than those in School I. All teachers used the 苏教版 Textbook published by Jiangsu Education Publishing House.

Because the CATs should be integrated in the teachers' teaching practice there had to be a close fit between the CATs and the mathematics content provided by the textbook. This implied that we could not simply take over the CATs we had developed for the Dutch project. In the Chinese textbook, students start to learn multiplication and division in the first semester of Grade 2. After learning the basic knowledge and skills of multiplication and division (the multiplication tables), students already learn the algorithms for multiplication and division near the end of the first semester of Grade 2. This means students become familiar with the standard digit-based algorithmic vertical notation of multiplication and division from a very early age on. The teaching/learning trajectory of these algorithms consists of problems with a progressively increasing number of digits. At the beginning of the second semester in Grade 3 the students have arrived at solving division problems in which three-digit numbers have to be divided by one-digit numbers (see Tab. 1 for the content of the Chapter 1 on division that is dealt with in Grade 3 in February, 2014).

Lesson	Type	Topic	Example problems
1	New	Quotient is a three-digit number	600÷3 = 200 986÷2 = 493
2	New	Quotient is a two-digit number	312÷4 = 78
3	Repetition of Lesson 1 and 2		
4	New	0 in dividend (and quotient)	0÷3 = 0 306÷3 = 102
5	New	0 only in quotient	432÷4 = 108
6	New	Two-step division problem	There are two bookshelves with four layers. When there are 224 books in total, how many books are on one layer?
7	Repetition of Lesson 4, 5 and 6		
8	Repetition of the whole chapter		

Tab.1: Lesson plan for teaching the standard algorithm for division of three-digit numbers divided by one-digit numbers in Chapter 1, second semester of Grade 3

When designing the CATs for this chapter, two requirements were taken into account. The CATs should be linked to the lesson objectives and they should provide teachers with information about their students' learning to help them to reach a deeper understanding than just knowing whether or not students have answered a problem correctly. In total, for this chapter 13 CATs were developed.

The CAT of our focus in this paper is *Solving division problems without standard algorithm*. This CAT was planned for Lesson 8, when the students have had extensive practice in using the standard algorithm. Normally, at this stage, most students are able to carry out the algorithm and solve division problems without making mistakes. However, this does not automatically mean that they have a deep understanding of the division operation. It is also possible that students just apply the procedure in a mindless, mechanistic way. To make decisions for further instruction teachers need to know how stable students' understanding is. When students only have superficial knowledge, they might get in trouble when they have to use the division algorithm with, for example, decimal numbers.

The main idea behind this CAT is to reveal whether students have a clue on how to solve a division problem without using the standard algorithm. Therefore teachers provided the students with a worksheet containing four division problems – 468÷2=, 594÷6=, 480÷3=, and 816÷4= – presented in horizontal number sentences.

To help the teachers understand the purpose and procedure of the CATs, four meetings were organized in which the CATs were discussed. To collect data about the use of the CATs, all of one teacher's lessons in which she used the CATs were observed and recorded, and after each lesson she was interviewed. The other five teachers were observed and video recorded for at least one lesson per week. In the end, all teachers wrote a short report about whether and why they liked or disliked the CATs.

Results

According to the teachers' reports, the students were given at most 10 minutes for solving the four division problems. When the worksheets were handed to the teachers, they quickly scanned students' solutions and their first conclusion was that the majority of the students answered most of the division problems correctly and that most students gave an explanation for how they solved them.

As the teachers reported, 468÷2 was not difficult for the students. But instead of solving the problem without using the algorithm, as was demanded, more than half the students did in fact use the standard algorithm. While the students' notations in horizontal number expressions suggest that they did carry out a number of sub-divisions based on splitting the dividend, what they really did was a step-by-step processing of digits, which is similar to an algorithmic approach (see Tab. 2a).

Although both students whose work is shown in Tab. 2a came to the correct answer, one might wonder whether they really have insight in the division operation. In contrast to this way of working, a solution that gives a better guarantee for having insight is using the number values of the dividend by splitting 468 into 400, 60, and 8, making three divisions, and adding the results. This solution is shown in Tab. 2b.

However, the real proof of having a good understanding of the division operation was delivered by 594÷6.

	468÷2	
a	Digit-based horizontal notations of sub-divisions	4÷2=2 6÷2=3 8÷2=4
		46÷2=23 8÷2=4 23+4=234
b	Splitting up the dividend and horizontal notation of sub-divisions	400÷2=200 200+30+4=234 60÷2=30 8÷2=4

Tab. 2: Two types of student solutions for 468÷2

	594÷6	
a	Verbal description of division algorithm*	先用百位上的5除以6不够除，用59除以6上9，再一用59除以6上9，所以商是99。
b	Horizontal notation of division algorithm (digit-based)	59÷6=9……5 54÷6=9
c	Horizontal notation of division algorithm (with number value)	59÷6=90……5 54÷6=9 90+9=99
d	Using a whole-number-based splitting strategy	540÷6=90 90+9=99 54÷6=9
		594 → 180, 414 180÷6=30 414÷6=69 69+30=99
e	Using a smart strategy	600÷6=100 600−594=6 100−1=99
		594=600−6 600÷6=100 6÷6=1 100−1=99

* Translation: Firstly, I used 5 in the hundreds place divided by 6, which was not enough. Then I used 59 divided by 6. I wrote down 9. In addition, I used 59 to be divided by 6, which equals to 9. So the quotient is 99.

Tab. 3: Five types of student solutions for 594÷6

According to all the teachers 594÷6 was very difficult for their students. To solve this division almost all students stuck to the standard algorithm, either by describing it in words (see Tab. 3a) or by writing down the algorithm in a horizontal digit-based way (see Tab. 3b). Yet, while still using a digit-based approach some students were also aware of the number value of the digits (see

Tab. 3c), indicating that they have a notion of what is going on when you have to divide a number. Notwithstanding this, their solution was still based on the standard algorithm. Teachers A and B discovered that many of their students solved the problems in this way, which they considered as “mixing up different strategies and notations”. Facing these –what they called– “seemingly right but wrong expressions”, they felt that they did not know how to explain to their students what they did wrong.

The students who split the dividend in two or more whole numbers and divided them each and expressed the division in a horizontal notation (see Tab. 3d) really applied an alternative for the standard digit-based algorithm. However, a few students came up with rather far-fetched splits which made the teachers unsure about how to react to these solutions.

Despite this uncertainty, the teachers were quite sure that the student work that best revealed students’ understanding of the division operation is the use of a smart solution, for example related to 600 to solve 594÷6 (see Tab. 3e). However, only about one or two students per class came up with such a solution. Teacher B was surprised that in her class two students, whom she considered as average (or even weak) students, now used such a smart strategy. In the teacher report Teacher B wrote:

“[This classroom assessment technique] expands students’ thinking. They are supposed to command how to use the algorithm, but that should not be their only tool. They need to think about the features of particular division problems in order to calculate flexibly, rather than immediately think about the algorithm to solve all problems.” (Teacher B, report)

All six teachers found it interesting to see their students’ thinking. All of them, however, were also initially unsure about what information they were supposed to find, and three reported that even when they saw the students’ responses they were still doubtful. In the interviews they also made it clear that they did not know how to deal with the results. As the main reasons they mentioned not being accustomed to asking students such questions or thinking about such questions themselves.

Conclusions

In this study we collected Chinese primary school mathematics teachers’ first experiences with using CATs. Based on lesson observations, feedback forms, interviews, and teacher reports we can conclude that CATs can enrich Chinese teachers’ assessment practice. All teachers agreed that the questions asked in the CATs were helpful to get to know more about students’ learning, because the questions focus on students’ mathematics understanding rather than on their skills or the accuracy of their calculations.

The CATs also gave teachers insight in what their students should learn and how to teach it. A first indication is that the teachers tried to redesign their instruction plan to assimilate the CATs into their lessons. Another sign is that some teachers taught their students to solve the CAT problems before using them in class; they probably wanted to prevent their students' bad performances in the CATs. Remarkably, some teachers even integrated parts of the characteristics of the CATs into their own teaching. This was, for example, illustrated by the fact that a teacher provided questions focusing on strategies rather than answers. In general, the use of CATs had a greater effect on instruction before class and in class, than after the class in which the CATs were used. In this sense, it looks more like teachers tried to merge the CATs with their previous instructional plan and use them as extra exercises than considering them as a turning point where instruction could be changed.

Notwithstanding, the teachers' reactions were overall positive; the teachers liked the CATs and considered them useful. So we think that CATs indeed can contribute to the improvement of Chinese teachers' assessment practice and consequently maybe also of their teaching practice in mathematics education.

Discussion

Of course, these conclusions should be taken with prudence. Only six teachers in only one grade and only one mathematical topic were involved. Further research is necessary to come to robust findings and generalisation. Moreover, despite the positive reactions, one might doubt whether the teachers really grasped the purpose of the CATs. In the case of the CAT *Solving division problems without standard algorithm*, it was remarkable that the first thing the teachers did was to check the correctness of the answers. This suggests that the teachers did not really see the CAT as a gateway to assess students' deep understanding of division. The fact that the teachers did not know how to react to their students' solutions, which they mentioned clearly in the interviews, can also be considered an indication of this.

What the students' solutions and the teachers' reactions also pointed at is how much mathematics education differs in different countries, even when it involves a rather straightforward topic such as division in the domain of whole number arithmetic. In this way the present study did not only give us information about whether classroom assessment practice in Chinese primary school mathematics classes can be improved by applying CATs, the study also brought another finding which we were not looking for at first to the fore. When starting this study, of course we were aware of the fact that students in China follow a teaching/learning trajectory for whole number arithmetic that starts with teaching students the digit-based algorithms from a very early age on. Therefore, we thought it would be helpful for the teachers to check whether their students can (still) solve division problems without using the standard algorithm. In the Netherlands, such a CAT would not be revealing for teachers because the Dutch trajectory for learning division is heavily grounded in whole-number-based calculation. So Dutch students will be quite able to not use the standard algorithm. When trying out this CAT in China, we did not realize that it would be virtually impossible for Chinese students to show their understanding of the division operation without using the fixed recipe of the digit-based algorithm.

Even more surprisingly, teachers also appeared to not comprehend the point of letting students try to use different solution strategies than the standard algorithm and had trouble in identifying whether they had used a different solution strategy or not.

In this way the CATs were not only an eye-opener for the teachers who were involved in our study, but also for us as researchers.

Acknowledgements

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THEME 5: WHOLE NUMBERS AND CONNECTIONS WITH OTHER PARTS OF MATHEMATICS

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Introduction

Broad concerns remain within mathematics education about mathematical areas being seen predominantly as discrete and bounded, rather than highly inter-connected, with these viewpoints highlighted across teachers, students and textbooks in the literature (e.g. Sowder et al., 1998). While the focus of this ICMI study is on whole number arithmetic, this theme explores whole number in terms of its interrelationships with the broader field of mathematics. A range of motivations drives interest in interrelationships between topics: mathematicians' emphasis on the power of generality, and the desire for mathematical virtues such as flexibility, efficiency and elegance within problem solving. A key driver for attention to broader connections comes from writing *within* the field of whole number arithmetic. Much has been written about the ways in which this field has moved beyond a sole emphasis on counting and arithmetic calculations (Anghileri, 2006), to emphasise attention to reasoning, structure and relation (Schifter, 2011).

The papers accepted for this theme address connections in a range of ways. Some look at using 'other' mathematical contexts as a route into well-founded and non-limiting ways of understanding whole, and rational, number. Others use work in the context of whole number to lay the ground for understandings of algebraic thinking, and other topics. Connections between topics also feature within considerations of curricula and pedagogy. In this introduction, we cluster the papers in these proceedings under summary headings that point to current areas of interest in the field, and provide brief comments on their foci.

Whole number arithmetic and multiplicative reasoning

An important antecedent for critiquing the traditional base of whole number arithmetic in counting lies in the approaches developed by Vasily Davydov and his colleagues. They proposed measuring activities as much more suitable bases for the acquisition of whole number understandings, with the benefit that both additive and multiplicative reasoning led naturally from situations structured in different ways, and rational numbers did not require the significant 'reconceptualisation of number' (Schmittau, 2003) that was necessary for children who had been inducted into (whole) number through counting activities. Related to this debate, some socioculturally oriented studies also note issues raised by the 'isolation' of representations within mathematical domains – e.g. the number line is increasingly used for whole number representation, but is largely discarded in the move to rational number (Saxe, Diakow and Gearheart, 2013), where part/whole diagram representations which are often related to discrete number quantities rather than continuous notions of number, continue to predominate.

A cluster of papers within this theme focused on aspects related to this issue. Venenciano, Slovin and Zenigami reported on a study located in Hawaii that drew directly on Davydov's measurement-based approach, but with specific attention to the ways in which place value understandings could be developed. Replacing the more traditional place value-based counting activities with measuring activities, they share excerpts of learner working that indicate place value understandings developing from intial attention to comparing quantities, that grow into awareness of the need for intermediate 'regrouped' measures for dealing with larger multiples of the unit that form the foundations for place value concepts.

Dole, Hilton, Hilton and Goos, and Larsson and Pettersson both present studies located in the problem of the inadequate experience that many students have in distinguishing between additive and multiplicative situations. Finding gaps in teachers' awareness of the breadth and sequence of proportional reasoning related ideas, Dole et al describe a curriculum analysis undertaken in Australia to broaden understandings of the scope, sequence and connections among topics that draw on proportional reasoning across the early grades through to grade 9. Larsson and Pettersson discuss the features noticed by Swedish learner pairs working on mixed sets of additive and multiplicative co-variation problems. The authors note that stronger performance was associated with pairs who were able to infer distance relationships from information based on speed relationships. Weaker performance, in contrast, was associated with reliance on single procedures and attention to superficial contextual differences in problem settings rather than on distance and speed differences.

Chen, van Dooren, Jing and Verschaffel explore relationships between task types and Chinese learners' performance on learning and assessment tasks on multiplication and division by rational numbers, and find some unexpected results: associations between computation/problem-posing learning task-based performance and assessment task performance and lack of associations between problem-solving task performance and assessment task performance.

Whole number arithmetic and early algebra

A selection of papers focused their attention on a range of aspects related to connections between whole number and algebraic thinking. These aspects included attention to representation, structure and generalisation. Mellone and Ramploud discuss their mathematical analysis of a representation commonly used in Russian and Chinese primary schools in teaching additive relation structure with young learners – the pictorial equation. In taking up and adapting this tool for use with somewhat older primary school Italian learners, the authors analyse the affordances seen in children's working through their 'cultural tool transposition'. The authors note increased visibility of structural and algebraic approaches to additive relation situations, contrasting with the numerical approaches that they describe as more traditionally predominant. Xin uses an approach based on modelling the 'grammar' of additive and multiplicative situations with children with learning difficulties in mathematics, through developing attention to the underlying algebraic structure of these situations. Results point to substantial improvements in performance through this pedagogic approach.

Comparing 5th and 6th grade Israeli students' working with visual-pictorial pattern representations and numerical pattern representations, Eraky and Guberman share data showing better generalisation performance when working with numerical forms. Noting that it appears to be 'easier' for children to generalise from numerical representations, the authors emphasise the need to push for more complex structural generalisations in these settings, while pointing also to the presence of multiple, rather than single stage, production of these generalisations.

Extending the lineage of studies focused on early algebra (Lins and Kaput, 2004), Ferrara and Ng present an approach to understanding the relationships amongst human, pattern and mathematics based on the idea of 'assemblage'. This refers to learning as an output of a distributed notion of agency that works between body and material. Analysing data based on two grade 3 Italian learners working with a figural pattern task, the authors highlight the insights gained on the role of pattern settings and whole number arithmetic awareness within the development of algebraic thinking.

Whole number arithmetic competence and language ability

Moving outside mathematics altogether, Zhang, Meng, Hu, Cheung, Yang and Jiang present a quantitative investigation of the extent to which early language ability correlates with Chinese kindergarten children's informal (e.g. counting) and more formal mathematical skills (e.g. addition and subtraction). While internationally, much attention has been focused on the ways in which language ability interacts with mathematical ability, this paper finds language ability more strongly associated with informal mathematics performance than formal mathematics performance. Zhang notes that this suggests nuances in the role of language proficiency in early number learning, with differential relationships on informal and formal mathematics. Findings such as these, emanating as they frequently do from non-English/non-western language settings, raise critical questions about the breadth of application of advice on inclusion of everyday contexts as an underpinning support for mathematical sense-making (e.g. Carpenter et al., 1999).

Whole number arithmetic and teacher education

A small cluster of papers considers issues related to teacher development beginning in the context of whole number arithmetic. Beckmann, Izsák and

Ölmert address the issue of 'isolation' of topics by building through from definitions and representations of multiplication into teachers' working with proportionality. Venkat presents data indicating that representational approaches in the context of whole number scaling up can simultaneously support teachers' mathematical learning and their mathematics teaching. Concerns with primary mathematics teacher development are at the fore of Baldin, Guimarães, Mattos and Mandarino's paper. They share data on an in-service teacher development model based on pedagogic content knowledge frameworks that was used to develop teachers' knowledge and practice related to whole number arithmetic.

Questions for discussion in the working group

The papers collated within this theme start to address a number of the guiding questions that were outlined in the Discussion Document that framed the work of this study. For example, a number of the papers outlined in this introduction address questions related to whether (and how) whole number understandings might be accessed and supported via other areas of mathematics. Avenues through which whole number arithmetic learning might be supported in teacher education are also represented in a cluster of papers in the collated set. This leads us to identify two broad questions that can guide the working group discussions, with a view to deepening and extending our collective thinking:

(1) Can WNA understandings be accessed and supported via other areas of mathematics? If so, what are potentially useful approaches?
(2) How can WNA learning be supported within teacher education across different countries? Are there commonalities in approaches? And what are the cultural specificities?

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Papers in Theme 5

Baldin, Y., Mandarino, M.C., Mattos, F.R., & Guimarães, L.C. A Brazilian project for teachers of primary education: Case of whole numbers.

Beckmann, S., Izsák, A., & Ölmert, I.B. From multiplication to proportional relationships.

Chen, L., van Dooren, J., & Verschaffel, L. Effect of learning context on students' understanding of the multiplication and division rule for rational numbers.

Dole, S., Hilton, A., Hilton, G., & Goos, M. Proportional reasoning: An elusive connector of school mathematics curriculum.

Eraky, A., & Guberman, R. Generalisation ability of 5th - 6th graders for numerical and visual-pictorial patterns.

Ferrara, F., & Ng, O-L. A materialist conception of early algebraic thinking.

Larsson, K., & Pettersson, K. Discerning multiplicative and additive reasoning in co-variation problems.

Mellone, M., & Ramploud, A. Additive structure: an educational experience of cultural transposition.

Venenciano, L., Slovin, H., & Zenigami, F. Learning place value through a measurement context.

Venkat, H. Representational approaches to primary teacher development in South Africa.

Xin, Y-P. Conceptual model-based problem solving.

Zhang, J., Meng, Y., Hu, B., Cheung, S.K., Yang, N., & Jiang, C. The role of early language abilities on math skills among Chinese children.

A BRAZILIAN PROJECT FOR TEACHERS OF PRIMARY EDUCATION: CASE OF WHOLE NUMBERS

Baldin, Yuriko Y[1], Guimarães, Luiz C[2], Mattos, Francisco R[3], Mandarino, Monica C[4]

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Abstract

During the years 2006-2009, the Ministry of Education of Brazil carried out a countrywide professional development project aimed at teachers of primary education, with focus on elementary mathematics. The authors of this paper were instrumental in the development, especially of the topics that focused on the arithmetic of whole numbers. The design of the didactic material has stressed the use of concrete materials and the problem solving activities to work out the concepts of decimal and place value representation, as well as the meaning of the operations/algorithms in word problems. The activities considered the integrated dimensions of the knowledge for teaching of in-service teachers.

The paper discusses the activities of the project that aimed to development of teachers' capacity to improve the learning environment in their classrooms.

Key words: decimal representation of whole numbers, interplay between positional abacus and place value chart, ludic approach to number line of whole numbers, problem solving, teachers learning through practice

Introduction

In Brazil, the Basic Education comprises 9 years of study for children of 6 to 14 years old. Although the law ensures the access to the school system for all, the quality of students´ achievement in mathematics, especially in arithmetic of whole numbers, falls far below the educational goals stated in official documents. One clear constraint to the improvement of the quality of students' knowledge is the education system of teachers for first years of primary education, precisely the 1st to 5th grades (children of 6 to 10 years old). By law, teachers for these grades are supposed to have a degree in 3 to 4 year course in Pedagogy, with a curriculum that rarely offers more than 60 hours of mathematics content. Worse still, many teachers of this level are not graduated. So, the adequate knowledge about content, didactics or training in teaching are not prevalent, especially in mathematics.

Therefore, initiatives involving mathematics educators and university researchers have been trying to develop professional training programs for primary level teachers in order to improve the quality of educational results. The project "Pro-Letramento in Mathematics" is one of the initiatives supported by Ministry of Education in Brazil. The authors of this paper have been part of this project as main researchers during the period 2006 – 2009.

The project grounded the design of the material for teachers on the reinforcement of specific content knowledge, developing it through didactical materials and activities that teachers could replicate directly with their students. This approach to the activities considered the framework of the pedagogical content knowledge - PCK (Shulman, 1986), in particular the dimension of the "knowledge for teaching" that includes the aspects of "teachability" (p. 9). In other words, the project activities have aimed at the development of "knowledge for teaching" of in-service teachers through the integration of the categories of knowledge pointed by Shulman (1986, p.9): content (subject matter knowledge), pedagogy (knowledge of different strategies of teaching and ways of learning of students) and curriculum (knowledge of instructional materials and their use).

Studies like (Stigler and Hiebert, 1999), (Ma, 1999), (Neubrand et al., 2009) show that the development of teachers' knowledge for teaching through practice and in practice can be one strategy to diminish the gaps as faced in Brazil. Ma (1999) states:

> "… the improvement of teacher´s knowledge (is not regarded) as necessarily preceding improvement of students´ learning. … both should be addressed simultaneously, and that work on each should support the improvement of other" (p. 143)

This paper focus one part of the project "Pro-Letramento in Mathematics" namely the content of whole numbers and basic operations, under the perspective that the professional development of teachers can be realised simultaneously with the use of their results, with the learning of teachers enhanced by the learning of their students. The paper presents examples of the activities, actually taken by participant teachers to their classrooms as part of mandatory task within the project.

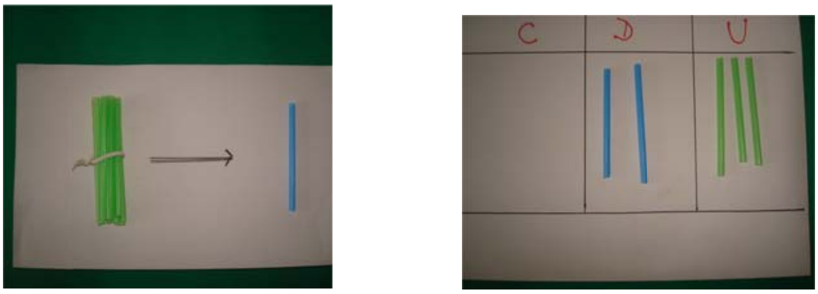
Materials and Methods

The project *Pro-letramento* consisted in a political action of public education, implemented in many federation states of Brazil, in a partnership between the Ministry of Education and Regional Secretaries of Education. The books (Belfort and Mandarino, 2008) presented activities for the learning of teachers on the conceptual ideas of Whole Number Arithmetic together with a variety of instructional materials with methodological orientation about their use with their students. Between the course sessions, the participant teachers had the task of reproducing the activities in their practices. Other important task of the participants was the constitution of a group of study in their community, in which each participant would act as tutor and supervisor to local school teachers who would replicate the activities and lessons in their classrooms. This aspect of the project has had the multiplication effect to spread among the local schools the content, the pedagogical methodologies and the instructional materials.

The teaching approach of whole numbers

The primary goal of the teaching material on whole numbers was the understanding of the decimal representation of whole numbers. As hands-on activities, the teachers experimented the logic of teaching materials, acting as if they were the children, performing the manipulation while interpreting its meaning, before taking them to the classrooms. We will not describe all the aspects of a workshop with teachers but will stress the main points of teaching approach.

The first hands-on activity is the grouping of the concrete materials (pebbles, seeds, straw cuts, buttons, etc.) which will represent the "units" ("**U**nidade" in Portuguese) in groups of ten that will be bundled. Each bundle represents and is replaced by a different object ("**D**ezena" in Portuguese, a "ten" in English) that can be bundled again in groups of tens, this time also represented and replaced by similar objects called "**C**entena" (a "hundred" in English), and so on, distinguishing each replaced object by its position in a chart. After understanding the process of counting and grouping in tens, the number of objects represented by their bundle status is registered in a Place Value Chart (PVC).



10 units = 1 ten representation 2 tens and 3 units positional

Fig. 1: Grouping the counted objects and decimal representation

Understanding this activity is fundamental to proceed to the registration of the numerals and numbers in decimal representation, and further to understand the algorithms, firstly of an addition. Baldin and Malagutti (2006) designed an activity with a positional abacus. The designed activity uses simultaneously this abacus and the straw cuts of different colors on a Place Value Chart (PVC) to register the numbers and the algorithms.

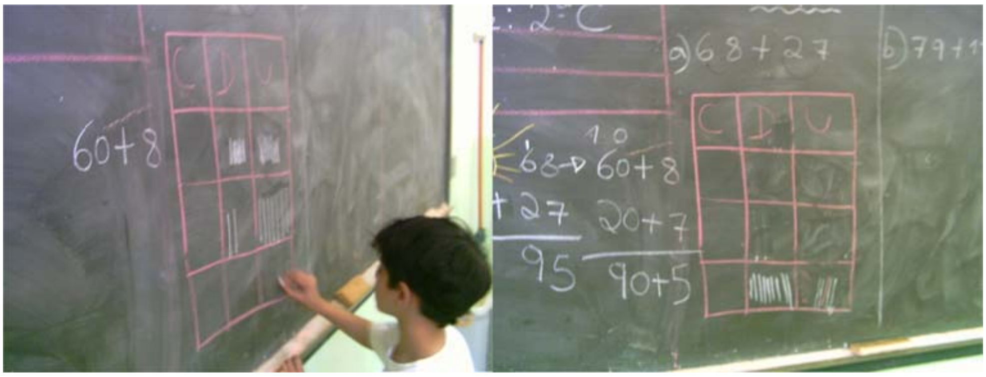
2 hundred, 3 ten, 4 units 200 + 30 + 4 = 234

C	D	U
2	3	4

Fig. 2: Simultaneous activity of decimal representation with abacus and PVC

The students themselves build the vertical positional abacus as part of grouping activity, with the agreement that the top to down levels of the bars represent the positional value, “Centenas”, “Dezenas” to “Unidades”. Movement of pieces in the abacus refers to the actions of adding (from left to right) and subtracting (from right to left). The students manipulated the straws on the spaces of PVC, with grouping and ungrouping according to the problem-situation, and they registered the numbers and the results of operations, conferring them with the abacus.

The Fig. 3 illustrates a student solving an addition of two-digit numbers, reproducing on the blackboard a Place Value Chart with the quantities of sticks representing the numerals in respective position spaces. He has shown understanding of the algorithm while his colleagues manipulated also the abacus.

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 1
 68
+27
 95
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Fig. 3: Place Value Chart, sticks and algorithm in a classroom

The principle that grounded the approach of positioning the teachers first to act as learners was that of pedagogical content knowledge (Shulman, 1986). During the project, it was largely successful for the development of teacher´s attitude in the classrooms, taking into account the integration between the conceptual mathematics ideas, multiple methodological resources that respect the pace of student, and the social context of collaborative learning environment in the classrooms, as of in the Fig. 3.

The same approach facilitated the study of subtraction. We remark that the project worked the operations of addition and subtraction as dual operations in a same situation-problem, interpreting each summand in an addition of two whole numbers as the difference in a comparison of the total sum with the other addend. A comment by a participant teacher regards this aspect enhanced by manipulation of the abacus.

> “I was surprised by the potential of abacus as learning instrument. The students understood the rules and the procedures of the addition and subtraction algorithms. The movement of the pieces corresponding to the operations facilitates the resolution of problems, and the students recognise the operations in the word problems not asking what is the operation they should do, as before.”

The introduction of the idea of order in the set of whole numbers also with concrete material and games facilitated the study of the decimal representation of whole numbers as discrete set in the oriented number line. A paper model that simulates a sliding ruler to follow the successor of a given number helped the idea of movement following the counter along the line, from left to right indicating the forward movement and right to left indicating backward, so helping to consolidate the concept of order of whole number set.

The concept of order is crucial for extending the curricular meaning of whole numbers beyond the simple counting procedure to develop mathematical thinking in first years of primary education, mainly to introduce the children to the fun and the challenges of the number theory, a curriculum content that includes even and odd numbers, the regularities in patterns, divisibility, etc. The representation on a number line as well as the deep understanding of the meaning of decimal representation play important role in this process, and teachers could live this experience through the project.

The understanding of the representation of whole numbers in a number line allowed one of successful approaches to the multiplication of whole numbers in the material of the project. In the Fig. 4, “skip” with fixed length “3” is performed twice (2 times).

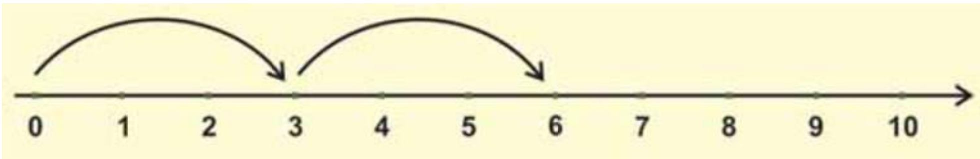

Fig. 4: Number line and multiplication: understanding 2 x 3 with “skipping”

In addition to the basic ideas of the operation division, an approach with number line as above helped to model the division algorithm, including a game where only steps of a given length were allowed: the concept of remainder was clearly apparent to the students in it. The distinction between exact division as dual operation of the multiplication of two whole numbers and the Euclidean algorithm of division had its meaning enhanced with the resolution of contextualised situation-problems and the interpretations on a number line. In this regard, the project also worked, as concrete material to assist a ludic activity, a rope with knots spaced at equal distance that represented a different scale fixed each time (as multiplier or a divisor, adequately) permitting progress in the learning of multiplication and division.

Participant teachers worked out other varied approaches with concrete material as well as ludic games and storytelling proposed by the project, besides analysing the students´ registers in notebooks to follow the results of learning. This made it possible to bring forth different contexts for arithmetic and the operations, in particular the combinatory approach of a multiplication as well as geometric configurations, which were not familiar to many teachers.

The interpretation of the multiplicative principle of the counting procedure in problems about the number of event possibilities develops the combinatorial thinking since 3rd grade, as part of curriculum about multiplication. Problems like the following were proposed using worksheets with figures in a ludic activity that introduced the representation of a tree of possibilities in the context of multiplication. The following example worked 2 x 3 as well as 3 x 2.

> “The balls of a shop A have two different sizes, Big and Small, and three different colors, Yellow, White and Red. How many different types of balls can I find in the shop A?”

To enhance the meaningful learning of operations, the “actions” interpreted from the verbs and the words of the problem texts have been studied. Some words often found in the context of Addition and Subtraction are: to put together (to join), to add, to compare, to take out, to complete, to be left, etc. However, in the context of a given problem, they are not to be taken blindly as key words, and even teachers may be caught by examples like the one shown below. Helping children (and teachers) to read and to interpret the word problems was one of advances we could observe, aggregating correct meanings to the algorithms to be performed as strategy of solution and to the validation of answers.

Examples like *“Maria has 12 toys more than Paulo. Both together have 20 toys. How many toys are Paulo´s?”* have been investigated through the steps of Problem Solving Methodology, with special attention to the interpretation of the given data that yielded the validation of the solution. Teachers first considered this kind of word problem very difficult, many claiming that the word “more” in the text could be a distractor. A careful interpretation of the context brought confidence to teachers who, as tutors to their group of study, could orient their colleagues to work the problem with children.

Discussions and conclusion

We have described some ideas so to support the claim that it is possible to develop a course to capacitate the teachers, aiming at the learning of mathematics content integrated to the methodologies of teaching. Although the project collected some evidences of the progress in the development of teachers’ knowledge for teaching, the authors observed some issues that accompany projects of this kind.

A critical observation concerns the methodology adopted in the project. The researchers must be wary of the possibility of participant teachers occasionally being “blind by procedure”, for instance, in the case of playing a game with a rope as a model of the number line, to understand the algorithm of division. To divide 29 by 7, we heard teacher’s complaints about “indiscipline of students” who would not make the full four “seven league steps”, but proceed directly to the position 28, recognising 4 as the quotient and 1 as the remainder! This is an example of the danger of "enticement by the instructional material" by teachers who might feel well in conducting a lesson "about the material" through "procedure to manage the material", shadowing the conceptual learning of mathematics.

The fact that the project has been carried out by different teams of researchers in different States with diverse socio-cultural backgrounds allowed rich analysis of the outcome in detecting the challenges faced in different regions. Belfort and Guimarães (2008) have pointed out examples of possible adverse constraints: the choice of tutors who are in direct contact with the teachers in the chain of collaborations inside the schools as well as the level of commitment and good will of local authorities that can be crucial. On the other hand, even under unfavourable conditions, when the meetings of tutors with local tutees can be carried out with some degree of success, the changes shown in children´s achievement and in the professional attitude of teachers are remarkable.

In 2005, the Ministry of Education in Brazil made a decision to test every student at the end of 5th, 9th grades and 3rd year (12th grade) in high school. In 2008, the data comparing the results of 2005 and 2007 were published for each of the states. Among the geographic regions in Brazil, the north-eastern region had shown very poor results in 2005. Therefore, the Pro-Letramento started in 2006 targeting four states of this region: Ceará, Piauí, Maranhão and Rio Grande do Norte. The Fig. 5, elaborated from the official data available in (//portal.inep.gov.br/basica-levantamentos-acessar), shows the percentual rate of growth between the average in mathematics test of students in 2007 compared to the results in 2005 of all the states of the north-eastern region.

It is interesting that the highest growth in students' average of 5th grade happens precisely in the four states that worked the project, the greatest increase rate being observed in Piauí. The result is compatible with a positive effect of the project. Although the project was not geared towards training for this test, since the test for 5th grade was strongly based in arithmetic, it is our feeling that the participant teachers of the project helped to focus their teaching strategies on whole number arithmetic, so the result of the test reflected their students' results.

It is also noteworthy that the results for 9th grade in the same states are not good, and even worse for the end of 12th grade of Basic Education. This analysis is further indication that a continuous attention to professional development of teachers is required. The Pro-Letramento material is currently at disposal at (//portal.mec.gov.br/), accessible to any educational systems, showing that the initial conception of the project is resilient enough to be useful in diverse backgrounds and circumstances, but the impact on educational results must be assessed continuously.

	Alagoas	Bahia	Ceará	Maranhão	Paraíba	Pernambuco	Piauí	R G do Norte	Sergipe
5th grade	0,67	0,66	0,89	0,87	0,58	0,58	1,00	0,75	0,42
9th grade	0,26	0,24	0,32	0,31	0,29	0,20	0,41	0,42	-0,09
High school	-0,21	-0,02	-0,02	0,18	0,20	0,06	0,19	0,09	-0,50

Comparing increase for the 5th , 9th grades and high school (12th grade)

Fig. 5: Percentual growth of improvement in students´ average 2007 compared to 2005

As a further indication of the basic effectiveness of the material and methodology adopted in teaching material, the first author implemented during the period 2007- 2009 strategic courses for development of in-service primary school teachers in Bolivia, with evidences of contribution to their professional demands. The Pro-Letramento material has shown itself quite adequate to be disseminated. Even in contexts it was not initially meant for, as in initial courses for prospective teachers, the materials proved to be accessible and suitable.

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FROM MULTIPLICATION TO PROPORTIONAL RELATIONSHIPS

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Abstract

This article summarizes a recently developed mathematical analysis that connects a quantitative definition of multiplication to 2 quantitative perspectives on proportional relationships. It then uses results of that analysis to identify 4 methods for solving missing-value problems. Finally, it presents results from an in-class test administered to 26 future middle grades mathematics teachers taking a content course on algebra that developed the 2 quantitative perspectives on proportional relationships. Results demonstrate that the 4 methods can be accessible to future teachers, suggesting that consistent and explicit use of a quantitative definition of multiplication can support teachers' developing proficiency with proportional relationships between quantities.

Key words: definition, multiplication, proportional-relationship, ratio

Introduction

The web of interconnected topics that include whole-number multiplication and division, fractions, ratios and proportional relationships, and others form one of the most central strands of school mathematics. At the same time, it is well-known that these topics present perennial challenges to many students and teachers (e.g., Lamon, 2007). In this paper, we offer preliminary evidence that future middle grades mathematics teachers can use a quantitative definition of multiplication as a key resource for developing multiple methods for solving missing-value proportion problems and for developing expressions and equations that relate two quantities in a proportional relationship.

Although there is a large literature on ratios and proportional relationships (see Lamon, 2007, for a review), empirical research has focused primarily on students. The relatively small number of empirical investigations into teachers' reasoning about proportional relationships suggests that their difficulties are often similar to students' difficulties.

Among other examples, teachers can have difficulty coordinating two quantities in a proportional relationship (e.g., Orrill and Brown, 2012) and distinguishing missing-value problems that describe directly proportional relationships from ones that do not (e.g., Cramer, Post, and Currier, 1993; Fisher, 1988). When solving proportion problems, teachers often rely on rote methods such as cross multiplying (e.g., Fisher, 1988; Harel and Behr, 1995; Orrill and Brown, 2012) and searching for key words (Harel and Behr, 1995).

Materials and Methods

The data for this paper come from a larger, on-going study in which we are examining the ecology of future middle grades (grades 4–8) mathematics teachers' multiplicative reasoning. In fall 2014, 24 future teachers completed an arithmetic course that used a quantitative definition of multiplication as a foundation for all subsequent topics related to multiplication, including fractions. In the spring semester of 2015, 23 of those future teachers and four additional future teachers completed an algebra course that used the same quantitative definition of multiplication to develop two perspectives on proportional relationships between two co-varying quantities. Both courses were taught by the first author. During class sessions, future teachers solved problems in small groups and then shared methods during whole-class discussion. Future teachers' reasoning was intentionally scaffolded by regular reminders to use the quantitative meaning for multiplication and the Common Core State Standards (Common Core State Standards Initiative, 2010) definition for a fraction, which is based on iterating a unit fraction. Data for this paper include the future teachers' written artefacts (e.g., homeworks and tests), lesson plans, and notes from the algebra course.

We address two research questions:

Research Question 1: What are methods for reasoning with a quantitative definition of multiplication to solve missing-value proportion problems?

Research Question 2: Can future middle grades mathematics teachers use a quantitative definition of multiplication to construct viable arguments explaining solutions to missing-value proportion problems?

Two perspectives on proportional relationships

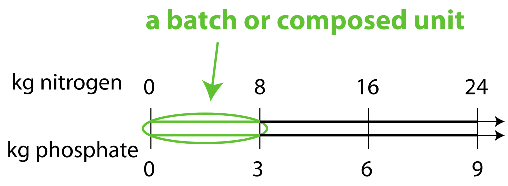
We follow Beckmann and Izsák (2015) in taking a quantitative definition of multiplication as a foundation for topics related to multiplication, including proportional relationships. Consider the multiplication equation

$$M \bullet N = P \qquad (1)$$

We interpret the multiplier, M, as a number of groups, the multiplicand, N, as the number of units in each/one of those groups, and the product, P, as the number of units in M of those groups. It is well-known (e.g., Greer, 1992) that the different roles played by the multiplier and multiplicand lead to two types of division—partitive, sharing, or how-many-units-in-each-group division and quotitive, measurement, or how-many-groups division. Beckmann and Izsák (2015) explained that the different roles played by the multiplier and multiplicand also lead to two distinct perspectives on proportional relationships. We briefly summarise these two perspectives on quantities in a fixed A to B ratio, where A and B are positive numbers.

The multiple-batches perspective

Given two quantities and measurement units for each, we view A units of the first quantity and B units of the second quantity as forming a composed unit (e.g., Lamon, 1993, 1994; Lobato and Ellis, 2010) or a "batch." From this perspective, the original batch (A units of the first quantity and B units of the second quantity) are fixed multiplicands, and the multiplier varies. More precisely, the proportional relationship consists of all pairs (r•A, r•B), where r can be any positive real number. Fig. 1 illustrates this perspective, which has been well-studied among children.

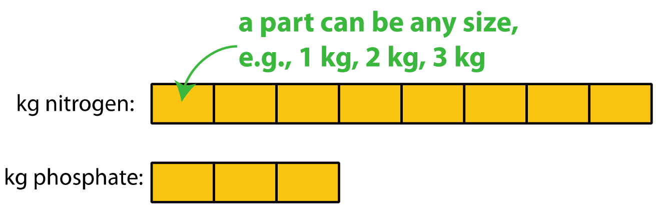


number of batches	kg nitrogen	kg phos-phate
1	1•8	1•3
2	2•8	2•3
3	3•8	3•3

Fig. 1: The multiple-batches perspective

The variable-parts perspective

Given two quantities whose size can be described with the same measurement unit, we view a "part" as a group that can vary in size. From this perspective, one quantity consists of A parts and the second consists of B parts, where all parts of both quantities contain the same number of measurement units. This time A and B are fixed multipliers and the multiplicand varies with the number of measurement units in each part. More precisely, the proportional relationship consists of all pairs (A•r, B•r), where r can be any positive real number. Fig. 2 illustrates this perspective, which has been largely overlooked in past research (Beckmann and Izsák, 2015).



kg per part	kg nitrogen	kg phos-phate
1	8•1	3•1
2	8•2	3•2
3	8•3	3•3

Fig. 2: The variable-parts perspective

Solving proportion problems with a quantitative definition of multiplication

In this section we illustrate four different methods for reasoning with a quantitative definition of multiplication to solve missing-value proportion problems. We illustrate the four methods with the following Fertilizer Problem:

> *Fertilizer Problem:* A type of fertilizer is made by mixing nitrogen and phosphate in an 8 to 3 ratio. If you use 35 kilograms of nitrogen, how many kilograms of phosphate will you need to make the fertilizer?

Two of the methods we will illustrate take a multiple-batches perspective and two take a variable-parts perspective. For each method, we express the solution as a product consistent with order of the multiplier and multiplicand in equation 1. A key point is that each method has a counterpart in the other perspective that reverses the factors in the product. Therefore, reasoning about proportional relationships from the two perspectives allows for close quantitative reasoning that distinguishes between numbers of groups and their sizes in multiplication. In this section, we present proficient explanations. Empirical results we present later will demonstrate that the future teachers could use a similar range of methods on a written test.

The "how many batches" method (multiple-batches perspective)

With this method we view 8 kg nitrogen and 3 kg phosphate as making 1 batch of fertilizer (see Fig. 3). Because we need to use 35 kg of nitrogen, we can ask how many batches of 8 kg are in 35 kg, and we can formulate this question with equation 2:

(? batches)•(8 kg nitrogen per batch) = 35 kg nitrogen (2)

The answer to this how-many-groups division problem is 35/8 batches (or 4 3/8 batches). Those 35/8 batches each require 3 kg of phosphate, therefore multiplication equation 3 expresses the total amount of phosphate needed:

(35/8 batches)•(3 kg phosphate per batch) = (35/8)•3 kg phosphate (3)

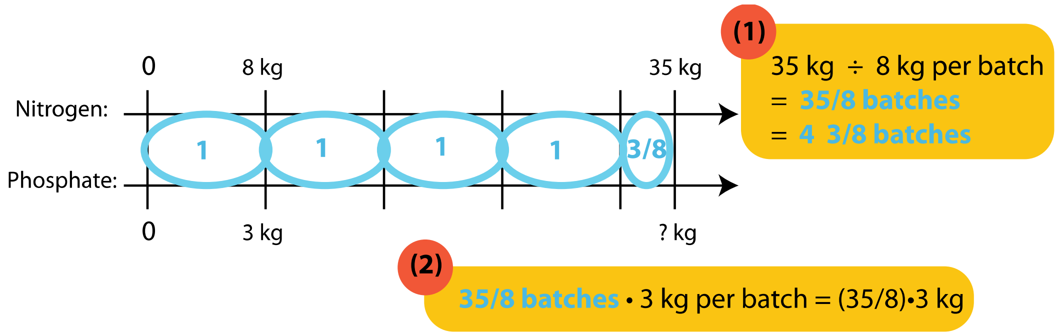


Fig. 3: The how-many-batches method (multiple-batches perspective)

The "how much in one part" method (variable-parts perspective)

With this method we view the fertilizer as 8 parts nitrogen and 3 parts phosphate (see Fig. 4). Because we need to use 35 kg of nitrogen we can ask how much nitrogen will be in each of the 8 parts, and we can formulate this question with equation 4:

(8 parts)•(? kg nitrogen per part) = 35 kg nitrogen (4)

The answer to this how-many-units-in-each-group division problem is 35/8 kg (or 4 3/8 kg). Because all parts are the same size, the 3 parts of phosphate each require 35/8 kg. Therefore, equation 5 expresses the total amount of phosphate needed:

(3 parts)•(35/8 kg phosphate per part) = 3□(35/8) kg phosphate (5)

Note that the multiplier and multiplicand in equation 2 and equation 4 are reversed and that the multiplier and multiplicand in equation 3 and equation 5 are reversed.

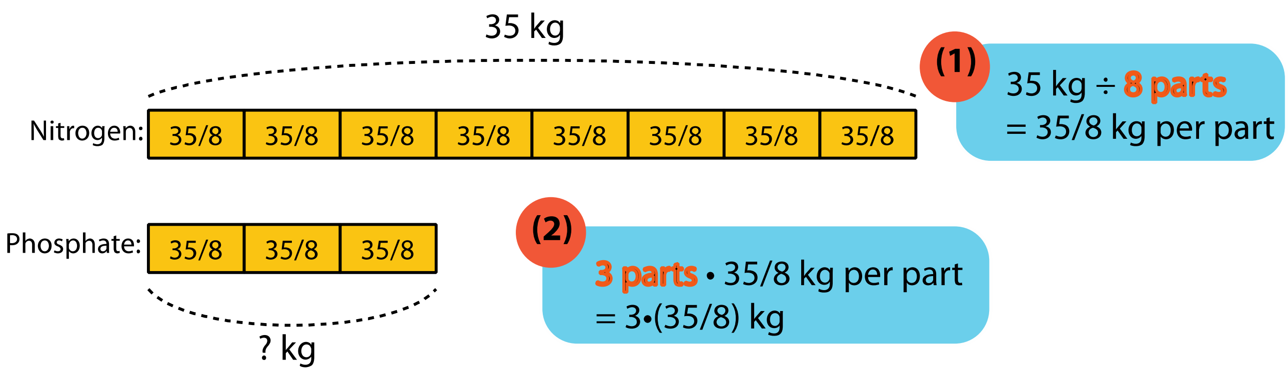


Fig. 4: The how-much-in-one-part method (variable-parts perspective)

The "how much of a measurement unit" method (multiple-batches perspective)

With this method we obtain the unit rate, 3/8 kg phosphate per 1 kg nitrogen, by dividing 8 kg nitrogen and 3 kg phosphate each into 8 equal parts (using how-many-units-in-each-group division). We can view 3/8 kg phosphate and 1 kg nitrogen as a group (see Fig. 5). Because 1 kg is the measurement unit for nitrogen, we need 35 of these groups. Therefore equation 6 expresses the total amount of phosphate needed:

(35 groups)•(3/8 kg phosphate per group) = 35•(3/8) kg phosphate (6)

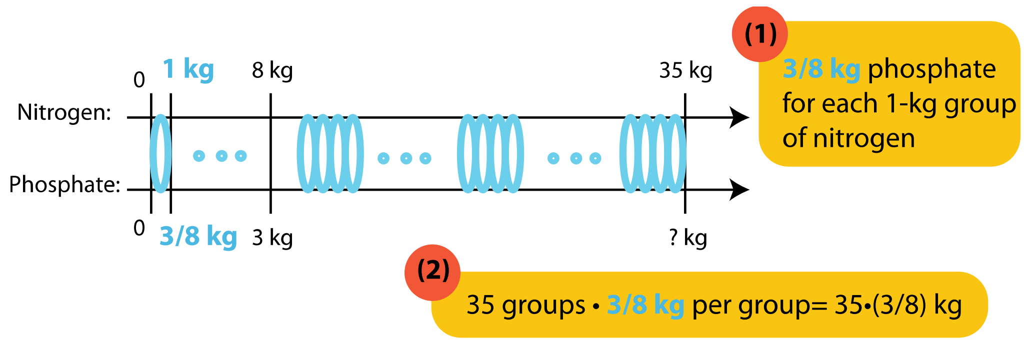


Fig. 5: The how-much-of-a-measurement-unit method (multiple-batches perspective)

The "how many total amounts" method (variable-parts perspective)

With this method we view the total amount of nitrogen as 1 group consisting of 8 equal parts (see Fig. 6) and the total amount of phosphate as 1 group consisting of 3 equal parts. Each part of phosphate is the same size as each of the part of nitrogen. This time we ask how many groups of 8 parts are in one group of 3 parts, which can we determine to be 3/8 by appealing to a definition of fraction or by using how-many-groups division. Because the 1 group of nitrogen consists of 35 kg and the amount of phosphate is 3/8 of a group of that size, equation 7 expresses the total amount of phosphate needed:

(3/8 groups)•(35 kg phosphate in one group) = (3/8)•35 kg phosphate (7)

Note that the multiplier and multiplicand in equation 6 and equation 7 are reversed.

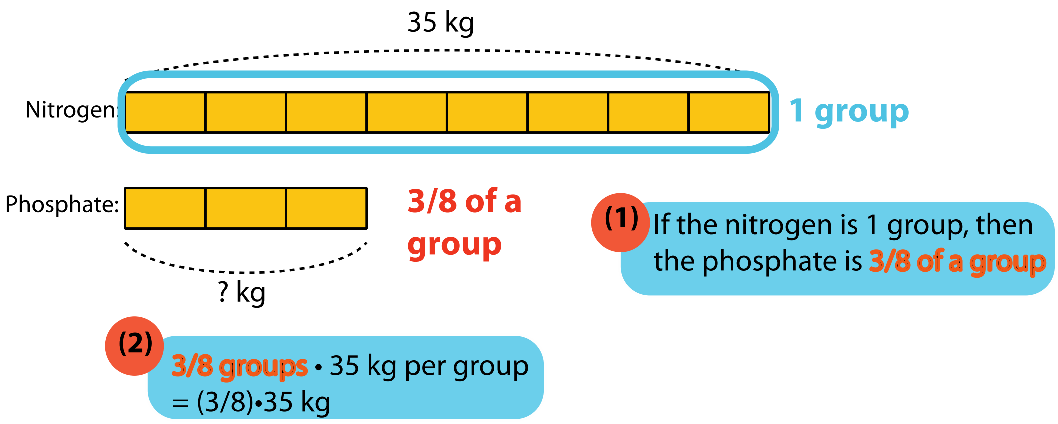


Fig. 6: The how-many-total-amounts method (variable-parts perspective)

Results

After 5 weeks of instruction during which all four methods presented above were developed by future teachers in small-group and whole-class discussions based on guiding questions and prompts provided by the instructor, we administered an in-class test as part of the regular business of the algebra course. One item asked for two solutions to the Fertiliser Problem with additional requirements as follows:

> *Problem 1*: Explain how to reason from a *multiple-batches* perspective to describe the number of kilograms of phosphate as a product $A•B$, where A and B are suitable whole numbers, fractions, or mixed numbers that you derive from 8, 3, and 35. Attend carefully to our definition of multiplication when discussing $A•B$. Use a math drawing to support your explanation.

> *Problem 2*: Explain how to reason from a *variable-parts* perspective to describe the number of kilograms of phosphate as a product $A•B$, where A and B are suitable whole numbers, fractions, or mixed numbers that you derive from 8, 3, and 35. Attend carefully to our definition of multiplication when discussing $A•B$. Use a math drawing to support your explanation.

In response to Problem 1, all but one student constructed a viable argument using the quantitative definition of multiplication and the multiple-batches perspective. Just over half of the students (15 out of 26) successfully used the how-many-batches method and just under half (12 out of 26) successfully used the how-much-of-a-measurement-unit method; these counts include 2 students who used both multiple-batches methods. The student who did not construct a viable argument mixed the two multiple-batches methods.

The statement of Problem 1 did not require students to identify the use of division. Some students discussed division explicitly, whereas others described a number of batches without explaining that number as the result of division, as in the example of student work in Fig. 7. This student explained 4 3/8 batches as follows: “We see in the double # line that 32 kg makes 4 batches + 40 makes 5, so there are 8 “parts” that make up one batch. … you see we have 4 total batches and 3 parts of one more batch. So we have 4 3/8 batches.”

In response to Problem 2, all but 2 students constructed a viable argument using the quantitative definition of multiplication and the variable-parts perspective. Nineteen of the students successfully used the how-much-in-one-part method and 5 used the how-many-total-amounts method; these counts include one student who used both variable-parts methods. Of the two students who did not construct a viable argument using the variable-parts method, one drew a strip diagram but viewed the 8 parts of nitrogen as making a total of 1 kg rather than 35 kg and used the how-much-of-a-measurement-unit method from the multiple-batches perspective. The other student attempted the how-much-in-one-part method but confused the number and size of the parts.

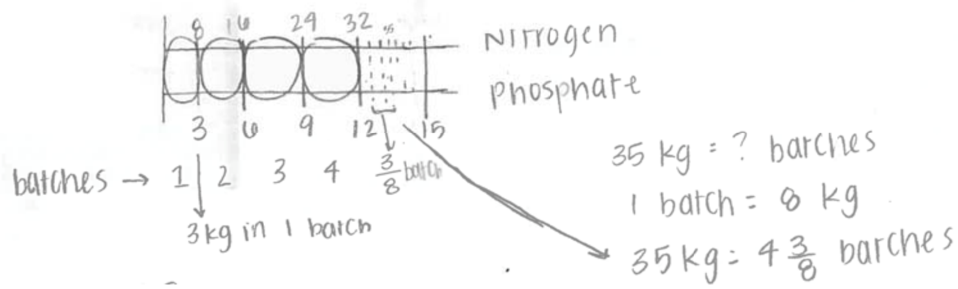


Fig. 7: Determining a number of batches

Although most students did not use the how-many-total-amounts method on Problem 2, Problem 3 on the same test was designed to elicit that method:

Problem 3: A type of fertiliser is made by mixing nitrogen and phosphate in an 8 to 3 ratio. Suppose you will use N kilograms of nitrogen and P kilograms of phosphate, where N and P are unspecified numbers of kilograms, which could vary.

Use the variable parts perspective and a math drawing to derive and explain an equation of the form (*fraction*)$\bullet P = N$, where "*fraction*" is a suitable fraction or mixed number. Attend carefully to our definition of multiplication when discussing (*fraction*)$\bullet P$.

In response to Problem 3, all but 2 of the 26 students produced a viable argument explaining the equation $(8/3)\bullet P = N$. Six students produced a viable argument but did not adequately connect the equation to the definition of multiplication. Another 6 students produced a viable argument that lacked clarity in some detail, such as omitting measurement units. The remaining 12 students produced viable arguments that attended carefully to the definition of multiplication as well as other details.

These data suggest that using a quantitative definition of multiplication to develop appropriate reasoning about proportional relationships is accessible to future middle grades mathematics teachers. All four methods we presented are accessible although the how-many-total-amounts method is the most difficult and was mastered by only about half of the class.

Discussion and conclusion

In this paper we reviewed a quantitative definition of multiplication that distinguishes between multiplier and multiplicand and we summarised two quantitative perspectives on proportional relationships that derive from this definition. We discussed four methods for reasoning with the definition to solve missing-value proportion problems and offered evidence that future middle grades mathematics teachers can use these methods.

Acknowledgements

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EFFECT OF LEARNING CONTEXT ON STUDENTS' UNDERSTANDING OF THE MULTIPLICATION AND DIVISION RULE FOR RATIONAL NUMBERS

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Abstract

In this study, two subsequent tasks (i.e., a learning and an assessment task) about rational number rules were administered to 172 fifth-graders from Liaoning province in China. The learning task, consisting of 8 items, was administered in one of three different formats (i.e., computation, problem solving, or problem posing), and, afterwards, all students were given the same assessment task, consisting of 16 items. Firstly, results revealed that the students did very well in the computation and the problem solving version of the learning task, whereas their performance on the problem posing version was rather weak. Secondly, the kind of the learning task being performed (i.e., computation, problem solving, or problem posing) did not produce significant differences in students' performance on the assessment task. Thirdly, there was a significant relationship between students' performance on the problem posing version of the learning task and the assessment task, and between their performance on the computation version of the learning task and the assessment task, but not between their performance on the problem solving version of the learning task and the assessment task.

Key words: arithmetic operations, learning context, natural number bias, problem posing

Introduction

In the 1990s, several researchers found that the type of number in multiplication and division problems is an important determinant of their difficulty level (De Corte and Verschaffel, 1996; Fischbein, Deri, Nello and Marino, 1985). To account for this finding, Fischbein et al. (1985) proposed a theory of primitive models of arithmetic operations that specified every arithmetic operation is associated with an intuitive or implicit model (e.g., multiplication is viewed as repeated addition), which intervenes in the process of selecting the operation needed to solve a problem. According to that theory, problems with a decimal multiplier/divisor smaller than 1 were more difficult than those with a decimal multiplier/divisor larger than 1; likewise, problems with a dividend smaller than the divisor were more difficult than those with a dividend larger than the divisor. As a further test of Fischbein et al.'s (1985) theory, De Corte and Verschaffel (1996) carried out a study to unravel the effect of arithmetic operations on rational numbers based on the problem-*posing* (instead of the problem-*solving*) methodology. That study used a paper-and-pencil test consisting of 12 multiplicative number sentences, 6 multiplications and 6 divisions, for example, 7.4 × 3.8[1] and 6 ÷ 4.8. Three different groups of students (upper elementary school, secondary school, and teacher trainees) were asked to generate for each number sentence a word problem that could be solved with the given operation. Generally in line with the results of Fischbein et al. (1985), it was revealed that students from the three groups tended to generate significantly more appropriate word problems for number sentences that are congruent with the primitive models of arithmetic operations than for the incongruent ones.

Recently, students' cognitive difficulties in the transition from natural numbers to rational numbers have been explained from a more general "conceptual change" perspective (Vamvakoussi, Van Dooren and Verschaffel, 2012; Van Hoof, Lijnen, Verschaffel and Van Dooren, 2013). It is argued that children create beliefs about what numbers are and how they should behave as a result of their experience with natural numbers in daily life and at the beginning of schooling. When rational numbers are introduced later on, the features of the rational numbers do not fit with the concept of numbers that children have developed so far. This inconsistency leads to children's inappropriate use of natural number knowledge in tasks with rational numbers, which is called "natural number bias" (Ni and Zhou, 2005). This "natural number bias" may be partly due to the nature of the concept of number itself, and partly to the didactical situations where the concept developed. In Brousseau's (1997) terminology, difficulties rooted from the inherent nature and development of the concept of number itself and that thus arise regardless of the instructional approach, can be considered as an "epistemological obstacle", whereas students' incorrect ways of thinking and misbeliefs about natural numbers that arise as a result of instructional choices, and therefore, are avoidable through the development of alternative instructional approaches, are considered as "didactical obstacles". In the research on the natural number bias, the effect of arithmetic operations on rational numbers also receives a lot of attention. Underlying the research is the claim that in the first years of elementary education, students acquire a lot of knowledge about arithmetic operations on natural numbers. They construct, among others, the rules that addition and multiplication will always lead to a larger outcome, while subtraction and division will always result in a smaller outcome. In the domain of rational numbers, these primitive models and their accompanying rules are no longer necessarily valid. However, students may still rely on them, which results in many mistakes, such as thinking that 0.39 × 5 is larger than 5 (Vamvakoussi et al., 2012; Van Hoof, et al., 2013).

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[1] In the Flemish educational setting, 7.4 and 3.8 respectively play the role of multiplier and multiplicand. However, in Chinese elementary schools, 7.4 and 3.8 respectively play the role of multiplicand and multiplier before the New Curriculum Reform happened in 2001. Since 2001, the role of the multiplier and multiplicand is not differentiated in multiplication.

As explained above, De Corte and Verschaffel (1996) explored students' behaviours in understanding the multiplication and division rules on rational numbers (i.e., multiplication makes larger when the operator is larger than 1 but smaller when operator is smaller than 1; division makes smaller when the operator is larger than 1 but larger when operator is smaller than 1 in the domain of positive numbers) by letting students pose word problems from multiplication and division number sentences. However, that study still leaves some open questions. Firstly, it did not provide any information on the effect of the contextualised activity on understanding the multiplication and division rules on rational numbers. Constructivist and situated cognition theories suggest that cognition is situated in, rather than isolated from, context, and that learning is optimised when students are engaged in a complex, realistic instructional context (Bednar, Cunningham, Duffy and Perry, 1991; Collins, Brown and Newman, 1989). Secondly, De Corte and Verschaffel's (1996) study did not address the effect of problem posing activities on students' later understanding of the multiplication and division rules on rational numbers. However, Silver (1994) and De Corte and Verschaffel (1996) suggest that problem posing may be a valuable vehicle for the improvement of students' mathematical understanding, especially their understanding of the mathematical meanings of number sentences. Based on this research, we argue that not only providing a proper context (i.e., giving students a word problem to solve), but also finding a context for a given operation (i.e., asking students to pose a problem starting from a given operation) might help students to better understand the multiplication and division rules on rational numbers compared to a decontextualised routine activity (i.e., merely computing the answers for given number sentences involving multiplication or division with rational numbers). So, the present study aimed to explore, in a group of Chinese students, the effect of three different learning experiences (i.e., computation, problem solving, or problem posing) on students' understanding of the above-mentioned multiplication and division rules on rational numbers, both during the learning task itself and during a subsequent assessment task.

Research questions and hypotheses

Firstly, we hypothesised that students will perform best on the computation version of the learning task, since Chinese teachers typically pay more attention to the development of students' computation skills, whereas they will perform worst on the problem posing version of the learning task given the relatively scarce attention being paid to this type of mathematical activity in a typical Chinese mathematics class (Chen, Van Dooren, and Verschaffel, 2011). Secondly, it was expected that posing problems starting from given number sentences, rather than merely computing the answers for given numbers sentences or even solving word problems, will help students to understand the multiplication and division rules on rational numbers better as evidenced by their performance on a later common assessment task. In other words, students who pose word problems from given number sentences in the learning task will perform best on the assessment task, whereas those who only solve the number sentences will perform worst, and those who solve word problems wherein the number sentence are hidden will perform in between. Thirdly, we hypothesised that there will be significant relationships between the three different versions of the learning task and the assessment task.

Method

In the present study, the learning and assessment tasks were administered to 172 fifth-graders. At the moment of participating in this study, the students had substantial experience with multiplication and division problems involving decimals (e.g., The price for the Chinese cabbage is 0.8 Yuan/kg. How much is 2 kilos of Chinese cabbage?), and had already some occasional experience in problem posing activities. Students were randomly assigned to one of the three learning task formats: computation (57 students), problem solving (57 students) and problem posing (58 students). In the computation task, students were required to compute 8 number sentences represented in different combinations of number types (i.e., combining a multiplier/divisor and multiplicand/dividend smaller and larger than 1): 4 decimal multiplications (i.e., 1.3 × 2.7, 2.4 × 0.9, 0.8 × 3.6, and 0.6 × 0.7) and 4 decimal divisions (i.e., 3.6 ÷ 1.2, 5.4 ÷ 0.9, 0.8 ÷ 1.6, and 0.6 ÷ 0.2). In the problem solving task, students had to solve 8 word problems on decimal multiplication and division containing the number sentences from the computation task (e.g., A kilo of bananas costs 1.3 Yuan. I buy 2.7 kilo. How much do I pay?). In the problem posing version, students were required to pose problems according to the same 8 number sentences as in the computation task. For example, students were required to pose problems according to the number sentence “1.3 × 2.7”. Shortly after the learning task, the same assessment task was administered to all students. In the assessment task, all students were asked to apply their learning experience from the learning task to select “×”, “÷”, “<”, “>” to make 16 number sentences valid. They were requested to do so without performing actual computations. These number sentences were also represented in different combinations of number types. For half of these 16 items, students were required to fill in blanks in statements such as “0.5 __ 4.1 < 0.5” with “×” or “÷”; for the other half they had to fill in blanks in statements like “0.9 ÷ 0.38 __ 0.9” with “>” or “<” (see Tab. 1).

Data analysis

For the computation version of the learning task, a numerical problem was awarded 1 point if it was scored as correct or 0 point if scored as wrong, which results in the total score was from 0 point to 8 points. For the problem solving version, each answer was awarded 1 point only with a correct arithmetic operation and the execution of the correct arithmetic operation; otherwise it was awarded 0 point. This also resulted in a total score from 0 to 8 points. For the problem posing version, to be considered correct, a problem, first, should satisfy the given numerical sentences, i.e., the numbers in parentheses should be computed by means of the given mathematical operations with the given numbers. Secondly, the problem should be stated in a word problem format (e.g., a pseudo word problem like "Mother had to divide 0.6 by 0.2. Can you help her?" was scored as wrong). Thirdly, the problem should accord with real world constraints (e.g., a problem like "If 5 watermelons are divided among 0.9 students, how many watermelons does one student get?" was scored as wrong because the number of students can't realistically be a decimal. A posed problem was awarded 1 point if it was scored as completely correct or 0 point as soon as it was scored as wrong on one of the above three criteria, which resulted in the total score from 0 point to 8 points. In the assessment task, if a correct operation symbol "×" or "÷" or relation symbols ">" or "<" was provided in the number sentence, the number sentence was scored as correct and awarded 1 point, otherwise it was scored as wrong and awarded 0 point, which resulted in the total score for the dimension of correctness was from 0 point to 16 points.

Results

As for students' performance on the three versions of the learning task, first, results revealed that the fifth-graders did very well in the computation version, which yielded a mean score of 7.4. Because the computation item "0.8 ÷ 1.6" violated the constraint that the divisor must be smaller than the dividend found by Fischbein et al. (1985), quite a few students (7%) gave a wrong answer "2" for that item. Second, students did quite well in the problem solving version, the mean score of which was 6.9. Similar with the result of the computation item "0.8 ÷ 1.6", in the problem solving item "1.6 kilos of carrot is 0.8 Yuan. How much is the carrot per kilo?", many students (33%) provided a wrong answer "1.6 ÷ 0.8 = 2". Third, students' performance on the problem posing version was much weaker, with a mean score of 5.2. Again, a substantial number of students (28%) provided a wrong answer such as "Xiao Hua bought 0.8 kg banana, and she spent 1.6 Yuan. How much is the banana per kilo?" in response to the number sentence "0.8 ÷ 1.6 = 0.5". In line with the result found by De Corte and Verschaffel (1996), some students posed non-realistic word problems from the given number sentences, such as the problem "If 5.4 watermelons are divided among 0.9 students, how many watermelons does one student get?" in response to the number sentence "5.4 ÷ 0.9 = 6".

As for the students' performance on the assessment task, the overall mean score was 12.7. The mean of each item of the assessment task is provided in Tab. 1. From Tab. 1, we can conclude that students did relatively well in the assessment task. In the task filling in blanks with ">" or "<", the items "0.9 × 4.7 __ 0.9 and 0.6 × 0.49 __ 0.6" among the 4 multiplicative items seem to had lower frequency of correct answers, the items "0.7 ÷ 4.6 __ 0.7 and 0.9 ÷ 0.38 __ 0.9" among the 4 division items had lower correct frequency. These observations seem to indicate that the multiplicative items with first number a decimal smaller than 1 are considerably more difficult. In the task filling in blanks with "×" or "÷", we found three results. First, it seems that the 4 items wherein students had to find the operation that leads to a number that is larger than the dividend were generally easier than those 4 wherein they had to find the operation that leads to a number that is smaller than that dividend. Second, both among the 4 items wherein one has to find a number larger than the dividend and among the 4 items asking for a number smaller than the dividend, the item consisting of two decimal numbers larger than 1 was by far the easiest one. Third, also in both cases, the item consisting of two decimals smaller than 1 tended to be the most difficult one, although the difference with the items involving one decimal smaller than 1 was much smaller (or, in one case, even non-existent). These results are generally in line with the results found in the Fischbein et al.'s (1985) study.

Filling in blanks with “>” or “<”	Correct frequency	Filling in blanks with “×” or “÷”	Correct frequency
2.6 × 1.4 __ 2.6	95.9%	9.2 __ 3.6 > 9.2	95.3%
1.8 × 0.7 __ 1.8	90.7%	0.8 __ 2.4 > 0.8	79.7%
0.9 × 4.7 __ 0.9	82%	5.2 __ 0.3 > 5.2	78.5%
0.6 × 0.49 __ 0.6	81.4%	0.9 __ 0.34 > 0.9	72.7%
7.2 ÷ 3.4 __ 7.2	86%	3.4 __ 2.5 < 3.4	79.1%
6.2 ÷ 0.3 __ 6.2	79.7 %	0.5 __ 4.1 < 0.5	64.5%
0.7 ÷ 4.6 __ 0.7	77.9%	1.6 __ 0.3 < 1.6	68 %
0.9 ÷ 0.38 __ 0.9	72.1%	0.7 __ 0.3 < 0.7	64.5%

Tab. 1: Presentation and Correct Frequency of Each Item of the Assessment Task

As for the impact of the learning task on students' performance on the assessment task, students' overall mean score on the assessment task was 12.5 after the treatment of the computation, 13.1 after the treatment of the problem solving, and 12.4 after the treatment of the problem posing. So, contrary to our expectation, the three treatments (i.e., computation, problem solving, or problem posing) did not produce significant difference in students' performance on the assessment task, $F(2, 169) = 0.76, p = 0.47$.

We also analysed whether there were significant relationships between the three different versions of the learning and assessment task. As expected, there was a significant relationship between students' performance on the computation version of the learning task and the assessment task ($\rho = 0.40$, $p = .01$), and between their performance on the problem posing version of the learning task and the assessment task ($\rho = 0.48$, $p = .00$). However, no significant relationship was found between students' performance on the problem solving version of the learning task and their performance on the assessment task ($\rho = 0.27$, $p = .09$).

Discussion and conclusion

The present study revealed (a) students' good performance in computation and problem solving, but not in problem posing in the domain of arithmetic, (b) no significant differences produced by the kind of the learning task in students' performance on the assessment task, and (c) a significant relationship between students' performance on the problem posing or computation and the assessment task, but not between their performance on the problem solving and the assessment task.

We end this contribution with some theoretical, methodological, and educational considerations. First, why did the kind of the learning task being performed (i.e., computation, problem solving, or problem posing) not produce the expected difference in students' performance on the assessment task? The main reason for this result might be that the treatment provided in the learning task was too short to produce a significant difference in students' performance on the assessment task. Moreover, the fact that students already had educational experiences with the learning and assessment materials may have jeopardised the possibility to find differential impact of the distinct learning formats on their performance on the assessment task. So, in future research, we will provide the three versions of learning task as more substantial forms of training at the moment that students actually start to learn the multiplication and division on decimals. Secondly, as far as the relationship between problem posing and problem solving is concerned (Chen et al., 2011), we found a significant relationship between students' performance on the problem posing version of the learning task and the assessment task, but not between their performance on the problem solving version and the assessment task, so in future research we will investigate why we did not find the expected correlation for the children from the problem solving condition. Thirdly, in the problem posing version of the learning task, we found that several students used the outcome in the parentheses in their posed word problems, even if they were told that that number in parentheses was the one that had to be asked for in the question. So, in future research, we will provide number sentences without outcomes in the parentheses in the problem posing version of the learning task.

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PROPORTIONAL REASONING: AN ELUSIVE CONNECTOR OF SCHOOL MATHEMATICS CURRICULUM

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Abstract

Proportional reasoning is widely acknowledged as a key to success in school mathematics, yet students' continual difficulties with proportion-related tasks are well documented. This paper draws on a large research study that aimed to support 4th to 9th grade teachers to design and implement tasks to foster students' proportional reasoning. Classroom data revealed limited initial teacher knowledge and awareness of the pervasive nature of proportional reasoning required in the mathematics curriculum. Teacher capacity to seize teachable moments for building students' proportional reasoning skills increased throughout the project. From this background, this paper presents an analysis of the proportional reasoning demands and opportunities of topics within the school mathematics curriculum in Australia. Implications for the study of whole number arithmetic (WNA) and other topics to promote proportional reasoning throughout the curriculum are discussed.

Key words: curriculum analysis, multiplicative thinking, proportional reasoning

Introduction

Proportional reasoning has been repeatedly identified as one of the most important goals of the school mathematics curriculum (e.g., Lamon, 2007; NCTM, 1989; Sowder 2007), encapsulated in the oft-cited words of Lesh, Post and Behr (1988) that "[P]proportional reasoning is the capstone of children's elementary school arithmetic and the cornerstone of all that is to follow" (pp. 93-94). Proportional reasoning is an understanding of the covariation that is inherent in the multiplicative relationship between two quantities (Lamon, 2007). Proportional reasoning is developed through the study of ratio and proportion typically in the middle years of schooling, but earlier through the study of fractions, decimals, and multiplication and division (e.g., English and Halford, 1995; Lamon, 2005; Sowder, 2007). It would seem reasonable to expect that, as students operate in the domain of whole number, there are opportunities for promoting students' proportional reasoning capacity. However, because students' difficulties with proportion and proportion-related tasks and applications are well documented (e.g., Behr et al., 1992; Misailidou and Williams, 2003; Dole et al., 2012), it would seem that this is not the case. In this paper, we address Theme 5 of whole number and connections with other parts of mathematics, exploring whole number arithmetic (WNA) teaching and learning and connections between other Mathematics topics in the curriculum.

The theoretical framework of our research draws on Lamon (2007) who described central core ideas for proportional reasoning as rational number interpretation, measurement, quantities and covariation, relative thinking, unitising, sharing and comparing, reasoning up and down. She emphasised how these are "recurrent, recursive and of increasing complexity across mathematical and scientific domains" (p. 9).

The essence of proportional reasoning is multiplicative thinking (Behr et al., 1992; Lamon, 2007). This involves the ability to determine situations of comparison multiplicatively rather than additively. The difference between multiplicative thinking and additive thinking can be seen by comparing two numbers, for example, 10 and 2. Multiplicatively, 10 is 5 times 2; additively, 10 is 8 more than 2. In situations of proportion, two quantities are related multiplicatively, and therefore additive thinking is inappropriate. The common application of incorrect additive thinking to proportional situations is well documented in the literature (e.g., Hilton et al., in press; Misailidou and Williams, 2003; Hart, 1981). Dole et al. (2012) highlighted students' ready-abandonment of multiplicative thinking through analysis of a large cohort of students' (approximately 700) responses on two proportional reasoning test items. The first item asked students to determine the amount of nectar required to feed 12 butterflies when every 2 butterflies require 5 drops of nectar. Fifty-three percent of respondents gave a correct solution based on multiplicative thinking. A second item asked students to determine the amount of flour required for a recipe consisting of 4 cups of sugar and 6 cups of flour when the sugar had been increased to 10 cups. A dramatic shift to additive thinking was evident as 66% of students stated that 12 cups would be required for the new mixture. The percentage of students who demonstrated multiplicative thinking for this item was only 14%, which is a considerable change from the 53% of students who adequately applied multiplicative thinking in the first item. The issue of 'awkward' numbers is also well-documented in relation to students' errors on proportional reasoning tasks (Lamon, 2007), but research by Dole et al. (2012) shows the power of additive thinking upon students who can reason proportionally, underscoring the need for greater examination of number relationships during whole number study.

Students' repeated failure to apply multiplicative thinking to proportional situations has been attributed to the limited capacity of primary school curricula to promote multiplicative structures, where multiplication and division are typically taught as extensions of addition and subtraction (Behr et al., 1992; Sowder et al., 1998). This criticism squarely lays the blame on a curriculum that does not adequately highlight the distinction between additive and multiplicative situations. School textbooks also may further compound the situation in their tendency to treat topics as isolated units with little connection to other units (Sowder et al., 1998). An analysis of middle school mathematics textbooks has shown the algorithmic way that topics requiring proportional reasoning are addressed with little or no connection made to related topics, such as decimals, ratio, proportion, percent, scale, and trigonometry (Shield and Dole, 2013). Clearly, curriculum documents do not make explicit mention of proportional reasoning, and are deficient in their capacity to support teachers in developing and strengthening multiplicative thinking, a necessity for success in so many topics in the school curriculum and the real world.

This paper draws on research conducted by the authors in this field working with teachers to support students' proportional reasoning in their classrooms. It also presents part of our analysis of the school mathematics curriculum in Australia, identifying a scope and sequence for the development of proportional reasoning. The aim of this paper is to consider how opportunities for promoting proportional reasoning are inherent in whole number study and throughout other topics within the mathematics curriculum, and mathematical ideas are connected through proportional reasoning. This paper addresses the following questions:

1. What is the impact on teachers' practice in building awareness of the pervasiveness of proportional reasoning throughout the curriculum?

2. To what extent are connections between WNA (whole number arithmetic) and other mathematics topics made explicit in the curriculum?

3. What are the implications of this research for teacher education and professional development?

Materials and Methods

This project involved approximately 90 middle school (Grades 4 to 9) teachers from five school clusters in geographical proximity. Each of the five clusters operated as separate entities but followed the same professional development seminars. Over the two years of the project, clusters met eight times, once per school term (four) per year. The program of workshops was informed by research for effective professional learning (e.g., Loucks-Horsley et al., 2010; Sowder, 2007). In between professional learning seminars, the researchers visited teachers in their classrooms, offering support and advice. The study aimed to investigate changes in teachers' knowledge and teaching associated with the development of students' proportional reasoning, as well as students' learning outcomes. It adopted a design-research approach (Cobb et al., 2003) for its ability to account for the complexity of naturalistic classroom settings. A large corpus of data was collected, with detailed analysis including statistical scrutiny of pre- and post-tests (see Hilton et al., in press), coding and classification of classroom observations, teacher and student interviews (see Hilton et al, 2013). Case studies of individual teachers were also developed. To answer the research questions for this paper, we present classroom vignettes of practice, teacher comments and curriculum document analysis.

Results

At the first project meeting, the presenters provided numerous examples of proportional reasoning both in the curriculum, in other subject areas, and in real life. The difference between multiplicative and additive thinking was explained. The teachers participated in a variety of activities designed to promote proportional reasoning suitable for middle school students.

Through informal discussions with teachers during classroom visits, the researchers gained insight into teachers' perceptions of the workshops and the effectiveness of the professional development programme. In all conversations with teachers, there was ready acknowledgement that they had limited understanding of the term proportional reasoning prior to commencing in the project: "Before I started in this project, I had no idea what it was at all. How ignorant was that?" (5th grade teacher of 28 years classroom teaching experience).

Some teachers said that they thought that the term proportional reasoning related most specifically to the topics of ratio and proportion and thus was something that teachers of eighth and ninth grade would be most familiar. However, teachers of those grades admitted that they had not considered the broader meaning of proportional reasoning before this project. As we made return visits to classrooms, there was general consensus of the impact of the project and increasing awareness of opportunities for proportional reasoning. "Absolutely, the project has made a huge difference. The practical things that you can do that make the children aware" (7th grade teacher of 15 years teaching experience). This teacher recounted an incidental teaching moment when a team of footballers visited the classroom and the students immediately started making comparisons to their own heights, to which the teacher commented: "Yes, you are using proportional reasoning". Another teacher echoed a similar sentiment: "I keep noticing more and more situations where I can emphasise proportional reasoning, and the children now also start noticing where they are using proportional reasoning" (4th grade teacher of 10 years teaching experience). A further example of the teacher drawing students' attention to proportional reasoning was through a whole class e-newsletter project with students resizing images to insert in the magazine. "The students had a lot of trouble reducing and enlarging photographs, because they would make the pictures very large or frog-faced, until we all discovered dragging at the corner. This was a moment to discuss proportional reasoning" (4th grade teacher of 33 years teaching experience).

Numerous examples of classroom activities and incidental teaching of proportional reasoning were observed. A fifth grade teacher gave her class individual, small packets of "Tiny Teddies" biscuits (small biscuits in the shape of a teddy bear) and asked them to create a cuboid box to hold the contents of the packet. She then asked them to scale up their cuboid to hold two packets of Tiny Teddies. The students all doubled the dimensions of their original cuboid and quickly came to realise that their new cuboid was much too big to hold two packets of Tiny Teddies. This led to intense discussion and exploration of scale in three dimensions. In another example, the teacher posed a question to her fourth grade students about the number of cabbages in a garden planted in a 3 x 2 array, which students readily determined as 6 cabbages in total. The teacher then asked how many cabbages would be planted if both sides of the garden bed were doubled. The immediate response was 12 cabbages. The teacher asked students to 'act out' the situation, with 6 students sitting on the floor as cabbages, and then other students sitting on the floor to model the doubling of each side. This was a powerful lesson in scale in two dimensions. A ninth grade teacher challenged his class to determine the area required to accommodate the world's population (7 billion) if they were all standing shoulder to shoulder. After making suggestions of what they thought the case might be, the students were astonished to realise that the population of the earth could easily fit within their state. In a third grade class, students were making fruit kebabs. They were challenged to thread apple and banana pieces in the ratio of 1 : ½ without further cutting any pieces of fruit. After some initial hesitation and much discussion and trial and error, the students realised that by doubling both numbers in the ratio to obtain whole numbers (i.e., 2 : 1), they were able to proceed.

Analysis of the curriculum

To respond to teachers' calls for a scope and sequence across the grades for proportional reasoning, we analysed the national mathematics curriculum from the Foundation Year (first year of formal schooling in Australia) through to Grade 10. Here, we present our analysis only of the 4th grade curriculum to provide a glimpse of how proportional reasoning may serve to connect mathematical ideas. In the summary below, we have embellished the intended curriculum through italicising our suggestions for promoting proportional reasoning in given topics.

Following from the 3rd grade, the 4th Grade curriculum continues to have a major focus on building place value knowledge as students work with numbers to 10 000. Consolidate place value: *opportunity to build proportional reasoning associated with base 10 relationships with 10 000 being 10 times 1000, and the two place value periods of Ones (ones, tens, hundreds) and Thousands (thousands, tens of thousands, hundreds of thousands)*. Place value extended to tenths and hundredths *and the multiplicative relationship between the places in the numeration system can be further emphasised*. Solve problems for multiplication and division with numbers to 10 000. Study number sequences from multiplication (multiples) of 3, 4, 6, 7, 8, 9. *This provides opportunity to move from repeated addition of skip counting to multiplicative thinking, e.g., 3, 6, 9, 12 skip counting arrives at the same number as 3×4.* Recall of multiplication and division facts for 2, 3, 4, 5, 6, 7, 8, 9, 10 for problem solving *further builds multiplicative thinking*. Basic fact exploration *can serve to build fractional thinking, e.g.,* $4 \times 3 = 12$; $12 \div 3 = 4$; $^{1}/_{3} \times 12 = 4$; $^{1}/_{4} \times 12 = 3$. Fractional thinking is extended to equivalent fractions, *which are inherently proportional. Equivalent fractions are based on fractional and multiplicative thinking and also ratio, e.g., cutting an object into thirds, then halving again results in twice as many pieces (*$^{1}/_{3}$ *is equivalent to* $^{2}/_{6}$*), and each piece (one-sixth) will be half the size of the each original parts (thirds).* Locating fractions, their equivalent fractions and decimals on the number line *requires scaling*. Developing money and financial mathematics *provides opportunity to consolidate fluency in money equivalences in dollars and cents*, *e.g., $1.25 is 125c; 2360 cents is $23.60; etc. This representation can be linked to decimal* *number representation: $0.25 is 2 tenths and 5 hundredths or 25 hundredths. Link back to fractions and that 25 cents is out of one hundred cents, and 100 cents is one dollar, which is represented as $1.00*. Recognising that Australian currency is decimal*, can be extended to currency in other countries further emphasising the multiplicative relationship*. In measurement, metric units of measure for length, mass, capacity, temperature are extended and consolidated *which includes the relationship between conversions*, and the capacity to *read scales of measuring instruments*. *Use a variety of measuring jugs to investigate scales – do they show 100mL intervals up to 1 litre, or just four intervals: 250mL, 500mL, 750mL, 1L? Determine what marked intervals represent and consider the part/whole relationship. Create number lines for measures with students determining the measures within intervals*. Compare objects using metric units: area and volume: *opportunity to draw out the 2D and 3D nature of these objects. Investigate what happens when we change one, two or three dimensions.* Convert between units of time: *opportunity to extend and apply multiplicative thinking to non-base 10 conversions e.g., 7 days in a week, 10 days is 1 week 3days, etc.* Simple scales for maps: *reading scales and converting to actual distances is an opportunity to extend multiplicative thinking, .e.g., explore a variety of maps and consider different scales; compare how many times bigger one country is to another; concepts of ratio can be addressed, e.g., for every 1cm on the map there is 5m in reality and vice versa.* Expressions of chance activities are extended from verbal to numeric *requiring fractional thinking (one in two chance; fifty-fifty chance; even chance)*. Data in picture graphs explored, where one picture can represent more than one: *an important step in proportional thinking, involving basic ratio; e.g., for every four cars in a survey represented by one picture on the graph; what would 2 ½ cars represent?*

Discussion and conclusion

Early in this project, it was apparent that teachers' understanding and awareness of proportional reasoning as an essential skill for mathematics and beyond was limited. The small but representative selection of interview and classroom vignettes provide evidence of increasing teacher capacity to extend students' proportional reasoning as their own awareness of proportional reasoning throughout the curriculum developed. The teachers were noticing and creating varied experiences for their students to reason multiplicatively, as suggested in the research literature as important for building students' proportional reasoning (Thompson and Bush, 2003). Teachers were engaging their students in activities of measurement, sharing and comparing, scale (reasoning up and down), unitising, exploring quantities and covariation, whilst developing rational number interpretation; all of which are key to proportional reasoning (Lamon, 2007). Building teacher awareness of the pervasiveness of proportional reasoning resulted in considerable change of practice, thus answering the first research question.

The analysis of the 4th grade curriculum presented here shows that building proportional reasoning is not explicit, but opportunities for enhancing proportional reasoning abound. Our classroom data shows that teachers from third through to ninth grade increasingly were using planned and incidental moments to explicitly teach proportional reasoning. Analysis of the 4th grade curriculum shows that topics can be revisited in other grades to continue to promote proportional reasoning. For example, the 'tiny teddies' box activity (volume) in the fifth grade aligns the fourth grade topic of *compare objects using metric units – area and volume*; as does the 'cabbages' activity (area) in the fourth grade. The 'world population' activity in the ninth grade aligns the topic of *[simple] scales and maps*, but at a higher level than that of the fourth grade, where it is located in our analysis here (this topic is also part of the ninth grade curriculum). Proportional reason threads, "recurs" and is "recursive" (Lamon, 2007) throughout the curriculum.

Our second research question asks: to what extent are connections between WNA (whole number arithmetic) and other mathematics topics made explicit in the curriculum? From our analysis of the fourth grade curriculum presented here, we can see that students continue whole number arithmetic working with numbers to 10 000 in this grade level. They continue to build place value knowledge and attain fluency in basic multiplication and division fact recall which they apply to problem solving involving multiplication and division. However, the curriculum makes no connection to proportional reasoning or emphasising the importance of multiplicative thinking.

Our work here is the beginning of the identification of connections between WNA and other mathematics topics in the curriculum, but also provides a glimpse of proportional reasoning as not only having its beginnings in WNA, as topics of measurement, data, and probability also require proportional reasoning. It is noted, however, that not every situation is proportional and being able to distinguish proportional from non-proportional situations is also an important part of proportional reasoning (Van Dooren et al., 2005).

Considering the third research question in relation to implications for teacher professional development, this research presents a model for promoting teacher knowledge and awareness of proportional reasoning. Explicit teaching and teachable moments can support strong foundations for proportional reasoning as the "capstone" of primary school arithmetic and the "cornerstone" of all that is to follow, as emphasised by Lesh et al. (1988). Commencing with whole number arithmetic associated with multiplication and division, the basis for developing multiplicative thinking and proportional reasoning presents itself, and continued emphasis throughout other topics may serve to enable students to make connections between WNA and other mathematical topics.

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GENERALISATION ABILITY OF 5TH - 6TH GRADERS FOR NUMERICAL AND VISUAL-PICTORIAL PATTERNS

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Abstract

The aim of the current study is to identify any difference between the generalisations students give when dealing with whole number patterns represented both numerically and visually-pictorially. The study participants were 91 students in 5th and 6th grade in schools in the centre of Israel. Analysis of the generalisations showed that students used more than one stage to reach the generalisation. The study shows significant differences in the correctness of task solutions in favour of numerical representation. From this we may conclude that dealing with numerically presented patterns requires less intervention on the part of the teacher as opposed to tasks presented pictorially: here the teacher should go deeper into the rules of building a sequence and also show tools to understand and analyse it.

Key words: algebraic generalisation, early algebra, number sense, patterns

Introduction

Natural numbers, their properties and operations on them are one of the key topics of elementary school arithmetic studies. Research examining various aspects of the study of the arithmetic of natural numbers and the curricula emerging in the wake such research refer to the need for computational proficiency alongside the cultivation of number sense. Number sense is a developed understanding of the meaning of numbers that enables the use of mathematical judgment of a given task in different ways, and the development of useful strategies for dealing with numbers and operations on them (McIntosh, Reys and Reys, 1992). The cultivation of number sense is not something attained or not by a student at a particular point in time, but rather a process that develops and matures with the acquisition of experience and knowledge (Veraschaffel, Greer and De Corte, 2007). One way to develop number sense is to investigate patterns of different kinds, starting from the pattern of building natural numbers, through to investigating sequences of different kinds in order to find properties shared by the numbers in the sequences or the regularity for building the sequences, using different arithmetic operations (Anghileri, 2000). One component of number sense is 'algebraic arithmetic': a type of activity that builds on "bridging arithmetic and algebra" (Pittalis, Pitta-Pantazi and Christou, 2014, p. 434). There are different definitions of this 'bridge' between arithmetic and algebra known as 'early algebra', but there is consensus among scholars on the following components (Blanton and Kaput, 2005; Cai and Knuth, 2011 ;Carraher and Schliemann, 2007; Pittalis et al., 2014):

- Thinking about the relations between unknown quantities, including solution of verbal problems using letters and equations;

- Generalisation of arithmetic patterns into algebraic expressions and understanding these patterns as a shortcut to calculations;
- Recognition of identities (including rules for arithmetic operations) as general properties in a given group of numbers;
- Use of a variety of representations of algebraic symbols to reach, formulate and justify generalisations.

Recent research on mathematics education has discussed the introduction of algebraic notions in the study of arithmetic in elementary level classes (e.g. Cai and Knuth, 2011; Stacey, Chick and Kendal, 2004). Supporters of this approach claim that the study of algebra has its roots in lower grades, when children notice the regularity of the behaviour of numbers in general, and natural numbers in particular. Moreover, observation of regularity enables learners to develop awareness of mechanisms of connection between numbers and arithmetic operations. Sequences of shapes presenting change (increase or decrease with a specific regularity) are suitable tasks for investigating regularity. They summon mathematical experience at different levels for each student (Smith, Hillen and Catania, 2007). The advantage of these tasks is that they may serve as a "bridge" from the physical world (i.e. cubes, toothpicks etc.) and prior mathematical experiences to formal algebra. This transition from concrete to abstract must be accompanied by the development of children's ability to generalise and reason. Blanton and Kaput describe it thus:

> *...a process in which students generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways* (Blanton and Kaput, 2005, p. 413).

Research shows that dealing with patterns affects several aspects of students' mathematical abilities: it enables them to develop a mathematical language regarding positing hypotheses and providing justification or refutation of those hypotheses (Moss and Beatty, 2006); it supports understanding of the relations between quantities (Carraher, Schliemann and Schwartz, 2008); it is a way to construct mathematical generalisations (Kieran, 1992; Carraher and Schliemann, 2007).

Scholars distinguish between different types of patterns: linear patterns, patterns in computational procedures, repeating patterns and so forth (Zazkis and Liljedahl, 2002). The current study deals with patterns that can be given two representations: numerical representation and visual-pictorial representation[2]. There are different kinds of tasks relating to these patterns. For this study we

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[2] Of course, there are other representations such as verbal representations (Carraher, Schliemann and Schwartz, 2007).

asked students to formulate a generalisation: what is the number (or number of circles) that will appear instead of *n* in the series? (Moss and Beatty, 2006). These generalisations may be expressed either in words or in symbols, according to the student's ability. An example of such a pattern is shown in Fig. 1.

Radford (2010) suggests relating to generalisation of patterns as both a result and a process; in other words, focusing both on the generalisation itself and on how the student reaches it. Harel (2001) mentions that dealing with patterns in each representation requires different strategies for reaching the generalisation, according to the pattern in question: for a numerical pattern, the student must identify the relations between two variables while for the visual-pictorial pattern the student must compare two adjacent elements, create an element that repeats itself in several initial elements and then identify the relations between the earlier shape and the next one. Harel claims that in both cases students must apply inductive thinking. The difference lies in the relations emphasised in these thinking processes: inductive thinking for numerical patterns uses algebraic concepts, whereas inductive thinking for visual-pictorial patterns relies on the relations that can be inferred visually from a given set of specific instances (Billings, Tiedt and Slater, 2008).

The next step is to identify whether there is a difference in the level of generalisation students reach for numerical and for visual-pictorial patterns. The aim of the current study is to identify any difference between the generalisations students give when dealing with whole number patterns represented both numerically and visually-pictorially. In particular, this study seeks to characterise the similarities and differences in elementary school students' abilities to generalise the regularity according to which sequences of natural numbers are built, where these sequences are presented either numerical or visually-pictorially.

Materials and Methods

The study participants were 91 students in 5^{th} and 6^{th} grade in schools in the centre of Israel. In regular math lessons they were given worksheets with tasks requiring recognition and generalisation of the regularity of a sequence. Each worksheet had two tasks – one for each form of representation.

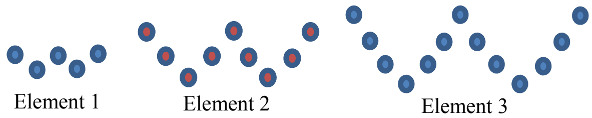
Among these worksheets there were two in which the same sequence was given – once in one format and once in the other (see Fig. 1). In this article, we relate to these two worksheets. The completion of each worksheet and the discussion about solution strategies took an entire lesson (45-50 minutes).

The two sequences on the worksheets are different in terms of their mathematical nature: the first is an arithmetic progression and the second is a sequence of triangular numbers. The rule for building the arithmetic progression is that the numbers are divisible by 4 with a remainder of 1. The other sequence is of triangular numbers, i.e. in which each element may be represented as a group of objects that can be arranged in the shape of an equilateral triangle. These worksheets were collected and analysed in two stages. First there was direct content analysis: the stages through which students progressed in order to reach the generalisation were classified according to the framework proposed by Radford (2010). The principle of this classification is the encoding of each stage of solution according to the typology of the forms of algebraic thinking following the level of generalisation students provided in their work. The details of the typology can be seen in Tab. 1 below. The second stage, involved statistical analysis of the data to compare the students' generalisation ability for the both: numerical and visual-pictorial representation (t-tests).

Worksheet 1

Task 1:

Look at the following sequence:



a. How many dots will there be in element 6? Explain.

b. Describe the sequence in your own words.

Task 2

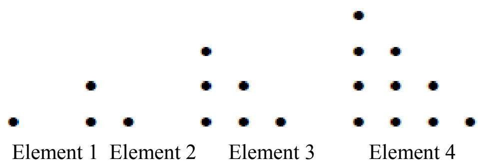
Look at the following series: 1; 3; 6; 10…

a. What is the fifth element in the series? Explain.

b. Describe the sequence in your own words.

Worksheet 2

Task 1:



Look at the following sequence:

a. How many dots will there be in element 6? Explain.

b. Describe the sequence in your own words.

Task 2

Look at the following series: 5; 9; 13; 17…

a. What is the fifth element in the series? Explain.

b. Describe the sequence in your own words.

Fig. 1: Worksheets given to the students

Results

We will relate to the method of generalisation for each sequence by presenting examples from the students' work (see Tab. 1 below). Analysis of the generalisations showed that students used more than one stage to reach the generalization. In other words, in order to formulate the generalisation, students underwent a **process** beginning with factual algebraic thinking and for some students this continued to contextual algebraic thinking.

Initially, the students examined the elements in each sequence in order to understand the nature of their construction. Here it is important to note that for numerical representation the students first looked for a connection between one element and the one either preceding or following it. In contrast, for the visual-pictorial representation analysis, students related to its "pictorial" structure:

Numerical representation	**Visual-pictorial representation**
Yuval: Here it is bigger by 1, here by 2, and then 4 and then 4. That's how we got to 10 (triangular numbers series).	*Alon: This shape has a pattern, everything here is equal and that's like the letter W that just grows 'from one generation to the next" by the addition of the circles (arithmetic progression).*

In other words, in the first stage, students objectified the regularity for both sequences and both representations. The difference is in how the objects were related to in each case: for numerical representation, students related to the difference between two adjacent elements, while for visual-pictorial representation they related to the external appearance of each element (the letter W, the triangles) and did not translate this into a numerical representation. As Tab. 1 shows, some students remained at this level of generalisation, with similar numbers for each sequence, but different numbers for each representation: for numerical representation, about two-thirds of the students went on to construct an 'in-action' formula, as opposed to about one half who went on to the next stage with the visual-pictorial representation.

The second stage of constructing the generalisation is to create an in-action formula. In this case, students gave an operative description how to construct the next element from its predecessor, a kind of 'recursive' description of the sequence, using words and not letters:

Numerical representation	**Visual-pictorial representation**
Daniel: Each time the sequences grows by two, then by three and then by four consecutively (triangular numbers series).	*Chen: It begins with 1 and then jumps by two to three and then by three to six (triangular numbers series*).

Some two-thirds of the students in each case (type of series and type of representation) remained at this level of generalisation, and only one third progressed to creating a formula with a generic example - building a formula based on a specific example that may be considered as a general case:

Numerical representation	**Visual-pictorial representation**
Noa: The number 73, for example, does not belong to this sequence because it is divisible by 1 and by itself and you can't take it as something multiplied by 4 plus 1(arithmetic progression).	*Daniel: if I need to know the number in 12th place, I will multiply 12 by 4 and add 1, because it is actually numbers with a reminder of 1 (arithmetic progression).*

The highest level of generalisation observed was a description in general terms, reached by about 12% when working on the task of the arithmetic progression in numerical representation. No students reached this level of generalisation for the triangular number sequence in numerical representation. Nor did any of them reach this level of generalisation for either of the two sequences in visual-pictorial representation.

Numerical representation	**Visual-pictorial representation**
Daniel: Element number X 4 and then I add 1 (arithmetic progression) *Noa: 4 × ▭ + 1 (arithmetic progression).*	No students reached this level of generalisation for this representation.

The statistical examination of the results obtained show that for the arithmetic progression there are no significant differences in the process of applying algebraic thinking between the two representations. In other words, the solution process undergoes the sane stages, with the same degree of success on the students' part. In contrast, the statistical examination of the other sequence shows significant differences: the correctness of the generalisation for numerical representation is higher for numerical representation ($t(91) = 2.12$, $p < .05$); recognition of the regularity is easier for numerical representation ($t(91) = 2.24, p < .05$); writing an in-action formula is easier for numerical representation ($t(91) = 4.03$, $p < .001$).

Level of generalisation	Sub-Level Description	Number of students reaching this level of generalisation of the sequence 5; 9; 13; 17…		Number of students reaching this level of generalisation of the sequence 1; 3; 6; 10…	
		Numerical	Visual-pictorial	Numerical	Visual-pictorial
Factual algebraic thinking	Objectification of regularity: Recognizing the connection between the number of the given form in a sequence and the number (or number of circles) in it.	*85*	*91*	*83*	*78*
	In-action formula: Building a 'recursive' formula/ no use of letters	*61*	*43*	*66*	*56*
Contextual algebraic thinking	Generic example: Student's formula is based on a particular case which can be related to as a general case.	*36*	*29*	*37*	*34*
	Description in general terms: Description of the regularity of building a series in words or symbols in general without relying on a specific example.	10	0	0	0

Tab. 1: Level of generalisation identified among students participating in the study

Discussion and conclusion

This study examined how students cope with tasks requiring generalisation of sequences. These sequences were presented both as numerical patterns and visual-pictorial patterns. The first conclusion of the study is that students are able to handle tasks dealing with patterns as early as elementary school. However, dealing with sequences students are familiar with (such as a sequences of numbers giving the same remainder when divided by a certain number) enables reaching a higher level of generalisation than dealing with patterns that most students are not familiar with (triangular numbers). This leads to the conclusion that dealing with patterns unfamiliar to students does not enable a high level of generalisation, but does oblige students to look at the pattern given very closely in order to identify different connections between the elements of the sequence. This method seems more reasonable for cultivating flexibility in "games with numbers".

The other finding emerging from the study is students' preferred form of representation: numerical or visual-pictorial. The study shows significant differences in the correctness of task solutions in favour of numerical representation. Tasks in numerical presentation are easier for students to solve and show higher rates of success. The qualitative analysis of the difference between the representations shows that students are busier with the visual properties of the patterns (looks like the letter W) as opposed to the more intensive focus on the arithmetic properties of the sequence presented numerically. From this we may conclude that dealing with numerically presented patterns requires less intervention on the part of the teacher as opposed to tasks presented pictorially: here the teacher should go deeper into the rules of building a sequence and also show tools to understand and analyse it.

The final point of note is the cultivation of number sense. As the study reveals, dealing with patterns is the right way to judge the data in the task: whether it is important to identify connections between two adjacent elements in a sequence, or to generalise the method for finding the element in a higher position (e.g. 100^{th} place), or to create an 'in-action' formula suitable to generalise the sequence and so on. Analysis of the generalisations obtained together with the students, alongside discussing the efficacy of each one will enable a teacher to deepen the students' understanding of the relations between the numbers and the flexibility of working with them.

In addition to all of the above, the study results support Radford's claim that algebraic thinking linked to the ability to generalise the patterns develops in stages. This claim seems correct not only in the case of the development of algebraic thinking over time, but also for a specific task, as emerged in this study.

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A MATERIALIST CONCEPTION OF EARLY ALGEBRAIC THINKING

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Abstract

This paper discusses an exploratory approach to understanding the relationship between Whole Number Arithmetic (WNA) and early algebraic thinking. In particular, the inclusive materialist framework offered by de Freitas and Sinclair (2014) is used to reconceptualise learning as occurring *in* the relations amongst human, pattern and mathematics. We examine a mathematical activity where children interacted with patterns in a classroom setting. We show that focussing on the relations amongst human, pattern and mathematics provides insights on the role of pattern and WNA for developing early algebraic thinking. In so doing, we call attention on the embodied nature of mathematical thinking and learning as well as its relationship with the learners' material surrounding.

Key words: early algebraic thinking, embodiment, inclusive materialism, material agency, pattern, Whole Number Arithmetic

Introduction

The development of early algebraic thinking has been widely studied in mathematics education (e.g. Carraher et al., 2008; Radford, 2014; Rivera, 2011). Traditionally, these studies focus on children's discourse when they engage in mathematical activities with patterns. From a theoretical standpoint, these studies tend to place children at the centre of the mathematical activities; hence, the pattern itself is taken as inert material that "represents" mathematics. Taking on a non-dualistic and materialist view, this paper conceptualises *algebraic thinking* as inseparable with *engaging* with the material surrounding. Within this exploratory approach, we aim to offer insights into children's experience as they engage with patterns, and their development from Whole Number Arithmetic (WNA) to algebraic thinking.

Interactions with patterns are necessarily embodied: at the very least, it is necessary for human perception and the body to coordinate with the pattern. These bodily experiences, such as pointing to a particular term in a sequence with one finger, also seem important in the learning process in that they are shaped by material encounters such as the design of the pattern in a figural and numerical sense. In other words, a child's interactions with the pattern are not merely initiated by the child, but are compromised between the child's body and his/her surrounding material world. This compromised *intra-action* between human and material is a move beyond the embodied cognition tradition (Lakoff and Núñez, 2000; Wilson, 2002), in which the centre of the activity is given to the learner, towards distributing agency across the learning situation (Rotman, 2008; de Freitas and Sinclair, 2013). In this paper, it is argued that this approach for understanding the body and materiality may better equip us for addressing the development of algebraic thinking through children's interaction with patterns.

In their book chapter: "When does a body become a body?", de Freitas and Sinclair (2014) offer a theoretical approach of unbinding the body from its skin to shed new lights on the ontologies of body and of mathematics. They redefine the boundaries of the body by considering the interactionist view that materials are not inert but are constantly interacting with each other and with the human body. Boundaries are re-created and *assemblages* emerge as the *body* and as the unit of analysis in the learning experience of a person working with a tool or a pattern. This approach rejects the conceptualist idea that materials have confined properties of their own; hence, agency is redistributed across the situation. Distributing agency could help theorise the role of materials in learning mathematics in ways that the traditional cognitivist could not. In particular, this approach argues that the functions and properties of a given pattern can only be captured in the human-pattern-mathematics assemblage. Assemblages possess emergent properties. De Freitas and Sinclair show how the material world—the chips, the goblets, and the limited surface of the desk—is implicated in an episode where children work with patterns: "The surface on which the students work are thereby also part of the material practice; it enables the visual sequentialising of actions. […] But had the students placed the chips differently, the assembling of matter and meaning would have changed" (pp. 28-30). The same chips and the goblets are seen as useful representations by the researchers, but students engage with them differently as part of their material assemblages.

Moreover, diagramming and gesturing are important mathematical processes in emerging assemblages. De Freitas and Sinclair call these "boundary-drawing apparatus", devices that reconfigure the world rather than representing it or coding it. "The boundary that it draws conjoins and separates the 'real' from the mathematical, the matter from the meaning. In Barad's terms, this is a discursive practice that actively engages with the matter-meaning nexus while enacting a cut or divide that also separates" (p. 51). Therefore, diagrams and gestures are not *only* iconic representations of the "real", they also effect its individuation. As a gesture is performed, its meaning becomes increasingly mathematical through a series of boundary-drawing. Located in the physical world, these movements potentially evoke mathematical meanings: "Does mathematics really just stand there, silently waiting for the breakthrough insight or shift in attention? Or might it somehow be much more implicated in the moving hands and the configuration of chips? If so, what do we mean when we say that the actions are concrete and the mathematical expression abstract?" (p. 30).

Materials and Methods

The data discussed in this paper is part of a larger classroom intervention study concerning the development of early algebraic thinking in grade 1-5 (age 6-10) children. The intervention study took place in a primary classroom in Italy, where the same researcher (the first author) consistently took on the role of a guest teacher and taught lessons in collaboration with the regular classroom teacher through a five-year period. In the selected data, the children were involved in pattern activities and were asked to look for mathematical structures behind some figural sequence. The structure of the sequence disclosed a direct relationship between any term of the sequence and its position in the sequence. Usually, the spatial configuration of the terms emerged first, and a numerical structure dealing with the numerosity of elements in the terms emerged later. For this paper, we report the filmed experience of two grade 3 students, Filippo and Lara, when they were working with a figural sequence for numbers of the type $6n$-2, n being the position (see Fig. 1).

Filippo and Lara were working on a pattern activity called "Do you remember Tobia?" with the other student pairs formed within the class. Tobia is the dog-character introduced in an activity that the children had already seen in their previous grade (Fig. 1). In grade 2, the children were asked to extend the sequence up to the sixth term and to complete it with the term "hidden" by the diagrammed track of Tobia. In grade 3, they were given the task of noticing any regularity in the sequence as well as the relationship between any figure and its position in the sequence.

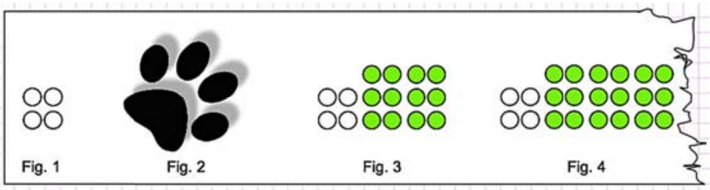

Fig. 1: The figural sequence of the activity "Do you remember of Tobia?"

Results

Dealing with the task in grade 3, Filippo and Lara started perceiving new properties and relations. In particular, they had just found that if one takes a term of the sequence, counts the circles on its bottom row and divides this number by 2, one gets the position of the term in the sequence. This introduces, in the children's discourse, the operation of division by 2. The researcher (R below) was near the pair when Filippo was talking about this while referring to Fig. 3 and 4 of the sequence with Lara, who sat in front of Filippo and turned around to face him in the discussion. The researcher asked the children to explain this to her.

> Filippo: For example, you look at this <*Points to term 3 of Figure 1 with the pen in his right hand*>, you do, you count the circles below <*Runs the bottom row*>, one, two, three, four, five, six <*Counts the circles* >, then you do six divided by two <*Looks up at the researcher*> that gives, oh.
>
> Lara: Three. <*Looks up at the researcher*>
>
> Filippo: And this is, is <*Moves the pen twice around term 3*> oh, the position.

Lara: Or you also take this <*Overlaps Filippo's voice. Points to term 4*>, you can also take this. <*Points to term 4 again*>

R: Oh, and does it also work here? <*Indicates term 4*>

Filippo: This one is equal. One, two, three, four, five, six, seven, eight <*Counts with the pen the circles on the bottom row of term 4. Lara joins him in counting*>. You do eight <*Looks up*> divided by 2.

Lara: It gives four. <*Looks at the researcher*>

Filippo: And this one <*Moves the pen twice around term 4*> is in position four. This one <*Shifts the pen to term 1*>, one, two, two <*Looks up at the researcher*> divided by two gives one. <*Points with the pen to expression "Fig. 1" below term 1, see Figure 1. Looks up at the researcher. Smiles*>

After this, the researcher along with the teacher gave the children a new task related to "position 25". Filippo explained the new task as follows.

Filippo: For example, in position twenty-five, you do, to discover that one <*Moves the pen many times around term 4 of the sequence*>, this one <*Runs the bottom row of the term*>, you do twenty-five times two, twenty-five times two, oh, then you put <*Points to term 4*>, oh, wait, twenty-five times two, and, this number, oh, wait, I do no longer remember <*Smiles*>. You do twenty-five times two <*Pauses. Looks around, beats his head*>, oh, what did I say? <*Looks at Lara, looks at the sequence*>

Lara: Twenty-five times two <*Pauses, looks at Filippo who points to term 4*> one, two, three, four, five, six, seven, eight <*Counts the circles on the bottom row of term 4*>, you do eight divided.

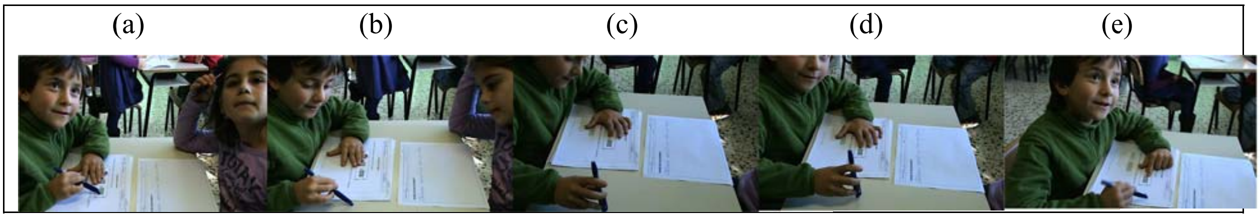

Fig. 2: (a-c) describing the bottom row of term 25;
(d, e) miming the other two rows

Filippo: Ah! You do twenty-five times two <*Looks at the teacher*>, and you put the result here below <*Mimes with the pen the arranging of the first circles on the bottom row of term 4. Looks up at the teacher. Figure 2a*>, you put the circles, all, of the result <*Continues the gesture outside of the paper. Figure 2b*>. When you arrive at the result with the circles, there <*Shifts the pen to a position towards the desk side. Figure 2c*>, you stop and you go above <*Shifts the pen to indicating the middle row of term 4, keeps reference to it with left index finger*> and you always put twenty-five, no, always the result <*Mimes the arranging of circles on the middle row, moving to the desk side. Figure 2d*>. Then here <*Points with index finger and pen to the initial empty space on the top row of term 4*>, you skip the first two and you go <*Keeps the finger as a reference, mimes the arranging of the circles on the row with the pen. Figure 2e*>, oh, and you put, you do <*Looks at the teacher*>, oh, you take two away from the result and put those ones! <*Repeats the miming of the top row. Looks at the teacher, smiles*>

The passages above show that the children perceived the first structural relations in the sequence, between the numerosity of the circles and the number of terms. This required more than noticing the overall spatial configuration of the circles and the recursive "always add 6" that the children already noticed in the previous grade. What Filippo and Lara were noticing here was of a very different nature. It introduced reasoning on the figural sequence in terms of whole numbers (number of circles), and even though the reasoning was related to one row at a time, it was no longer strictly related to the spatial disposition of circles. So, operations came to the fore as necessary means to manage relations between numbers, like the number of circles in a row and the term number (its position). This is interesting from the mathematical point of view, since it pointed out a property of the sequence and a functional way of looking at it.

When Filippo and Lara were in the process of writing down their discoveries, the written production required a slow elaboration of their thinking. At a certain point, the teacher asked what they were doing, and Filippo responded as follows.

> Filippo: Twenty-five times two, it gives fifty, then you put <*Points to term 3 of the sequence*> fifty circles below <*Mimes with right index finger the arranging of circles on the bottom row of term 25 starting from the bottom row of term 3 and moving outside of the paper towards the desk side*>, fifty circles in that [row] above <*Repeats the gesture in correspondence with the middle row of term 3*>, and here you leave <*Points to the first block of white circles*>, in the row of white <*Points to the block with both hands*> you leave two empty [spaces] and you make forty-eight <*Repeats the gesture with both hands miming the top row on the desk, and then running away*>

In the previous passages, Filippo recalled the operation twenty-five times two as an operative means to solve the task. Instead, here he talked about position 25, directly using the number of circles, through the result of the operation. He was giving a precise explanation for the shape in term 25, which were two rows with fifty circles and one with forty-eight circles. The numerosity of elements and the numerical relationships were more and more emerging from the spatial structure of the sequence. Similarly, the children were more and more distancing from the four given terms of the sequence, as they started to reason with more generality on its characteristics.

The children were almost finished writing down their discoveries, when the researcher intervened to ask whether the previous relationship was applicable in a different case, prompting them to think algebraically.

> R: But what if it were… you made the case of the twenty-five, of position twenty-five, didn't you? But, what if it were position…
> Lara: One hundred?
> R: One hundred, or position Pippo, what, what would you do?
> Filippo: I would do, oh, I do… but, position Pippo?
> R: Ah ah. <*in affirmative sense*>
> Filippo: I do Pippo times two!
> R: And what do you find?

Filippo: Two Pippos!!!
R: What do you find, what does making Pippo times two tell you about the figure?
Filippo: Eh? Oh. <*uncertain*> What? In what sense?
R: When you do twenty-five times two for the figure in position twenty-five, what do you find?
Filippo: A number!
R: Well, the number of what?
Filippo: Of the circles that I arrange above <*Runs the middle row of term 4 of the sequence*>, that I arrange below <*Runs the bottom row*>, and, skipping two <*Points to the empty space on the top row*>, that I also arrange here. <*Runs the top row and moves over it to mime the rest of the circles of term 25*>
R: So, when you do Pippo times two, that calculation, that operation, what does it give you about the figure? <*Filippo keeps silent*> Why do you do it, to find what?
Filippo: It gives you the circles to arrange below and then also above. <*Repeats again the previous gesture along two rows of term 4*>
R: Ah! So, how is the figure in position Pippo made?
Filippo: In position Pippo it is made: two rows of Pippo <*Mimes the two rows again running the rows of term 4*>, you take two away from Pippo <*Runs the top row*> and you put another row with the result of Pippo minus two <*Looks at the researcher*>. You do Pippo minus two <*Points to the empty space of the top row*>, you get a result.
R: Didn't we say that Pippo is the position? <*Points to expression "Fig. 4" on paper*>
Filippo: <*Keeps silent*> The position. You put, oh, the result of Pippo below, above.
R: The result of Pippo?
Filippo: Of Pippo times two, oh, I put the result of Pippo times two in a row <*Runs four times the bottom row of term 4 and extends it over, Figures 3a and b*>, in another row <*Mimes the imaginary middle row, Figure 3c*>, and I take two <*Points to the two missing circles of term 4, Figure 3d*> away from the result of Pippo times two, and I put those ones here. <*Mimes the imaginary top row*>

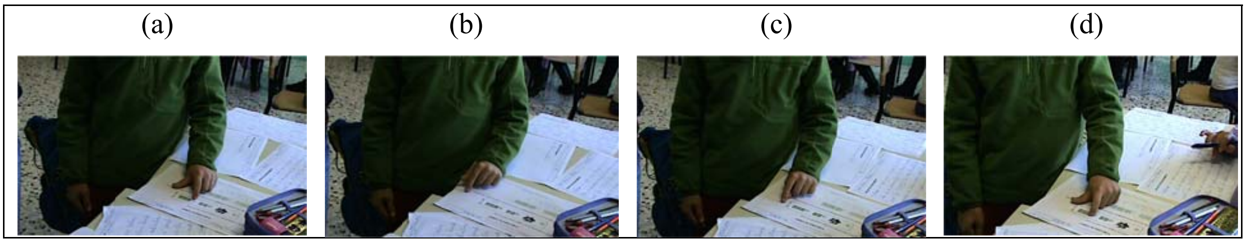

Fig. 3: (a-c) miming the bottom and middle rows; (d) pointing to missing circles

The passage above shows that new elements like Pippo entered the assemblage, which enriched Filippo's algebraic thinking and helped move his discourse to a more functional one compared to one previously discussed. The researcher introduced Pippo as a parameter to move the original problem to the highest generality. By introducing Pippo, she was suggesting to the children that it did not matter what the term number was, therefore inviting the children to focus on the operations on Pippo rather than the numerical results upon the operations. In response, Filippo talked of the algebraic expression "Pippo times two" and referred its result to "Two Pippos"—an interesting plural use of Pippo that also evidenced Filippo's way of how to perform operations on Pippo.

When asked to explain what the operation of "Pippo times two" meant for the second time, Filippo took much time to think about the question and was not successful in his first attempt. Some tension can be observed between Filippo and Pippo when Filippo tried to communicate the meaning of Pippo in an arbitrary figure. Having mixed up Pippo with the number of circles on the bottom row, he was reminded that Pippo was the "position" by the researcher, after which he clearly explained the meaning of Pippo by using it as a parameter to formulate algebraic expressions in the form of $2p$ and $2p$-2 for the number of circles in the respective rows in the "Pippo" position.

Discussion

The inclusive materialist framework highlights the emerging relations between the children, pattern, and mathematics as Filippo and Lara engaged in the task. It shifts the centre of the mathematical activity from the children to the children's body-material assemblage. After noticing the relationship between the term number and the number of circles on the bottom row, Filippo and Lara's discourse shifted from a recursive one, to one that involves the structural relations in the figure. We note that this discourse was only possible because of the way the pattern was arranged. Namely, the figural sequence was arranged in such a way that when the term number is multiplied by 2, it yields the number of circles on the bottom row. Had the pattern been arranged differently, the children's discourse would have developed differently. Hence, the arrangement of the figural pattern was a part of the children's material-body assemblage and their emerging conception. The thinking of "multiplying and dividing by 2" emerged in the assemblage, which allowed them to consider any one term independently from its previous term and moved their discourse from a recursive to a functional one.

As much as the task of "position 25" was significant, the opportunities for the children to work within WNA were also crucial for developing the children's algebraic thinking. We could say that WNA helped move the assemblage forward to the 25th term of the sequence. However, this does not mean that the children moved from concrete to abstract thinking. For the children, there was nothing abstract about the 25th term in the figural sequence; they could have found it by repeated addition. Their algebraic thinking began to develop as they engaged in finding a generic way to solve for the number of circles in a remote term like 25. WNA was instrumental in this process because it enabled the children to work with the relationship between the number of circles and the term number as well as verify that their method worked for the first several figures. Hence, this meant that they could use the relationship that they found and verified, and generalise it for any term. At the same time, the children's discourse of using WNA was increasingly developed and their confidence of working with the figural and numerical structure strengthened having interacted with the pattern for quite some time. Their emotional state, the specific arrangement of the pattern, their experience of applying WNA to the pattern, and the teacher's questioning were all part of the assemblage when Filippo made the generalisation of "twenty-five times two" to obtain the number of circles on the bottom row and to reason about the other two rows. In saying "twenty-five times two", the whole number operation was also becoming an algebraic operation because Filippo seemed to be talking about it not only in its particularity (the 25^{th} term) but also in general sense (for other remote terms).

By unbinding the "body" of mathematics as an assemblage of the children, pattern, and mathematics, we see that Filippo's gestures were an essential part of his learning. Through his gestures, Filippo mobilised what was a static figure, and this allowed the creation of new mathematics as new terms were gestured figuratively. He did so without the reference of terms like the fifth or sixth term, but he used his hands and fingers to enact the exact shape in the 25^{th} position as well as in the "Pippo" position.

When Filippo got stuck on the "Pippo" task, he was reminded of what he did in the "term 25" task. He then gestured near term 4 on his paper to explain what he did as if the fourth term was the 25^{th} term. At other times, he would gesture outside of the paper to talk about a term that should appear after the sequence. According to de Freitas and Sinclair, these gestures were mobile boundary-drawing devices that conjoin and separated the "real" with the mathematical. Once the children found the logical and structural relations that existed in the figure, they created new mathematics. Filippo's hand gestures of running the three rows and his other gestures were not merely iconic representations of the figure. Rather, they were a part of the children's emerging body-material assemblage that gave rise to both WNA and algebraic thinking. In the study, the observation about Filippo and Lara's development of generalising a structural relationship was not unique to them but was consistent with other children in the classroom. This suggests that assemblages as well as the body of mathematics were located within a broader classroom setting.

In this paper, we offered an approach to understanding the relationship between WNA and algebraic thinking within the material entanglement of children, pattern, and mathematics. In so doing, we called attention on the embodied nature of learning mathematics in relation to the learners' material surrounding.

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DISCERNING MULTIPLICATIVE AND ADDITIVE REASONING IN CO-VARIATION PROBLEMS

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Abstract

In this study of arithmetical reasoning, which extends earlier work, we explore what properties students, when working in pairs, discern in additive and multiplicative co-variation problems that help them to distinguish between problem types. Results showed that pairs who solved each problem appropriately discerned mathematically significant properties such as speed, starting time and distance. Pairs who over-used additive reasoning focused on the distance difference without considering speed. While speed is considered to be a difficult quantity, here it seems to help students distinguish between multiplicative and additive situations.

Key words: additive reasoning, co-variation problems, multiplicative reasoning

Introduction

The study presented in this paper extends work undertaken by Van Dooren and colleagues at the University of Leuven on students' reasoning when solving multiplicative and additive co-variation problems. Co-variation problems deal with two phenomena that change at the same time. An example of a multiplicative co-variation problem is two persons starting at the same time and running laps at different (but constant) speeds. If they run at the same speed but one person has started before the other it is an additive co-variation problem. The Leuven group has undertaken several studies on the ways in which students understand and exploit multiplicative reasoning showing that students have difficulties determining when multiplicative reasoning is appropriate (e.g Van Dooren et al., 2008; Van Dooren et al, 2010a, 2010b). We extend this line of research by investigating those features that students discern when solving co-variation problems that underpin the forms of reasoning they adopt. What problem features are discerned by those who reason correctly and how do these differ from the features discerned by students who over-use multiplicative or additive reasoning, i.e. employ multiplicative reasoning to problems where additive reasoning is appropriate and vice versa?

Multiplicative reasoning, which typically develops slowly (Clark and Kamii, 1996; Van Dooren et al., 2010a), underpins many topics in mathematics such as fractions, ratio and functions (Vergnaud, 1994), emphasising the importance of distinguishing between additive and multiplicative reasoning. Proportional problems, such as calculating how many marbles Tina has if she has 3 times as many marbles as Ann, who has 5, are typically employed to distinguish additive and multiplicative reasoning. A child who reasons additively will sum the numbers and declare that Tina has 8 marbles, while a child who reasons multiplicatively would multiply 5 by 3 and answer 15 marbles (e.g. Clark and Kamii, 1996). When students receive instruction about proportionality they tend to over-use not only multiplicative reasoning but also additive reasoning simultaneously (Van Dooren et al., 2010a). Explanations for such inappropriate reasoning processes are many and frequently draw on students' tendency to notice superficial features, such as numbers and formulations, rather than the deep-level properties of a problem (Verschaffel et al., 2000).

In problems employed to assess multiplicative reasoning the numbers as well as the quantity need to be carefully considered to avoid other sources of inappropriate reasoning. For example, numbers in word problems usually represent two different types of quantities, extensive and intensive. Extensive quantities, such as mass, can be measured and magnitudes sensibly added; the weight of two boxes of apples is the sum of each box's weight. Intensive quantities are usually ratios between two extensive quantities, such as meter per second, and have been defined as quantities "not susceptible to actual addition" (Piaget, 1952, p. 244). Thus, it is nonsensical to say that two persons' running speed is the sum of each person's speed. Extensive quantities generally pose fewer problems than intensive (Nunes, Desli and Bell, 2003). However, familiarity is important as well-known quantities like price per item are easier to manage than weight per unit volume. Speed is also considered to be an accessible intensive quantity (Van Dooren, et al., 2010a) due to the perceptual experience students have of the concept of speed (Nunes et al., 2003).

Two studies by the Leuven group are of special interest to this paper. In one study written tests, involving both additive and multiplicative co-variation problems, were given to 325 students to investigate the development with age of their ability to interpret additive and multiplicative problems (Van Dooren et al., 2010a). From this study we learn that numbers have greater influence than the mathematical situation. Integer ratios and differences tended to be handled as multiplicative problems while non-integer ratios and differences more often were treated additively (Van Dooren et al., 2010a). In the second study, 75 students were set two tasks. The first involved their categorising but not solving word problems, while the second involved their solving a set of similar word problems (Van Dooren et al., 2010b). The problems reflected additive and multiplicative co-variation problems. Students who worked on the categorisation before solving problems were significantly more successful in distinguishing between additive and multiplicative problems than students who solved problems before categorising. These studies show that many students cannot discern additive from multiplicative co-variation problems and that the act of categorising may help in the process of distinguishing additive from multiplicative problems. However, since the studies exploited written tests students' reasoning processes were difficult to identify.

Our aim was to extend our knowledge with respect to students' multiplicative and additive reasoning through our addressing the following research question; what features do students discern in additive and multiplicative co-variation problems that help them to distinguish the two problem types?

Materials and Methods

In order to study what features of a problem students discern we invited ten pairs of 6th grade students to solve problems in pairs, allowing us to follow students' problem solving discussions in a natural setting. We constructed two co-variation problems, one multiplicative and one additive, using the same context, children swimming lengths in a swimming pool. Problem 1 was a multiplicative problem, (MP), and problem 2 was an additive problem, (AP).

Gustav, Martin, Sofia and Elin are swimming in a pool.

1. Martin swims faster than Gustav.
   Martin and Gustav start swimming at the same time.
   When Martin has swum 6 lengths, Gustav has swum 2 lengths.
   How many lengths has Martin swum when Gustav has swum 10 lengths?
2. Sofia and Elin swim equally fast.
   Sofia starts swimming before Elin.
   When Sofia has swum 12 lengths, Elin has only swum 6 lengths.
   How many lengths has Sofia swum when Elin has swum 10 lengths?

We chose numbers in both problems to make it easy to apply both additive and multiplicative reasoning. All differences and ratios were integers, each question was posed using the same number of lengths (10) for the child who had swum the fewest and focused on the child who had swum the most. Such an approach might prompt an over-use of multiplicative reasoning (Van Dooren et al., 2010a), but our main reason was to ensure that all computations were simple and did not draw attention from student's understanding of the problems. The choice of context, swimming lengths in a pool, is well-known to Swedish students and hence would make it easy to grasp the meaning of the described situations. The problems' formulations were similar to those found in the two Leuven studies that we intended to extend (Van Dooren et al., 2010a, 2010b).

The students who participated were already enrolled in a research project and had been interviewed several times about different aspects of multiplication. They were paired according to information concerning their multiplicative reasoning from individual interviews earlier the same term. We paired students who had shown similar reasoning in order to form as homogeneous pairs as possible. The interviews took place in a room adjacent to their normal classroom and were video and audio recorded. Afterwards, any written materials were collected and full transcriptions were made. Students were instructed to collaborate on an agreed solution to each problem. The order in which the two problems were presented was decided arbitrarily and then used consistently for all pairs - there were too few pairs to warrant investigating whether the order made a difference. The problems were written on a card, which was presented to the students, before the interviewer read the problems aloud. After they had solved both problems students were asked to describe what was similar and what was different about the problems.

An iterative process, involving both authors, of reading and comparing students' arguments yielded qualitatively different categories of reasoning processes relating to the features students discerned. These categories referred to *speed, starting time, distance, context* and *numbers*. Context and numbers were considered to be superficial features, while speed, starting time and distance are features of mathematical significance in these two problems.

Results

Six pairs solved each problem with appropriate reasoning (i.e. the MP with multiplicative reasoning and the AP with additive reasoning). Four pairs over-used additive or multiplicative reasoning (i.e. used the same type of reasoning to both problems), three by additive and one by multiplicative reasoning.

All six pairs who solved each problem with appropriate reasoning discerned mathematically significant differences, which they discussed with each other as they were working on the solutions. All six pairs discerned that the *speeds* were different in the MP and the same in the AP, as exemplified by Anna's statement.

> Anna: Here one swims faster [pointing at the MP] and here they swim equally fast [pointing at the AP].

Four of the six pairs explicitly concluded that the difference in speed led to an increased *distance* in the MP or that the same speed led to a fixed distance in the AP, as exemplified by Jonathan's and Marcus' statements.

> Jonathan: Because he swims faster [Jonathan moved two fingers simultaneously along the table with one finger moving faster]

> Marcus: If they are equally fast then of course she keeps that distance. [Marcus holds his hands on a fixed distance from each other and moves them forward at the same pace.]

The importance of the speed was also repeated when they were explicitly asked to describe the similarities and differences they had discerned in the two problems. While speed is an intensive quantity, it is familiar to students and all six pairs were able to see how speed affected the distance between swimmers.

One of the six pairs with appropriate reasoning, Jonathan and Hugo, first solved the MP by additive reasoning but then, having started on the second problem, stopped and returned to the first problem and changed their solution to multiplicative reasoning.

> Hugo: I think about the first [problem] as well, how that was. He had the same distance ahead [of the other swimmer] all the time.

> Jonathan: He was four lengths…

> Hugo: He was four lengths ahead all the time, and here it is, here they start. When Sofia has swum twelve, and then six, then she is six ahead. Six lengths ahead.

> […]

> Jonathan: Martin swims faster. […] Then Martin must have kind of much more [lengths]. For two lengths Gustav have swum, six times, must on four [lengths] then twelve.

The four pairs who over-used additive or multiplicative reasoning found a procedure when they solved the first problem, which they reused with the second. The pair who used multiplicative reasoning to both problems discerned the speed difference when solving the MP then reused the procedure they had developed with the AP, as presented in the excerpts below.

> Sebastian (on MP): Because for each length he does […] he does, six [divided by], two [is], three lengths.

> Sebastian (on AP): Because each time she swims one length, she swims two lengths.

This pair's discussion on the MP is similar to the six pairs who reasoned appropriately to both problems; they discerned the *speed* difference and concluded that it led to an increasing difference in *distance*.

The three pairs that reasoned additively focussed on and calculated the *distance* between the two swimmers and then added the distance to the ten lengths mentioned in the question, as exemplified by Alva's statement.

> Alva (on MP): Because ten, he is four [lengths ahead], then it is fourteen.

None of these three pairs discerned speed while solving the first problem (MP); they all discussed the distance in lengths, which prompted a correct solution to the AP, but not to the MP.

When explicitly asked to describe differences all four pairs that over-used additive or multiplicative reasoning described *contextual* and *numerical* differences, as exemplified by the excerpts below.

> Felicia: It is girls there and boys there, that is different.

> Sebastian: The numbers are different but the way to calculate is the same.

In the end of the interview Hanna and Matilda, as well as Julia and Alva, also pinpointed significant mathematical differences; *speed* and *starting times*. We exemplify this by presenting what Matilda and Hanna said.

> Matilda: They start at the same time and they do not start at the same time.

> Hanna: And those two do not swim equally fast and those two swim equally fast.

However this did not lead any of them to revise their solutions or reasoning.

In summary, students who solved the problems by appropriate reasoning discerned and used the information about *speed* being different or the same and sometimes also that *starting time* being different or the same and inferred that this had implications for the *distance* between the swimmers. When these students were asked to explicitly describe differences and similarities they talked about these mathematically significant features. Some of these pairs also noted superficial differences such as boys in MP and girls in AP, but this was said with laughter and gestures indicating that it was not of importance.

In contrast the students who over-used additive or multiplicative reasoning found a procedure for the first problem, which they applied to the second. The additive procedure was based on differences in *distance* and the multiplicative on differences in *speed*. When they described the ways in which the problems differed they talked about *numbers* and other superficial *contextual* differences. Two of the pairs noticed not only that the speeds were different in the MP but the same in the AP but also that the start times were the same time in the MP but different in the AP without reflecting that these differences had any influence on how to solve the problems.

Discussion and conclusion

In this study we find evidence that when students discern significant features in a problem they can reason appropriately. That is, what students discern as significant determines what they do. Students, who discerned the significance of the swimmers' *speed* were able to infer that the distance between the two swimmers was relational in the MP and absolute in the AP. This can be construed as the nature of their discernment also determines their reasoning process. Hugo and Jonathan's reasoning bore evidence of this when they reacted to having used the same reasoning to both problems even though they discerned significant different features and went back to revise their solutions. They used the speed difference as grounds for thinking more about how to solve the multiplicative problem. Even though speed is considered to be difficult to handle for young students (Nunes et al., 2003) the notion of speed as being different or the same was discerned by all successful pairs and led them to appropriate reasoning. This can be explained by the familiarity of the situation as well as the fact that the students did not work with any quantities in the two problems; the speed was expressed as being different or the same and measured by the number of lengths the swimmers had swum at two points of time (i.e. extensive quantity). The appropriately reasoning pairs could infer that the speed being different lead to increased difference between the swimmers while the same speed lead to a fixed difference. This might be an important clue for how to design instruction to help all students to make such inferences.

We also confirmed earlier findings (Van Dooren et al., 2010b) that to ask students, after solving problems, to describe what is different did not help them to discern significant differences. If students have not discerned the difference in swimming speed, it is of course not possible for them to describe that. But two pairs, when questioned after solving the problems, identified both speed and start time differences without realising their significance with respect to the solution processes. Perhaps asking them to first read both problems and then discuss what was similar and different before solving them might have had an impact on their solutions. However, this was not the intention with our study, we wanted to see what features that students pay attention to and act upon when they solve co-variation problems of multiplicative and additive nature. This confirmation of the findings from Van Dooren et al. (2010b) has implications for instruction; comparing and categorising word problems before solving them enhances students' ability to discern mathematically significant features in them.

With respect to the first problem, all pairs engaged with the context of children swimming without suspension of meaning, extracted numbers and applied an arithmetical operation, as has found regularly in many different contents and cultures (e.g. Verschaffel et al., 2000). This may be due to the familiarity of the context, as swimming lengths in a pool is meaningful for these particular students. However, after finding a procedure (correct or incorrect) for solving the first problem four pairs re-used that procedure with the second problem without further discussion or reflection. In particular, Ebba and Sebastian, the only students who over-used multiplicative reasoning, worked collaboratively on the MP until they had an agreed solution they thought was appropriate. When they worked on the AP they did not discuss but immediately applied the procedure they had employed on the MP. One simple explanation may be that they just did not pay sufficient attention to the presentation of the problem. Another, more significant explanation, may be that in the game of school mathematics instruction typically addresses one category of problem at a time (Verschaffel et al., 2000). Such matters also call for instruction, inspired by the study where students had to categorise problems rather than solve them (Van Dooren et al., 2010b), inviting students to discuss with both peers and teachers rationales for their categorisations.

The most important part of the results is that the students who discerned the significance of speed being different or the same were able to reason appropriately to both the multiplicative and the additive co-variation problem. Even though speed is considered to be a difficult quantity it might be the key to grasp this type of situations and more research is needed in this area.

We believe that the methodology of our study, where we had student pairs discussing and solving problems together, as well as discussing the similarities and differences of the problems, might be useful in shedding light on how to move students' focus from superficial features to significant mathematical features. The study reported in this paper could be used as a pilot for a larger study, involving more students, focussed on generalisation. In such a study the order of problems could be random for students pairs to rule out any differences from starting with a MP or an AP as well as giving the question about similarities and differences about the problems before asking them to solve the problems.

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ADDITIVE STRUCTURE:
AN EDUCATIONAL EXPERIENCE OF CULTURAL TRANSPOSITION

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Abstract

In this article we present a "western" reflection on a particular graphical representation of the additive structure, the pictorial equation, used in Chinese and Russian primary school educational practices. We will discuss the possibility of using the same representation also in Italian primary schools by presenting an educational experience of *cultural transposition*. Looking at educational practice in another culture is seen, in this perspective, as an opportunity to rethink our own.

Key words: educational cultural transposition, pictorial equation

Introduction

The unstoppable phenomenon of globalisation is permeating social and cultural spheres bringing both positive and negative effects. The numerous international assessment projects (such as PISA and TIMSS) are among the most visible promoters of this process in the world of education, particularly in the field of Mathematics. This growing opportunity to share and reflect not only on the results of these tests, but also on the educational practices affecting these results, creates new space for reflection in the educators debate. Transposing educational practices between countries, or even continents, is becoming a reality involving several issues. For example, it is increasingly acknowledged that the different educational choices and tools are inseparable from the history and the culture of the places where they were born and developed: each time that an educational path is looked at from the "outside" its rationale and nature risk being misunderstood if the whole cultural context is not taken into consideration (Bartolini Bussi and Martignone, 2013). Furthermore we think that whilst comparing two or more cultural-educational backgrounds, it is crucial to maintain their differences without "translating" them from one culture to another, but rather highlighting these very differences in order to review their meaning processes and daily use. We call this process *cultural transposition* of educational tools and we will define it better the further ahead in this paper trying to show how it can be a powerful tool to review our own educational practices. In particular, in this study we will present an example of cultural transposition of the *pictorial equations approach* of the Chinese curriculum[3] (Cai and Knuth, 2011; Sun, 2011). Our "western" viewpoint allows us to link

[3] Due to space limits, in this study we don't deal with the use of *pictorial equation* in the Singaporean curriculum (*Cai et al., 2011*).

the Chinese pictorial equation approach to the use of the *intermediate strategies of graphic representation* described in Davydov's Activity Theory approach to number (Davydov, 1982). In this article we will propose a reflection on some cultural aspects of the use of pictorial equations (see an example in fig. 1) in both Chinese and Russian (Davydov's) curricula. This reflection will be developed as an attempt to better understand the ways of perceiving and using this representation in the two different cultural environments. In this sense we are not proposing a comparative study of mathematics education, but rather a dialogue between different educational practices, where each teaching choice, in meeting another one, is considering its own un-thought practices (Jullien, 2006). In particular we will argue that in both educational practices, even if for different reasons, the use of pictorial equations during the first years of primary schools seems to promote a particular approach to the arithmetic of whole numbers, in particular to addition, with a specific emphasis on its structural aspects, in other words with an algebraic perspective.

Finally we will present an educational experience built on the use of this representation developed in an Italian fifth grade classroom.

Davydov and the intermediate strategy of graphic representation

The representation shown in Fig. 1 is one of the key elements of the arithmetic approach proposed by one of the most popular exponents of the Russian Activity Theory: Vasili V. Davydov. The spread of Vygotskij's sociocultural reflections and the Cold War influence are some of the components of the cultural milieu in which Davydov's mathematics curriculum was developed. In those years the desire to strengthen education, science education in particular, grew alongside the desire to train new generations of scientists and engineers capable of beating the advance of Western technology. It is interesting to underline that while Davydov's mathematics approach was born in this particular cultural and historical context, it has many followers all around the world even today (e.g. Cai and Knuth, 2011; Iannece, Mellone and Tortora, 2010; Slovin and Dougherty, 2004).

According to this theory, the genesis of the number concept is rooted in the experience of measuring continuous quantities (Davydov, 1982). The notion of quantity comes from comparison of elements of a given class (e.g., lengths of segments, amounts of water, weights, etc.) and measuring means relating a given quantity to a part of it, assumed to be a unit. In this perspective, counting itself may be conceived as the particular measuring process of discrete objects, whence the sequence of natural numbers appears as just an example of quantity.

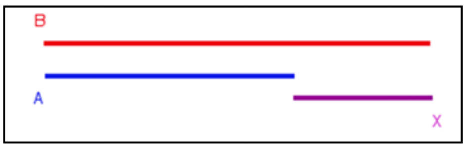

Fig. 1: Example of a graphic representation

But a deeper acquaintance with quantities allows children to widen their knowledge of numbers to include, in turn, integer, rational and real numbers.

Davydov suggests that since their very early years, children should manipulate the properties of quantities and use algebraic language to treat them well before practice with natural numbers.

He proposes activities where, firstly, an order relationship between two physical quantities is recognized and expressed, and then the attention is focused on the quantity that has to be added to the smaller quantity to obtain the larger one. In order to help children to move to a more abstract level, he suggests transforming qualitative aspects into physical segmental quantities (without paying too much attention to their lengths), as in Fig. 1: "*To prepare children for this shift, an intermediate strategy of graphic representation is used. The children represent the physical quantities with two segments A and B. How the difference in measures of the quantities can be determined is then discussed. Line segment A is superimposed on the line segment B. The difference expressed in the form B-A is defined equal to x*" (Davydov, 1982, p. 234).

In accordance with this perspective the graphic representation of the segments helps children to recognise the relationship between the two quantities A and B and in using subtraction as a formal description of the process of comparing A and B, rather than as a decrease. It is important to notice that the representation is seen as a bridge between the physical and the concept worlds. As Davydov said: "*In a school subject, intermediate means of description have crucial significance because they mediate between a property of an object and a concept*" (*ibid.*, p. 237). In this sense the children, after focusing on the relationships between the physical quantities, are invited to use the graphic representation of Fig. 1 (named also pictorial equation) as an intermediate step between the concrete experience and the more formal representations by algebraic symbols.

The representation of additive structure in Chinese variation problems

In a multicultural frame like ours, it is important to note that while written and spoken English, like, for this, Russian and Italian, are phonetic and symbolic languages to a greater or lesser extent, the Chinese language is built on ideograms that link it closely to reality, and for this reason it is essentially *immanent* (Jullien, 2006). In order to better explain this claim and its implications, let us look at Fig. 2 where the development of three ideograms from "pictograms" to ideograms is sketched. Looking at the first row, for example, we notice that in the ideogram 马 [mā, horse] there is a progressive moving away from the iconic drawing, but the nature of the ideogram is still in keeping with its original meaning and some concrete elements, such as the mane or the paws, are still recognizable even the modern day ideogram. This feature is so evident, that even today in order to teach ideograms Chinese teachers connect some elements of the ideogram to parts of the real object. In this sense we can talk of the immanence of the Chinese language.

Fig. 2: Development of ideograms

According with the hypothesis of Boroditsky (2011), language contextualises human's perception and organisation of reality thereby deeply affecting individual meaning processes. In particular in western culture we can recognise the process of categorisation, influenced by language, directed towards distinguishing and building precise boundaries. We refer to the Principle of Non-contradiction principle or to Aristotle's categories and their evolution into Kant's ones, with two examples.

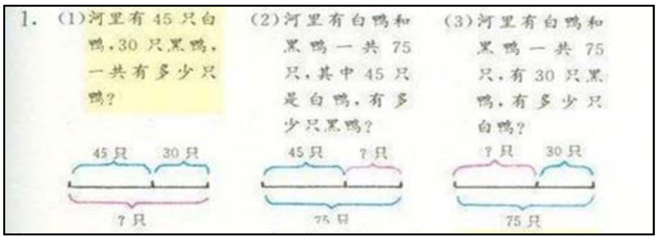

Fig. 3: Variation problem

On the other hand in Chinese culture we can recognise a characteristic way of looking at reality in which the act of distinguishing is only done in order to find unifying relationships: (以法通類, 以類相從) "to categorize in order to unite categories" (Bartolini, Sun and Ramploud, 2013), in other words categorisation is a process oriented towards building relationships and uniting concepts. Recent studies (Sun, 2011) put this aspect of the Chinese thought in relation to the development of mathematics teaching methodologies. Indeed, this perspective has been recognised by many authors in one educational tool used in the first years of primary school used in Chinese mathematics education practice: the variation problems (see Fig. 3; Sun, 2011; Bartolini, Sun and Ramploud, 2013; Ramploud and Di Paola, 2013). This is essentially a method to solve word problems by searching not for the solving operations, but rather for the arithmetic relationships involved. As we can see in fig. 3, several word problems with the same arithmetic structure are presented in the same page. The pictorial equation is posed below the different word problems and serves as an aid to recognise the common arithmetic structure underlying the problems. What is essential is this shift from the purely arithmetical domain to a relational one oriented towards – in Cai's words – "informal algebra" (Cai and Knuth, 2011). Unlike "pre-algebra", in which the name itself highlights the fact that it is something prior to a formal introduction to algebra, the name "informal algebra" suggests something parallel to formal algebra. Indeed they talk of a smooth transition from informal algebra to formal algebraic language (Cai and Knuth, 2011). Consequently, unlike Davydov's approach, in the variation problems the pictorial equation is not seen as support to move from the concrete plane of operations with real quantities toward the symbolic plane of algebra, but rather it is seen as holistic summary of the written text, keeping in this way the immanent feature of the Chinese language.

The paradigm of cultural transposition

Now we develop our framework, that is the *cultural transposition* of educational tools. First of all, it is important to underline that this framework cannot be considered as a meta-framework, but rather, that it acquires its meaning precisely from our western point of view. Here, specifically, we will use it to describe and read an educational experience of the use of the pictorial equation given in the next section.

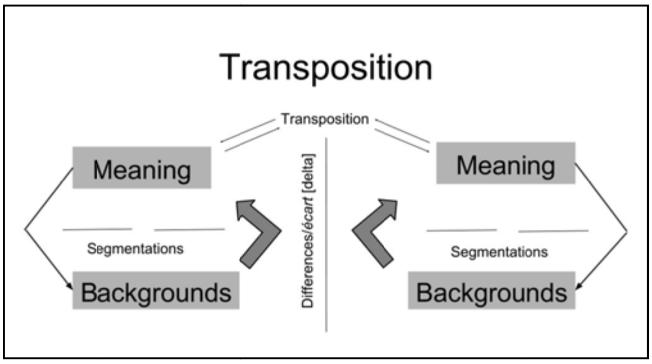

Fig. 4: Cultural transposition process

We try to make clear the cultural transposition process using the schema in Fig. 4. The arrows going from the backgrounds to the meaning represent the process through which the cultural contexts generate the meaning (Bateson, 1972). The presence of arrows going also from the meaning to the backgrounds suggests the dual nature of the relationships between backgrounds and meaning. Indeed if the context implicitly influences the meanings, there will always be a back-action of the meaning that "shows" the context, segmenting and organising it. In our discourse the different cultural backgrounds generate possibilities of meaning and of mathematics education perspectives that, in turn, organise the contexts and school mathematics practices in different ways. This interpretation allows us to describe the process of cultural transposition as something that, differentiating two cultural processes of meaning, reconnects the different cultural backgrounds consistently with the theoretical construct of *"pattern which connects"* (Bateson, 1972). In this way the differences between cultural contexts are maintained, without any attempt to "translate" elements from one culture to another, but rather to carefully review the different daily meaning processes in order to become more aware of our own ones.

Starting from the reflections of the previous paragraphs we can argue that the use of different representation registers is present in Davydov's approach as in Chinese problems with variation. We think that for different reasons, in China for reasons inherent to the language while in Russia for pushing mathematics education towards advanced mathematics areas since the beginning, the approach to Whole Number Arithmetic (WNA) is definitely geared towards structural and algebraic visions of the additive structure, often inhibited in western primary teaching practice. Indeed, while pictorial equations appear in Chinese and Russian (Davydov) textbooks from the first year of primary school, in Italian schools they are found from the 6th grade to the 8th grade and they completely disappear in the following grades. In general we can say that in the Italian curriculum, as in America, the study of algebra starts from the 7th grade and the Western and the Eastern cultures offer two different segmentations of reality in this specific mathematics education choice.

An educational experience of cultural transposition

Here we report an experience of cultural transposition of the pictorial equation developed in an Italian fifth grade classroom. The class teacher is an expert teacher researcher who has been working for several years in a research group with the semiotic mediation framework (Bartolini and Mariotti, 2008) as the main theoretical reference for her work. For this reason she recognised in the pictorial equation a real cultural artefact with particular features: on the one hand arte-factual characteristics, like its manipulability and, on the other hand, its abstract features like its flexibility and its independence from the context described in the specific word problem.

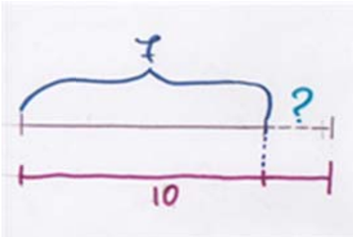

Fig. 5: Pictorial equation

For this reason, in the first part of the educational path, the teacher decided to offer the pictorial equation to pupils without any word problem text connected to it, as in Fig. 5. Starting from it she asked pupils to describe a problem without a concrete reference and orchestrated a mathematical discussion. The pupils produced many interesting reasoning like the following one: *"In this case I take off three and find the difference. It is the same when the question mark is in a different point of the graph"*. This pupil is recognising one of the fundamental features of the pictorial equation, its holistic flexibility that allows the use of the same drawing even with different "position" of the unknown. This idea was expressed by another pupil with the following words: *"The problem changes according to the position of the question mark, but you can use the same drawings"*. The possibility to observe these kinds of pupil reactions, even starting from an activity different from the variations problem, led us to think that the pictorial equation, if used with awareness, can be used as a word problem-free tool able to summarise in itself different relationships (according to the position of the unknown). On the base of this latest solicitation concerning the shift of the unknown, the teacher decided to design a series of activities to engage pupils in the choice of a pictorial equation and in the consistent creation of possible problems modelled by it (something similar to the variation problems). During this individual work, one of the pupils, in devising the variation problems on the base of a chosen pictorial equation, introduced new variables. As we can see by looking at two pages of his book notes (Figs. 6-7), even though the classroom teacher did not do explicit work with letters, the pupil invented his problem directly using the letters. We proceeded to interview him asking whether he had in fact already seen this presentation or had ever heard of it outside of the school environment. The responses were negative and he argued: *"You can use the letters, because in this case they mean all the numbers"*. From his invented problems and this observation we can appreciate how the use of the pictorial equation allowed him to become aware of the importance of the structure of a problem, instead of relating a problem to specific numbers. At this point, in order to move pupils' attention from the pictorial equation as a structured model of problems to the pictorial equation as a tool suitable for facing the word problems that were more typical for these pupils (that is, problems more common for the western tradition), the teacher proposed the following word problem: *Grandma gives 618 euros to her grandchildren Franca, Nicola and*

Stefano. Franca will receive twice what Nicola gets and Stefano 10 euros more than Nicola. How much will each grandchild get? The pupils solved the problem by working in small groups and using the pictorial equation, now familiar to them, during this mathematics activity.

A very interesting episode happened when one of the groups reported its solution on the blackboard (it is documented by a video-registration). The movement of the pupil's left hand (Fig. 8) suggests that he is using the segment representing Nicola's amount as a unit of measurement of the representation (in this sense we can see a link with the Russian cultural tradition with its strong emphasis on measure and unit of measurement). Moreover, the numerical data of the problem are practically abandoned in favour of using only the literal and graphical representations. Thanks to the use of the pictorial equation they easily explain to the rest of the class that "*it is enough to remove ten and Stefano' and to divide by three, in order to have Nicola'* " (Fig. 9). The simultaneous and immanent presence of all the elements and relationships involved in the problem seems to be a supporting feature.

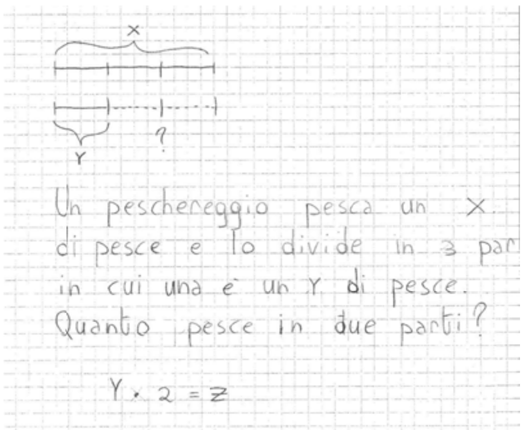

Fig. 6: A fishing boat fishes x amount of fish and divided it into three parts one of which one part of fish is y. How much fish is there in two parts?

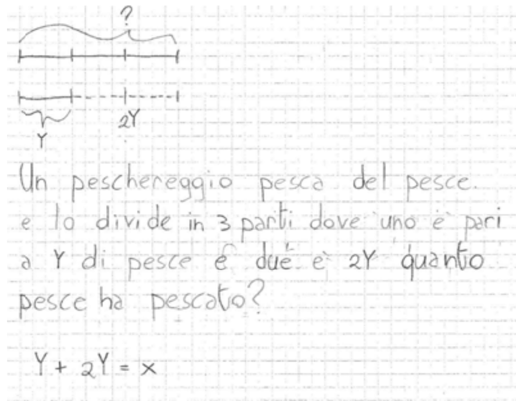

Fig. 7: A fishing boat fishes some fish and divides it into three parts one of which is y of fish and the other is 2y how much fish was fished?

Fig. 8: Movement of the pupil's left hand

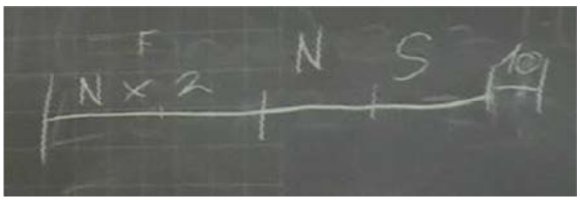

Fig. 9: Pictorial explanation

Conclusion

Our reflections stem from the belief that observing foreign cultural school practices can be an opportunity to re-examine, with western eyes, our teaching practice in schools. In this case the reflection on the pictorial equation (Fig. 1) and on its meaningful use linked to specific historical and cultural reasons in

Russian and Chinese primary schools, gave us the opportunity to explore its use in a more conscious way in an Italian fifth grade class. Indeed, even if it is not present in the tradition of the Italian school curricula, in the experience presented we recognised in it the opportunity to develop an approach towards Arithmetic with primary pupils which pays more attention to the structural features than to the numerical ones. Indeed we observed pupils' natural and flexible recourse to algebraic language in a context built on the pictorial equation. Moreover, we recall that algebraic thinking is recommended as one of the primary mathematics education goals by the most recent Italian national curricula (MIUR, 2012). In this sense the didactic choices for teaching mathematics favouring the structural aspects of Arithmetic rather than the numerical ones, as embodied in the Chinese "ways of seeing" and used in Russia since the last century, seem more timely than ever also in Italian primary school practice.

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LEARNING PLACE VALUE THROUGH A MEASUREMENT CONTEXT

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Abstract

Children's everyday measurement experiences serve as the basis for developing mathematical properties about number. In the *Measure Up* program, this context is applied to first develop foundational properties about mathematical relationships, and later to place value and number magnitude. The mathematical concepts are introduced through learning activities in a measurement context. Measuring activities with nonnumeric quantities are designed to embody the magnitude associated with the place value. We argue that this approach develops understanding of the structure of multi-digit numbers. This curricular approach contributes to the discourse laying foundational ideas about number in the primary mathematics curriculum. In this paper we describe how number structure is developed from measurement contexts using everyday experiences to define unit and number relationships in different bases. We share student works and discuss how these give evidence for this curricular approach.

Key words: Davydov, Measure Up, number systems, place value, unit

Introduction

The concept of place value has a critical impact on young students' mastery of arithmetic and the comprehension and production of multi-digit numbers. Studies have shown that one's language and the names for multi-digit numbers can affect the acquisition of the number system (i.e., Ng and Rao, 2010; Van Luit and Van der Molen, 2011). Yet simply training children to have good number naming skills does not necessarily lead to understanding place value and the base-ten system (Cheng and Chan, 2005).

Beyond a semantic issue, the ability to compare the magnitudes of multi-digit numbers has been found to be strongly related to mathematical achievement (Fazio et al., 2014). Thus it follows that if early misconceptions about place value are allowed to perpetuate throughout children's mathematics education, these could contribute to common errors in later work with algebraic expressions. Students' incorrect transfer of arithmetic representations of verbal text can affect the use of algebraic notation. For example, students may learn that "5 more than 20" can be combined to create 25 and then respond to "10 more than *h*" in a similar manner, 10*h* (Weinberg et al., 2004). Developing a rigorous understanding of place-value in the early years could have long-reaching effects on later mathematical learning.

Prior studies on the development of number skills have focused on students working solely in base ten. In addition to number naming strategies, other indicators of place-value understanding have focused on the application of strategic counting skills (i.e., Chan, Au, and Tang, 2014; Ho and Cheng, 1997) and the ability to compare magnitude differences between number pairs

(Moeller et al., 2011). Rather than focus on base-ten counting to teach place value, we propose a conceptual basis for instruction of place-value understanding. We describe the development of number as it emerges from carefully designed experiences with continuous quantities. We argue that children learn whole number by first recognising and articulating the theoretical basis that defines the mathematics. This is verbalised and demonstrated by the children through instructional supports in classroom learning activities. Measurement serves as the context to develop place value as a consequence of the base while maintaining the developed mathematical properties. We maintain that this approach, as proposed by Davydov (1975a, 1975b), promotes a robust understanding of place value and enables children to carry out and justify quantitative comparisons.

In this paper we present a foundational topic of the Measure Up (MU) curriculum (Curriculum Research and Development Group, 2006), the development of place value. MU introduces place value as a way to record the measure of quantities and make comparisons among those measures. Students construct a set of units such that the relationships among those units are powers of the base. The numeration system provides a structure to record each unit in a manner that can be extended to include larger and smaller supplementary units. Following this development, the decimal system is accepted as a particular instance of this numeration system.

Our objective in this study is to describe a measurement approach to number structure and place value and share evidence about the effectiveness of this approach. We conclude by discussing how later topics, such as rational number and algebraic expressions, are introduced in ways that maintain the integrity of the mathematical properties developed in these critical first experiences. We investigate the following question, *How does a measurement context support student understanding of place value?*

Background

The MU Grades 1–5 program uses a measurement context to develop critical mathematical topics. This curriculum, developed in the U.S. by the Curriculum Research and Development Group, is built from the experimental curriculum derived by the El'konin-Davydov team (Davydov et al., 1999). MU is grounded in the premise that mathematical structures constitute the foundation for mathematical knowledge. Knowledge is developed through everyday experiences with common measureable quantities of area, length, volume, and mass. The Grade 1 work begins with generalised ideas (e.g., describing how one quantity compares to another) and progresses toward an application of the generalised ideas in specific cases (e.g., using number values to solve a problem). This is in contrast with the more typical U.S. counting approach to school mathematics (Devlin, 2009) that begins with a focus on specific cases (e.g., number facts) and builds toward an understanding of generalised cases (e.g., properties of mathematics described with variables). MU presents this in reverse, properties before the number facts. This reversal becomes accessible to young children in the study of equality, for example, as they trace, cut, tape, pour, and weigh continuous quantities. When prompted to determine a specific characterisation of a difference, a unit is used to measure and determine a count. This focus on quantities provides an appropriate means to consider the theoretical foundation of the mathematics and gives relevance to the work with whole number.

The concept of place value develops when measuring a large quantity requires using the unit a number of times greater than the base number. Once the count reaches the base number, students are prompted to create a supplementary unit in order to numerically represent that quantity. For example, in order to measure a large volume of liquid in base six, students start with a small container, name its capacity unit E, and iterate E six times to create a supplementary unit, referred to as a place II or E_{II} unit. This new unit can then be used to measure the quantity beyond 5 E units. For larger volumes, students can extend the process by iterating the E_{II} unit six times to create a place III or E_{III} unit. By using the different units, students have the physical tools to measure and record a given quantity. In this context, whole numbers are not viewed as merely counting numbers but as the representation of a measured quantity. This approach develops a perspective of place value as measurements structured by the base.

Multiple experiences of such regrouping in different bases provide students with opportunities for considering the notation, “10,” (read “one–zero”) as a representation that can be generalised to quantities in any base, thus, representing quantities other than “ten.”

Study the table.

	dm	cm	
Peter	4	1	*A*
Nick	3	11	*A*
Mary	2	21	*A*

Who do you think was able to measure length *A* quickest? Explain in words why you think so.

Fig. 1: Learning task about relative magnitude using “cm” and “dm” units

The learning task shown in Fig. 1 is characteristic of an introductory task where students are asked to think about the relative magnitude of length units and the efficiency afforded when using the longer unit, “dm,” to measure length A. Given three different measurement recordings, students are asked to decide which was the “quickest.” One student decided Peter would be the quickest, explaining that “dm” takes more space (i.e., length) than the shorter “cm” unit, and that this would therefore take less time. The efficiency afforded by the longer unit helps to justify the need for place value. The design of this activity reinforces the property of equality (i.e., equivalent measurement results obtained for Peter, Nick and Mary) while leading students to attend to numeric structure (i.e., regrouping units).

Materials and methods

For this study we are interested in identifying indicators of Grade 2 students' generalised understanding of place value. In MU, place value learning activities are structured in a measurement context using different base systems where only a certain number of digits are available. For example, in a base-three system only digits 0, 1, and 2 are used to record a quantity. In this case, when children measure a quantity larger in magnitude than two units, a new supplementary unit is required.

Data were collected from 2002–2008, where ultimately thirty students' work were analysed for this study. The students were ages 7–8 years, and they were grouped in cohort classes of ten. They had 40–45 minutes of daily instruction in MU mathematics. The students attended an urban research laboratory school, where the population at the school reflects the state's public school population with respect to ethnic diversity, socioeconomic background, and standardised achievement scores. Although different teachers were involved in this multi-year study, they employed a consistent pedagogy in the MU instruction. The lessons and assessment described in this study were used approximately halfway through the second year of the curriculum. The data sources for this study are student written responses on assignments and assessments.

Responses to two problems from a five-problem assessment used to assess student understanding of the number system are described here as indicators of student understanding about place value. These problems provide students with the opportunity to show place-value understanding from a non-counting approach. Problem 1 (see Fig. 2) is a performance assessment, similar to learning activities in the MU lessons. This problem provides students with information about the composition of an area. In this task, a generalised unit (i.e., unit E) is used to create supplementary units (i.e., E_{II} and E_{III}) in order to construct area *J*.

Use your units and the information from the table to draw area *J*.

III	II	I	
1	3	2	(four)

Fig. 2: Assessment problem 1, composing an area in base four with different units

In Problem 2 (see Fig. 3), students compare similar numbers with different bases and explain how they decided on their response. The task, $4_5 \square 4_7$, invites students to write a relational symbol to show how the two quantities compare. The task, $21_6 > 21_\square$ invites students to select a base for the second two-digit number to make a true statement. The last task $3\square_9 > 38_9$, has students create a two-digit number in order to make a quantity greater than the second. This last task, purposefully designed with a structural flaw, is consistent with the tradition of the Davydov curriculum. It confronts students with a situation that is intended to stir protest and affirm the rules guiding place value.

Complete the statements below and explain your answers.

$4_5 \square 4_7$

$21_6 > 21_\square$

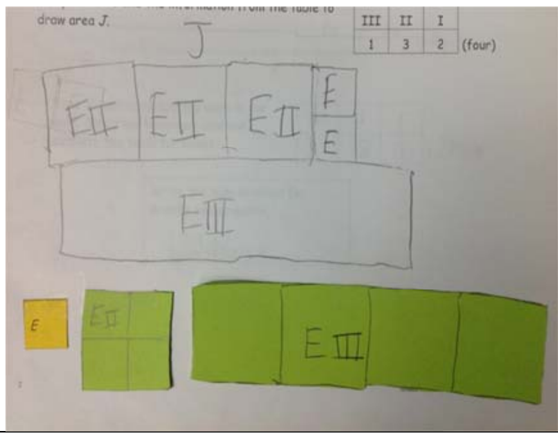
$3\square_9 > 38_9$

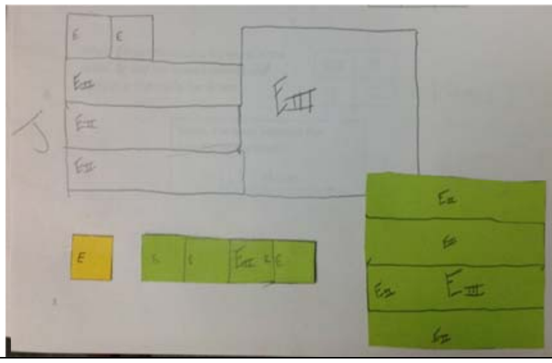
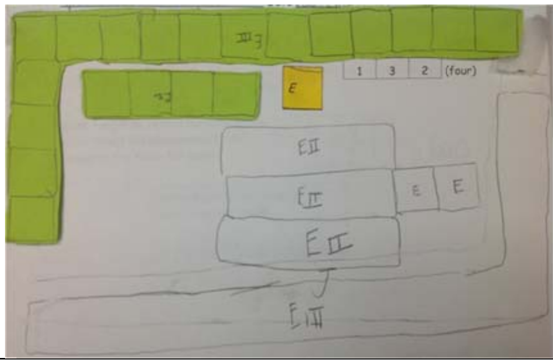
Fig. 3: Assessment problem 2, comparing numeric quantities

The three tasks in Problem 2 assess students' abilities to compare numeric representations of quantities, complete statements, and explain. This set of tasks also provides students with the opportunity to carefully examine place-value and base relationships among the digits in each statement.

Results

Responses on the two assessment problems (Fig. 2 and 3) show points in MU students' development of place value. In Tab. 1, we share two acceptable responses and a third response that shows partial development of the concept. Examples A and B show the use of the E unit to create supplementary units. Four iterations of E are used to create a new area, a supplementary unit named E_{II}. Four iterations of E_{II} are then used to create the next larger supplementary unit, E_{III}. The three units are used to create the desired area, 132_4. The students' constructed areas are taped to the paper with their tracings.

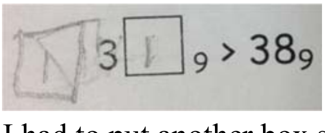
Example	Quantities of area created in 132_4 four.
Example A: Acceptable response	

Example B: Acceptable response	
Example C: Developing response	

Tab. 1: Responses to the assessment problem 1

Example C shows the iteration of the unit E to create the second place unit E_{II} and, instead of using the E_{II}, the continued use of E to create the third place unit, E_{III}. Although this approach results in an accurate area, it neglects using the supplementary unit to iterate by four, a critical embodiment of the base system.

Responses to the third task of Problem 2 show insights about the number system. Eighty percent of the responses address the impossibility of solving the task in the manner the first two first tasks were solved (see Tab. 2). Recognising the lack of available digits greater than 8 (in base nine) and the relationship that needs to be preserved, students respond by pointing out the flaw in the task. Jordan's response is particularly insightful; in order to preserve the relationship between the quantities, he decides to change a condition of the problem by creating another place.

Student	Responses to the assessment, $3\square_9 > 38_9$
May	I am stuck It says > but they took the big anser [*sic*] I can't think.
Kit	It dosent [*sic*] make sense because the other statement is 38_9 and so I cant write anything
Ren	It's eight so I can't use digit 9
Jordan	 I had to put another box so the statement would be true
Mic	Because if I put in answer [*sic*] it will be lesser and it spote [*sic*] to be greater

Tab. 2: Responses to assessment problem 2

Discussion and conclusion

The ability to compare the magnitude of numbers based on their digits has been studied extensively in terms of a unit-decade perspective. Such studies (e.g., Mann et al., 2011) examine the way students interpret individual digits regardless of their place value, assumed in a base-ten number system. Our study helps to reveal students' perceptions of magnitude as influenced by a measurement approach, first with generalised quantities and later using numbers in multiple bases.

Prior research about the role of syntax in the learning of numbers has shown that ease with number naming does not ensure understanding of the relevant number system. For instance, students often learn to count beyond 9 without pausing to consider the system of base-ten counting or the magnitude of the unit in each place. In MU, the physical actions of creating and measuring with units provide quantitative context for the structure of the written number representation. The concrete activities are designed to provide a quantitative context for the structure of the numeric representation.

The data reported here indicate how place-value understanding is supported through learning experiences in a measurement context. This has serious implications for later learning, such as in Grade 4 where the understanding of number structure expands to include rational number. The unit of measure continues to serve as a critical tool for both the conceptual and the physical development of partial units (e.g., thirds in base three or tenths in base ten) and partial-partial units (e.g., ninths in base three or hundredths in base ten).

The MU approach to place-value introduces students to a robust perspective for learning mathematics as sensible and coherent. Furthermore, mathematical modelling with concrete, iconic, and abstract representations provides students with multiple opportunities for making connections. Evidence for this is noted when students are introduced to algebra and interpret an expression (e.g., $2a+5b$) as in its simplest form because these are *a*-units and *b*-units that cannot be combined (Venenciano and Dougherty, 2014).

We suspect that MU students develop the capacity to focus on the supplementary measures, (e.g., E_{II} and E_{III}) as units themselves rather than as counted collections of discrete pieces. We believe this notion is particularly developed when students work with quantities such as volume where the distinction of E units is no longer obvious. Based on our work with the later grades of the MU curriculum, we hypothesise that students who develop this understanding can transfer this perspective, an inherently multiplicative one, to other topics such as ratios, exponents, and work with variables.

Acknowledgements

The authors wish to thank the students, teachers, parents, and administrators of the school for their generous support of our research. We also thank the Curriculum Research & Development Group at the University of Hawai‘i for their support of the research and development activities of the Measure Up project.

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REPRESENTATIONAL APPROACHES TO PRIMARY TEACHER DEVELOPMENT IN SOUTH AFRICA

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Abstract

While representational flexibility and efficiency in mathematical learning are widely accepted as important, some formulations of professional knowledge locate work with representations within the pedagogic content knowledge domain rather than within disciplinary learning per se. In South Africa, broad evidence points to significant mathematical content knowledge gaps among primary teachers at, or close to, the level of teaching. This situation suggests the need to simultaneously direct attention to representations as pedagogic objects and mathematical tools, in order to support teachers' mathematical learning and their mathematics teaching within teacher education. Findings from a South African in-service primary mathematics for teaching course incorporating this simultaneous orientation are reported in this paper. Analysis based on teacher responses to an integer-scaling item suggests that emphasis on representation can potentially be simultaneously productive for both mathematical learning and teaching.

Key words: primary mathematics, representations, teacher development, South Africa

Background

The importance of representational flexibility and efficiency in mathematical working and learning is widely accepted. Within some formulations of professional knowledge though, the need to work with representations has tended to be located within the pedagogic content knowledge domain rather than within disciplinary learning per se, with Shulman's (1987) oft-cited reference to 'the most useful ways of representing and formulating the subject that make it comprehensible to others' providing a key point of departure for this view. The shift here is from mathematical problem-solving for oneself through the use of representations to supporting the mathematical learning of others through providing representations that bring mathematical objects into being. One consequence of this view in teacher education is the potential for representations as object being the focus in methodology courses, while content courses focus on mathematical content with representations as tools within problem-solving.

In South Africa, broad evidence continues to point to significant gaps among primary teachers in relation to mathematical content knowledge at, or close to, the level of teaching (Taylor, 2011; Venkat and Spaull, 2015). This situation suggests the need to simultaneously direct attention to representations as object and tool, in order to support teachers' mathematical learning and their mathematics teaching within teacher education, beginning in the context of whole number arithmetic. This simultaneous orientation to representations was at the forefront of the 20-day in-service primary mathematics for teaching course, set within the broader Wits Maths Connect – Primary (WMC-P) longitudinal research and development project working with ten government primary schools to develop mathematics teaching and learning. Connecting between representations formed a central pillar of the work in the 20-day course alongside the need to 'explain' choices and steps in order to communicate the idea that mathematics is reason-able. Across primary and secondary mathematics teacher education, this has led to our development of the idea of 'mathematical discourse in instruction' (MDI) (Venkat and Adler, 2012; Adler and Venkat, 2014), with the emphasis on discursive development in teachers' relationship with mathematics.

A further contextual point to note in South Africa is that while early grades (Grades R-3, age appropriate learners aged 5-8) teaching does occur in home languages in some schools, this is not the case in all schools, and national policy switches all teaching to English in the overwhelming majority of schools from Grade 4 onwards in primary schools. For most teachers in the system, teaching and learning thus occur in teachers' second or third language. This context provides a further rationale for emphasising the discursive base within primary mathematics teacher education.

Data sources

In this presentation, I present a small slice of data illustrating shifts that suggest that emphasis on representation and explanation can be simultaneously productive for both mathematical learning and teaching. The data presented are drawn from teachers' responses (n = 39) to one item on the pre- and (repeat) post-test set in 2013. The item in focus involved whole number scaling up of quantity in the context of petrol prices in South Africa:

1 litre of petrol costs R10.75

Provide a method AND an explanation for working out the costs of:

a) 3 litres of petrol

Teachers' responses were coded for a) correct numerical answer across the pre- and post-tests, and b) for expansion in terms of MDI, in relation to representation and explanation. Expansion was considered in terms of several dimensions, all of which have backing from a range of literature bases, including Ball, Thames and Phelps (2008) notion of 'unpacking', writing in systemic functional linguistics pointing to cohesive ties emanating from elaborations within and across semiotic modes (O'Halloran, 2000) and Duval's (2006) noting that 'conversion' moves between semiotic registers tend to be harder within mathematical learning than 'treatment' transformations within registers.

Findings and analysis

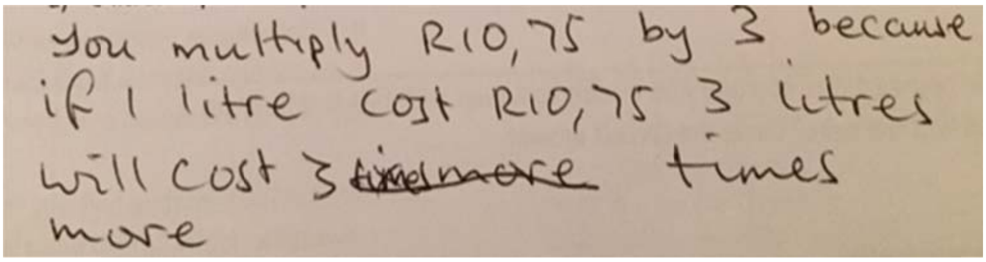
Correct numerical answers were given on 17 of the 39 pre-test scripts for the stated problem. In the post-test, this figure had increased to 31 correct answers across the 39 scripts. While this is important from the perspective of fundamental content knowledge for primary mathematics, a robust body of evidence points to the need to go beyond doing mathematics for oneself for good quality mathematics teaching. Thus, the second analysis looked at representations and explanations provided within the production of answers. Of interest here was the fact that 27 of the 39 post-test responses indicated expansions in terms of representations and/or explanations in comparison to individuals' pre-test responses. In the response excerpts below, I discuss some of these elaborations noting that some expansions do relate to representations as pedagogical objects aimed at supporting the learning of others, while other teachers have used representations as tools to develop their own understandings of the underlying proportionality idea in this problem.

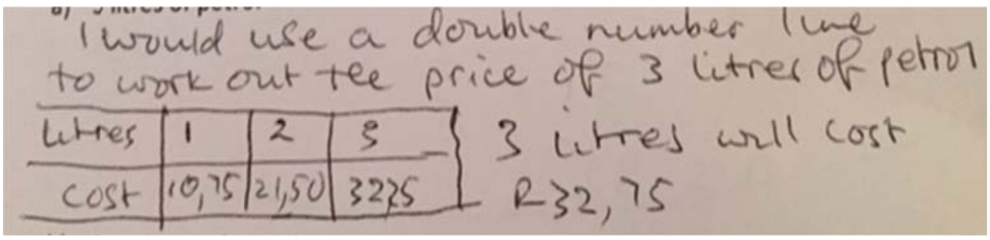
The first two examples of expansion presented below are drawn from teachers who produced correct numerical answers in the pre- and post-tests, suggesting that their expansions of use of representations were pedagogically, rather than mathematically motivated. The next two examples are drawn from teachers who moved from giving no answer or an incorrect answer in the pre-test to a correct answer in the post-test, with representational shifts figuring within this move. These two examples show representations being used as tools for enhancing their mathematical problem-solving. In all instances, pre-test responses are provided before post-test responses.

Category 1: Correct answers in pre- and post-tests

Below, responses from two teachers in this category are presented and discussed.

Teacher 1

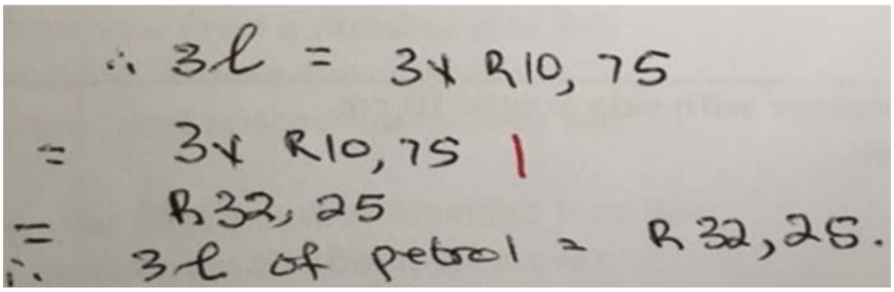

You multiply R10,75 by 3 because if 1 litre cost R10,75 3 litres will cost 3 times more

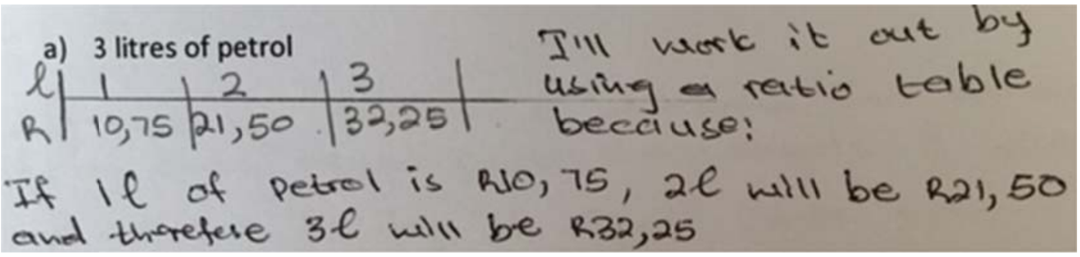

I would use a double number line to work out tee price of 3 litres of petrol

Litres	1	2	3
cost	10,75	21,50	32,25

3 litres will cost R32,75

Teacher 2

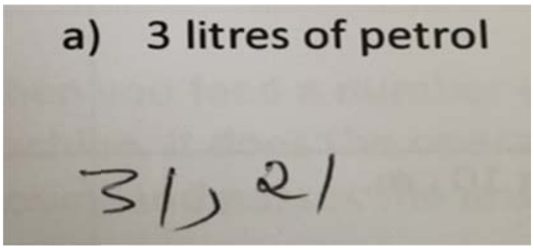


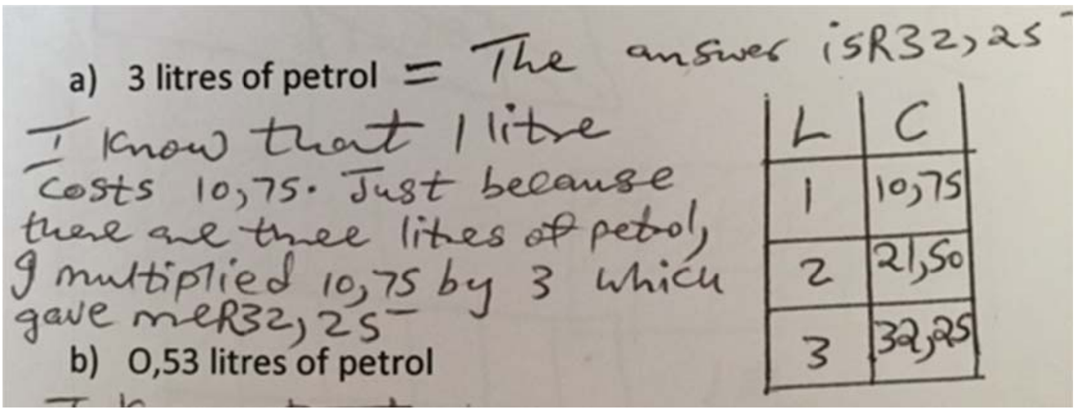


Both teachers' responses here indicate awareness of how to calculate the answer, but the post-test response makes more elements of the process overtly visible – for example, the scaling up moves for 2 litres and then 3 litres, with a supporting narrative attached. In a context of low learner performance, awareness of these kinds of 'decompressions' (Ball, Thames and Phelps, 2008) is likely to be critical. There is also the 'naming' of tools for use within these kinds of problems: the 'double number line' and the 'ratio table', which have become an object of their pedagogic attention while simultaneously pointing to the provision of a broader range of explicit representational problem-solving tools within their teaching.

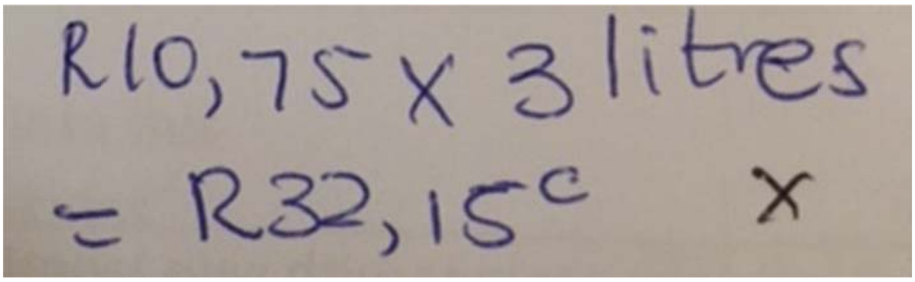
Category 2: Moving from incorrect or no numerical answer to correct answers

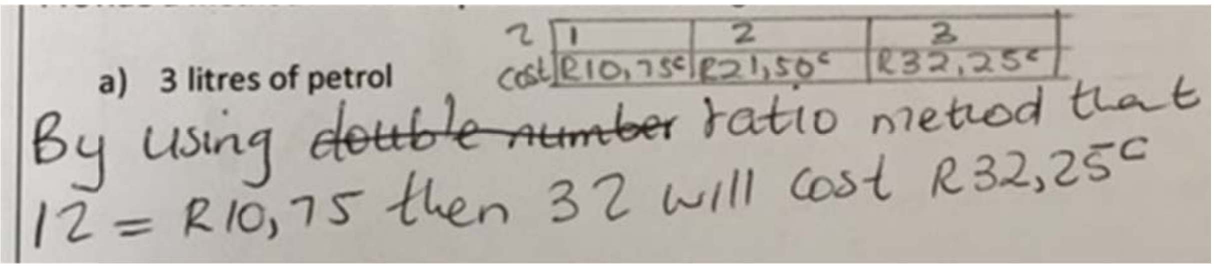
Teacher 1





Teacher 2





Here, an expanded representational repertoire appears to have aided production of correct numerical answers, while at the same time, adding to a skill base that has been noted as important for teaching. Ball et al's (2008) identification of 'selecting representations for particular purposes' (p. 400) as one component of specialised content knowledge provides a potentially useful way of thinking about how teachers whose pre-test responses showed correct answers used expanded representations to unpack ideas. At the same time, for teachers showing moves to getting this item correct in the post-test, an expanded representational repertoire had certainly provided tools that supported extensions in their common content knowledge. Whether this use of expanded representations was actually at the level of specialized content knowledge is not ascertainable from this data set.

Some teachers' responses showed expansions at the level of explanatory narrative only, while others included the use of number line representations of scaling up. Of interest here, is the similarity of responses across the first and second categories, suggesting that attention to representational competence can provide a bridge that allows for concurrent attention to teachers' learning of mathematics and their teaching of mathematics.

Concluding comments

The last point made above is important, and particularly so in the South African context where the need to attend to primary teachers' mathematical knowledge and primary mathematics teaching are both urgent concerns. Adler and Davis (2006) have noted that teacher education mathematics content courses in South Africa have tended to privilege 'compressed' mathematical working, and further, that courses have tended to either foreground mathematics or teaching within their working. The overview findings of gains at the level of mathematical performance in the setting of expansions in representational competence, suggest that an approach based on emphasising representations and explanations may be productive for bringing these two endeavours together.

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CONCEPTUAL MODEL-BASED PROBLEM SOLVING

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Abstract

This paper will introduce a Conceptual Model-based Problem Solving (COMPS) approach that aims to promote elementary students' generalised word problem-solving skills. With the emphasis on algebraic representation of mathematical relations in cohesive mathematical models, the COMPS program makes connections among mathematical ideas; it offers elementary school teachers a way to bridge the gap between algebraic and arithmetic teaching and learning. The COMPS program may be especially helpful for students with learning disabilities/difficulties who are likely to experience disadvantages in working memory and information organisation. Findings from an empirical research study are presented and implications for elementary mathematics education are discussed.

Key words: conceptual model-based problem solving, elementary mathematics, students with learning disabilities/difficulties, whole number arithmetic, word problem solving.

Introduction

About five to ten percent of school-age children have been identified as having mathematics disabilities (Fuchs, Fuchs and Hollenbeck, 2007) and students whose math performance was ranked at or below the 35 percentile are often considered at risk for learning disabilities or having learning difficulties in mathematics (LDM) (Bryant et al., 2011). Students with LDM lag behind their peers very early on in their educational trajectory. According to the most recent National Assessment of Educational Progress (NAEP) results, from 2011 to 2013 mathematics achievement score gains were seen only for higher performing students at the 75th and 90th percentiles, but there were no significant changes over the same period for lower performing students at the 10th and 25th percentiles (NAEP, 2013).

In conjunction with this lack of growth in mathematics learning among students with disabilities, expectations for all students, including those with LDM, have been elevated in today's educational climate. In particular, the Common Core State Standards for Mathematics (CCSSM, Common Core State Standards Initiative [CCSSI], 2012) emphasise conceptual understanding of ideas and the connections between mathematical ideas. The CCSSM also emphasises mathematical modeling and algebra readiness throughout elementary mathematics. There is a need to explore potential intervention support that addresses this new emphasis to facilitate *all* students' access to higher-order thinking and meeting the Common Core Standards.

Traditional instructional practice

One of the distinctive features of traditional instruction (TDI) is its focus on the *choice of operation* when dealing with problem solving. To determine the choice of operation, it is not uncommon to see that students rely on the "key word" strategy (e.g., the word "altogether" in the problem would cue an operation of addition) for making a decision on the choice of operation. The key word strategy, which has been the practice in the United States for generations (Sowder, 1988), directs students' attention toward isolated "cue" words in the problem. The key word strategy might be a "quick and dirty" way to "fix" word problem solving; however, it is at odds with contemporary approaches to word problem solving that stress conceptual understanding of mathematical relations in a problem *before* attempting to solve it with an operation (Jonassen, 2003). In particular, the key word strategy does not orient students' attention to a problem's underlying mathematical structure and relations or encourage mathematical modeling that is emphasised by the CCSSM. Further, applying the key word strategy might contribute to students being prone to "reversal operation" errors when encountering the so-called "inconsistent language" problems (e.g., "Tara solved 21 problems. She solved 3 times as many problems as Pat. How many problems did Pat solve?"), where students might mistakenly multiply, when they need to *divide*, for solution due to the key word "times" (Xin, 2007). Other strategies commonly used in teaching word problem solving include "draw a picture," "guess and check," etc. It should be noted that when the numbers in the problem are small, it might be manageable to correctly solve the problem using the "guess and check" or "draw a picture" strategy. However, when the numbers become large, such problem-solving processes may become cumbersome or inefficient.

COMPS Program

The conceptual model-based problem solving (COMPS) approach (Xin, 2012) focuses on prealgebraic conceptualisation of mathematical relations in model equations. The COMPS program represents a pedagogical shift from traditional problem-solving instruction that focuses on the *choice of operation* for a solution, to a *mathematical model-based problem-solving* approach that emphasises an understanding and representation of mathematical relations in mathematical model equations. Findings from empirical studies involving elementary students with LDM (e.g., Xin, Wiles and Lin, 2008; Xin and Zhang, 2009; Xin et al., 2011) indicate that COMPS has shown promise in improving students' problem-solving skills as well as pre-algebra concept and skills. The objective of this paper is to present one of the empirical studies (Xin et al., 2008) that support the effects of the COMPS program on additive and multiplicative word problem-solving performance of elementary students with LDM.

Materials and Methods

In Xin et al. (2008) study, I designed a set of *word problem (WP) story grammar* self-questioning prompts to facilitate conceptual understanding of mathematical relations in word problems and represent such relations in mathematical model equations. Fig. 1a and 1b present conceptual models for additive word problems and Fig. 2 presents conceptual models for multiplicative word problems. The *word problem (WP) story grammar* facilitates comprehension of the story problems. Students' mapping of the numbers onto the diagram is on the basis of their conceptual understanding of the mathematical relations in the problem, rather than a mindless exercise or random procedure.

Although *story grammar* has been substantially researched in reading comprehension (Boulineau et al., 2004), *WP story grammar* has never been explored in math word-problem understanding and solving. By definition, *story grammar* in reading comprehension literature refers to a typical structure shared by most narrative stories. Similarly, a word problem story structure that is common across a group of word problem situations can be defined as *WP story grammar*. Borrowing the concept of story grammar from reading comprehension literature, I developed a set of *WP story grammar* questions (see Fig. 1 and 2) to guide students' extracting of mathematical relations from the problem and symbolic representation of the relations for solution.

Part-Part-Whole (PPW)

A *PPW* problem describes multiple parts that make up the whole

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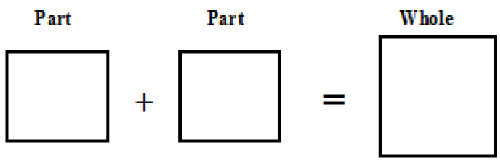

PPW WP Story Grammar Questions

☐ **Which sentence (or question) tells about the "whole" or "combined" amount?** Write that quantity in the big box on one side of the equation by itself.

☐ **Which sentence (or question) tells about one of the parts that makes up the whole?** Write that quantity in the first small box on the other side of the equation.

☐ **Which sentence (or question) tells about the other part that makes up the whole?** Write that quantity in the 2nd small box (next to the first small box).

Fig. 1a: Conceptual Model of *Part-Part-Whole* Word Problems (Xin, 2012, p. 47)

Additive Compare (AC)

An AC problem describes one quantity as "more" or "less" than the other quantity

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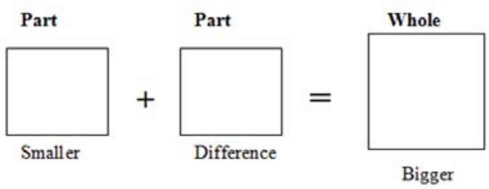


AC WP Story Grammar Questions

Which sentence (or question) describes one quantity as "more" or "less" than the other? Write the *difference* amount in the diagram.

Who has more, or which quantity is the bigger one?

Who has less, or which quantity is the smaller one? Name the bigger box and smaller box.

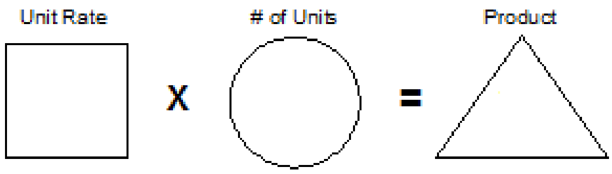
Which sentence (or question) tells about the bigger quantity? Write that quantity in the bigger box on one side of the equation by itself.

Which sentence (or question) tells about the smaller quantity? Write that quantity in the smaller box next to the *difference* amount.

Fig. 1b: Conceptual Model of *Additive Compare* Word Problems (Xin, 2012, p. 67)

Equal Group (EG)

An *EG* problem describes number of equal sets or units



EG WP Story Grammar Questions

Which sentence or question tells about a Unit Rate (# of items in each unit)? Find the unit rate and write it in the Unit Rate box.

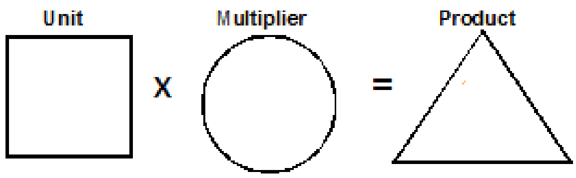
Which sentence or question tells about the # of Units or sets (i.e., quantity)? Write that quantity in the circle next to the unit rate.

Which sentence or question tells about the Total (# of items) or ending product? Write that number in the triangle on the other side of the equation.

Fig. 2a: Conceptual Model of Equal Groups Word Problems (Xin, 2012, p. 105)

Multiplicative Compare (MC)

A MC problem describes one quantity as a multiple or part of the other quantity



MC WP Story Grammar Questions

Which sentence (or question) describes one quantity as a multiple or part of the other? Detect the two things (people) being compared and who (the compared) is compared to whom (the referent UNIT). Name "whom" and "who" in the diagram. Fill in the relation (e.g., "2 times") in the circle.

□ **What is the referent UNIT?** Write that quantity in the referent unit box.

△ **What is the compared quantity or product?** Write that quantity in the triangle on one side of the equation by itself.

Fig. 2b: Conceptual Model of Multiplicative Compare Word Problems (Xin, 2012, p. 123)

Design and Participants

An adapted multiple probe design (Horner and Baer, 1978) across participants was employed to evaluate the functional relationship between the intervention and students' word problem-solving performance. Single-subject research design was chosen because the design provides a methodological approach well suited to the investigation of single cases or groups (Kazdin, 1982). In particular, with the multiple probe design, intervention effects can be demonstrated by introducing the intervention to different participants at different points in time. "If each baseline changed when the intervention is introduced, the effects can be attributed to the intervention rather than to extraneous events" such as history, maturation, testing, etc. (Kazdin, 1982, p. 126). Participants were five 4th and 5th grade students with LDM. On the basis of students pre-tests' performance, three students were identified as needing intervention in additive word problem solving and were engaged in solving *part-part-whole* (PPW) and *additive compare* (AC) problems; two students were identified as needing intervention in multiplicative word problem solving and were instructed to solve *equal groups* (EG) and *multiplicative compare* (MC) problems.

Intervention Procedure

Participating students received intervention in COMPS three times a week, with each session lasting for approximately 20-35 minutes. Each student received three to six sessions of instruction on PPW or EG, two to three sessions on AC or MC problem instruction, and one to two sessions on solving mixed word problems including both PPW and AC or EG and MC types.

Students were assessed on either an additive problem-solving criterion test, which involved 14 variously constructed addition and subtraction word problems; or a multiplicative problem-solving criterion test, which involved 12 variously constructed multiplication and division problems. Calculators were allowed throughout the study to accommodate participants' skill deficit in calculation.

Results

As for additive word problem solving, during baseline condition (prior to the intervention), each of the three participants performed at an average of 21%, 28%, and 28% correct respectively on the criterion test. Following the intervention, the two students who completed COMPS instruction on additive word problem solving performed 79% correct during post-intervention assessment (a 58% point increase from the baseline performance of 21%

correct) and 86% correct (a 58% point increase from the baseline of 28% correct) respectively.

As for multiplicative word problem solving, during the baseline, each of the two participants performed at an average 3% correct and 0% correct respectively. After the intervention, both participants' performed at 100% correct, which indicates a 97% and 100% point increase, respectively, from the baseline.

As for the effect on prealgebra concept and skills, two pre-algebra probes were used to assess potential improvement of students' performance. The *solve equations* probe required students to find the value of an unknown quantity (i.e., letter *a*) that makes the equation true (e.g., $93 = 79 + a$; $196 = a \times 28$). Positions of the unknown were systematically varied across three terms in the equation (i.e., the augend, addend, and sum; or the multiplicand, multiplier, and product). Six items were included in either the addition/subtraction probe or multiplication/division probe. In addition, the *algebraic model expression* probe was designed to test students' algebraic expression of mathematical relations or ideas. Twelve items (e.g., "Write an expression or equation. Choose a variable for the unknown. Shanti had some stamps. She gave 23 to Penny. Shanti has 71 stamps left") were included in the addition/subtraction probe; five items (e.g., "Antoni has collected 84 autographs. He filled 14 pages in his news autograph album. Each page holds an equal number of autographs. Write an equation with a variable to model this problem") were included in the multiplication/division probe. These items were directly taken from commercially published mathematics textbooks that had been adopted by the participating schools (Maletsky, et al., 2004).

Findings from this study indicated that (a) on the *solve equations* probe, from pre- to post-intervention, the two participants who completed the *additive* problem-solving intervention improved from 33% to 67% correct and 0 to 100% correct, respectively. The two participants who completed the multiplicative problem-solving intervention improved their performance from 0 to 67% correct and from 0 to 100% correct, respectively; (b) on the *algebraic model expression* probe, the participants of this study had no knowledge of what they were asked to do and made no attempts during the pretest. After the intervention, the two participants who completed the additive problem-solving intervention scored 71% and 83% correct, respectively, on the corresponding *algebraic model expression* probe. The two participants who completed the multiplicative problem-solving intervention both scored 100% correct on the corresponding *algebraic model expression* probe.

Discussion and conclusion

The word problems included in this study represent "the most common form of problem solving" (Jonassen, 2003, p. 267) in elementary school mathematics curricula. Learning to solve variations of these word problems is the basis for solving more complex real-world problems (Van de Walle, 2004). It should be noted that the way word problems are classified in the COMPS program is directly linked to the underlying mathematical models. It is different from other classifications that are on the basis of semantic analyses of the story situations (e.g., Cognitively Guided Instruction).

Given the generalised mathematical models for the additive and multiplicative problem structure (see Fig. 1 and 2), a range of arithmetic word problems involving four basic operations can be represented and modeled. In addition, the COMPS (with the assistance of *WP story grammar* in representation) emphasises symbolic or algebraic expressions of mathematical relations in model equations that directly links problem representation to solution; it has the potential to innovatively bridge the gap between learning arithmetic and algebra.

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THE ROLE OF EARLY LANGUAGE ABILITY ON THE MATH SKILLS OF CHINESE CHILDREN

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Abstract

This study investigated the role of early language ability in the development of math skills among Chinese kindergarteners. The participants were 2012 6-year-old children from 60 kindergartens in South China. They were tested on both informal and formal math skills. The informal math testing focused on basic number concepts such as object counting, while the formal math testing involved numerical calculations such as addition and subtraction. The children's language and non-verbal reasoning abilities were also assessed. Correlational analysis showed that the children's language ability was more strongly associated with informal than formal math skills. Hierarchical regression analyses revealed that the children's language ability could uniquely predict both informal and formal math skills, with age, gender, and non-verbal intelligence statistically controlled. However, language ability predicted more variance of informal math skills than of formal math skills. The findings indicate that children's language ability may have different roles on the development of formal and informal math skills.

Key words: formal math, informal math, language ability

Introduction

Increasing attention is being paid to the development of academic skills of children at an early age, including knowledge of letters and numbers, understanding of magnitude, and counting, because these skills form the cornerstones of children's future learning (Geary, Hoard and Hamsun, 1999; Miller et al., 2013). To date, researchers have consistently found that children's early academic skills are not only predictive of their academic achievement in later school years (Duncan et al., 2007; Early et al., 2007) but are also important for the children's adaptation to school (Blair, 2003). Of these academic skills, mathematics and language are the most important domains. Both skills have been found to be associated with and predictive of each other, concurrently and longitudinally (Duncan et al., 2007; McClelland et al., 2007; Purpura et al., 2011). A more important and intriguing finding of Duncan et al. (2007) was that children's math skills in kindergarten could better predict their language development longitudinally than could their language ability itself. As math development is stable over time, undeveloped early math skills could hinder children from learning more advanced math and other skills. It has been found that children who perform poorly in kindergartens usually continue to fall behind their peers (Aunola et al., 2004). It is thus important to theoretically and practically explore the cognitive factors that may influence children's early math development, in particular language ability.

Although substantial research has been carried out in this field, most focuses on the math skills of children speaking alphabetic languages. To the best of our knowledge, few or no studies have explored the predictive factors in the early math development of Chinese children. Studying Chinese children is important for the following reasons. First, Chinese culture is different from Western culture in that Chinese society places high value on students' academic achievements (Ho, 1986; Tseng and Wu, 1985). Second, many Chinese preschoolers devote more effort to math and reading because of the expectations of their parents and of society. As a result, Chinese students have been found to perform better academically than their age-matched American counterparts, especially in math tests (Chen and Stevenson, 1995; Stevenson, et al., 1990). For example, the excellent performance of Chinese students in the OECD's most recent PISA math literacy test has attracted much attention from researchers and educators. Finally, the Chinese language is different from alphabetic languages such as English in many ways. The relationship between language and math skill may thus be different from those found for children speaking alphabetic languages.

This study aimed to fill the abovementioned research gaps by exploring the influence of Chinese children's early language ability on their math skills.

The current study

We investigated the role of early language ability on the math skills of Chinese children at kindergarten. Following Purpura and Ganley (2014), we conceptualised early math as two components: informal math and formal math. Informal math is related to numerical concepts such as comparison of magnitudes and counting of objects. Formal math involves numerical calculations or whole-number arithmetic such as addition and subtraction. As informal math is more closely related to the concept-based knowledge of whole numbers while formal math involves more manipulation/calculation of numbers, we expected language ability to be more highly correlated with informal than with formal math.

Materials and Methods

Participants

The participants were 2012 K3 children (958 girls and 1054 boys, mean age = 6.65 years, SD = .14) recruited from 60 kindergartens in three cities in the Guangdong province of China. All of the children were assessed for math skills, language ability and non-verbal intelligence.

Measures

Math

The following eight tasks were designed to test early math skills. The first six tested informal math skill and the last two tapped formal math skill.

Object counting. In each trial of the task, the children were instructed to count the animals presented within a circle on the test paper and write the number down.

Forward counting. In each trial, three numbers were presented in ascending order with one number being absent. The children were asked to write down the missing number.

Comparison of numerical magnitudes. In each trial, two numbers were presented visually and the children were asked to judge which number was the larger.

Backward counting. This task was similar to the forward counting task, except that all of the numbers were presented in descending order.

Pattern. In each trial, a series of regular figures (e.g. star, square) were presented in an open-ended row and following a certain rule or pattern. The children were asked to discover the pattern and choose one of several figures to best fit the end position in the row.

Missing number. Three numbers were presented in ascending order with one of them missing. The children were instructed to write down the missing number.

Addition. The children were asked to do some simple arithmetic additions of whole numbers and write down the answers.

Subtraction. The children were asked to do some simple arithmetic subtractions of whole numbers and write down the answers.

Language

Language ability was tested from both the receptive and the productive perspective, through listening comprehension and Chinese character dictation tasks respectively.

Listening comprehension. In each trial, a simple sentence was aurally presented and the children were asked to choose from five pictures the one that best depicted the sentence.

Chinese character dictation. In each trial of the task, the children were asked to write down a certain character in a multi-character word.

Non-verbal Intelligence

Raven's Progressive Matrix (Set A and Set B) was used to measure the children's non-verbal reasoning ability (Raven et al., 1994).

Procedure

The children completed all of the tasks in their kindergarten classrooms under the instruction of experimenters who were master students at the University of

Macau. The testing was carried out in three sessions, with each session lasting about 30 minutes.

Results

Preliminary analysis

Tab. 1 shows the correlations for all of the tasks. The averaged scores of the first six and the last two math tasks were used to index informal and formal math, respectively. The addition of listening comprehension and character dictation was used as an index of overall language ability.

As shown in Tab. 1, overall language ability was significantly associated with both informal and formal math at a high level (r = .75 and .67 for informal and formal math, respectively). Further statistical analysis revealed a significant difference between the two correlations, $t = 8.33$, $p < 0.001$, suggesting that language is more closely related to informal than to formal math.

Task	1	2	3	4	5	6	7	8	9	10	11	12
OC	-											
2. FC	0.39	-										
3. MC	0.25	0.53	-									
4. BC	.34	.67	.54	-								
5. P	.29	.48	.51	.50	-							
6. MN	.38	.66	.56	.68	.51	-						
7. A	.37	.60	.54	.62	.47	.64	-					
8. S	.31	.55	.49	.60	.44	.62	.69	-				
9. IMS	.46	.87	.71	.90	.69	.82	.71	.67	-			
10. FMS	.37	.63	.56	.67	.50	.69	.91	.92	.75	-		
11. LC	.30	.52	.56	.55	.57	.53	.53	.50	.66	.56	-	
12. D	.39	.58	.44	.53	.43	.57	.55	.55	.64	.59	.46	-
13. OLA	.41	.64	.57	.63	.57	.64	.62	.61	.75	.67	.79	.89

Legend: OC- object counting; FC – forward counting; MC – magnitude comparison; BC – backward counting; P – pattern; MN – missing number; A – addition; S – subtraction; IMS – informal math skills; FMS – formal math skills; LC – listening comprehension; D – dictation; OLA – overall language ability

Note: N = 2012. All of the correlations were significant at p < 0.001.

Tab. 1: Correlations among variables

Hierarchical regression analysis

To test the unique contribution of overall language to informal and formal math skills, hierarchical regression analysis were conducted. The results are shown in Tab. 2 and Tab. 3 for informal and formal math skills, respectively. Gender, age and non-verbal intelligence were entered in step 1 as control variables and the overall language score was entered in step 2. Tab. 2 shows that overall language ability significantly predicted a 31% unique variance of informal math skill ($\beta = 0.66$, $t = 37.69$, $p < .001$) when gender, age and non-verbal intelligence were controlled for, with the model as a whole accounting for 61% variance. A similar analysis was conducted for formal math. As shown in Tab. 3, language ability uniquely accounted for a 26% variance in formal math skill when the same variables were controlled ($\beta = 0.61$, $t = 29.80$, $p < .001$), with the model as a whole accounting for a 47% variance in formal math. In other words, language ability is more predictive of informal math skill than of formal math skill.

Variable	*B*	*SE (B)*	*β*	*ΔR2*
Step 1				0.30***
Gender	**-.14**	**.45**	**.01**	
Age	-.02	.04	-.01	
Raven test	**1.25**	**.04**	**.55*****	
Step 2				0.31***
Gender	1.45	.34	.06***	
Age	-.03	.03	-.01	
Raven test	.45	.04	.20***	
Language ability	0.66	0.02	.66***	

*Note: ***p<.001*

Tab. 2: Hierarchical regression predicting informal math skill from overall language ability

Variable	*B*	*SE (B)*	*β*	*ΔR2*
Step 1				0.21***
Gender	**-.01**	**.13**	**-.002**	
Age	-.004	.01	-.01	
Raven test	**.28**	**.01**	**.41*****	
Step 2				0.26***
Gender	.37	.11	.06***	
Age	-.01	.01	-.01	
Raven test	.08	.01	.14***	
Language ability	0.16	0.01	.61***	

*Note: ***p<.001*

Tab. 3: Hierarchical regression predicting formal math skill from overall language ability

Discussion and conclusion

The study generated two main findings. First, language is important for developing math skills in Chinese children. Specifically, language ability is able to significantly predict both formal and informal math skills. Second, language ability is linked differently to different math skills, being more closely associated with informal math than with formal math. Future studies should explore the mechanisms underlying these findings, which may lead to effective interventions for improving children's math skills through enhancing their language ability.

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THE ICMI STUDY 23 PANELS

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Since the meeting of the International Program Committee in Berlin (January 2014) the issue of panels was addressed: How many panels? Which themes? Which participants? The IPC agreed already in that meeting on three themes that were timely to be addressed for different reasons.

Traditions in whole number arithmetic was considered important. The ICMI Study n. 10 on *History in Mathematics education* contained some examples of the ways of representing whole numbers in different ages and regions, but about 15 years had passed since the publication of the Discussion Document and updating was timely. In most cases secondary school subjects only were addressed and, moreover, the discussion had not yet considered the recent development of cultural anthropology, cultural psychology and neuroscience. The comparison / contrast between Eastern and Western traditions was addressed by the ICMI study 13 on *Mathematics education in different cultural traditions: a comparative study of East Asia and the West,* but we had different ambitions and aimed at involving also regions like Africa and Central America, with the limited focus on whole number arithmetic.

Teacher education too was considered very important, as the deepest and most significant findings about whole number arithmetic in the literature on mathematics education may be thwarted by the scarce preparation of teachers. We regretted that the volume of the ICMI study 15 on *The professional education and development of teachers of mathematics* did not pay much attention to primary school teachers: as we wrote in the Discussion Documents in many countries they are generalists and, in the general opinion, because of the weak nature of their training/expertise, are not considered as true professionals, as if every layperson could teach children the basic arithmetic.

Last but not least, special education. This issue might have deserved a study on its own, to address, for instance, specific teaching approaches for students who are sensually impaired (e.g. blind students and deaf students). The literature about specific learning disorders (e. g. dyscalculia) is growing, but the presence of mathematics educators in the field is still limited, at least at the research level. On the contrary, it is the mathematics teacher who first comes in touch with learning difficulties which sometimes are mistaken for learning disorders. Collecting and analysing supportive learning environments which could reduce learning problems might help also to reduce the diagnoses of apparent learning disorders that are, on the contrary, the effect of not effective teaching.

The rationale for the three panels has been explained above. What is interesting to highlight is that the whole IPC felt strongly involved in the panels, offering their expertise to start the discussion with the audience. The three chairs (Fedinando Arzarello, Jarmila Novotná, Lieven Verschaffel) worked in strong cooperation with the other members of the IPC. Some plenary speakers were willing to accept to serve as discussants. Some other panellists were chosen among the participants who have submitted papers potentially addressing the above issues.

In this chapter only the introductory documents are offered to the participants of the conference, in order to help them to focus on the chosen issues. It is our hope that the panels may develop in full chapters in the volume, exploiting also other suggestions that might come from the audience.

PANEL ON TRADITION

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Abstract

The main topics to be discussed in the panel are introduced with some bibliographic references and pointing out consequent questions to answer. They concern tradition with respect to the following issues: verbal and non verbal representations of numbers; numbers and artefacts for arithmetic; the role of nowadays technological devices when emulating traditional abaci or allowing a direct interaction between fingers and the screen of multi touch devices in counting activities.

Key words: artefacts, culture, linguistics, neuroscience, semiotic representations, tradition

Introduction

Encyclopaedia Britannica defines tradition as:

> "a : an inherited, established, or customary pattern of thought, action, or behaviour (as a religious practice or a social custom);
> b : a belief or story or a body of beliefs or stories relating to the past that are commonly accepted as historical though not verifiable".

Moreover it points out that tradition concerns the "handing down of information, beliefs, and customs by word of mouth or by example from one generation to another without written instruction" and represents a "cultural continuity in social attitudes, customs, and institutions".

It is apparent from this definition that the way whole numbers are spoken, written, thought, taught, and learnt sums up in what we can address as (part of) tradition. Hence researchers and teachers must consider them under many different instances and perspectives, which entail tradition in its multifaceted aspects: cultural, epistemological, psychological, and neurological.

Some of these components have a more or less strong "local" connotation, linked as they are to the different cultures and traditions of peoples. Others are more general and seem to have universal traits. Hence the so-called Near-Universal Conventional mathematics (NUC: the definition is by Bill Barton, 2008, p. 10) may be in more or less deep conflicts with such local instances. This possible contrast can indeed represent a main problem for teachers: a reasonable learning trajectory for whole numbers (and possibly for the whole mathematics) from the one side cannot avoid their traditional roots, but from the other side its main goal must address the NUC.

This general background should shape the discussions in the panel: its aims are to scientifically deepen the analysis of some of these different roots, considering old and new findings from research and practice, and making explicit the main consequences for possible concrete didactical trajectories.

I will sketch here some items to focus in the panel, with some questions I think relevant. The members of the panel are asked to possibly extend this list and to deepen the analysis of its items.

Different semiotic representations of numbers

Whole numbers representations across space and time encompass a large variety of semiotic systems, including language of course, but not only it.

Numbers and words

The way numbers are said in different languages raise a complex issue, faced in many old and new researches: from the pioneering book of Menninger (1934) to more recent works (Zaslavsky, 1973; Ifrah, 1981), all these show what Bishop called the *mathematical enculturation* (Bishop, 1991) of numbers: see also Asher (1991), Selin and D'Ambrosio (2000), and Barwell et al. (2015).

From the one side, the way whole numbers are spoken and written is one of their most important features, which can reveal their different cultural aspects, and this issue must be considered in teaching early arithmetic. I list here some well know examples.

In many languages, numbers from 11 to 20 are spelled according to different rules and words than the next ones, e.g. from 20 to 30, and this may keep hidden the mathematical structure of those numbers [12 Vs/ "twelve"; 14 Vs "quattordici" (~ four-ten), but 17 Vs "diciassette (~ten-seven)"]; French numbers from 60 to 99 are spelled according to an old base 20 root, typical of some Celtic languages; for example to say 97 a French girl/boy must learn to say "quatre-vingt-dix-sept", that is "four (times) – twenty – ten - seven", while a German child must learn "Siebenundneunzig" (seven and ninety), an Italian child must say "novantasette" (ninety-seven), and so on. On the contrary, in Chinese language the grammar of numbers is more regular, and this may be an advantage in learning numbers: an Italian teacher, Bruna Villa, has made a nice learning design for her children in grade one to teach them how to enter into the machinery of whole numbers. She based her teaching on what, following a proposal of Brissiaud, Clerc and Ouzoulias (2002), she called the method of the "small Chinese dragon" (Villa, 2006), where numbers are said with a uniform Chinese-like structure (e.g. 11 is "ten-one" and not "undici"; 21 is two (times)-ten- one and not "ventuno"), and passing to the Italian system only afterwards. In such a way she has been able to shorten times necessary to kids for mastering whole numbers from 1 to 100 (in Italian words and standard arithmetic representations), and doing the first arithmetic with them.

A further fascinating example, which shows strong differences between the way numbers are spelled in a language and their mathematical structure is illustrated in Barton (2008), where he discusses the way numbers are said in Maori language. Before the contacts with Europeans, numbers in Maori language were like verbs: they expressed actions, e.g. saying that "there were two persons" was something similar to say that "those persons two-ed". This difference was even more dramatic when the negation was involved: "To negate a verb in Maori the word kaore is used. […] Unlike English, where negating both verbs and adjectives requires the word 'not', in Maori, to negate an adjective a different word is used, ehara". Hence, when this verbal feature of Maori numbers was ignored, the mathematics vocabulary process, translated from English, acted against the original ethos of the Maori language.

Other researches point out possible interferences of the ways numbers are used in everyday language with respect to the mathematical meaning of numbers. In a nice book, unfortunately available only in Italian, a researcher in linguistics, Carla Bazzanella, points out that numbers in everyday language can assume also an indeterminate, more or less vague meaning and not the canonical cardinal denotation: for example in Italian we can say "that tie may have costed 100 euro" ("quella cravatta sarà costata 100 euro") to express an approximate value, or "would you like to eat 2 spaghetti?" ("vuoi mangiare 2 spaghetti?") to mean "would you like to eat some spaghetti?" (Bazzanella, 2011). I will face this issue from another point of view below.

Non-verbal representations of numbers

From the other side, many researches have pointed out the way numbers are represented in different non linguistic ways in different cultures all over the world (Gheverghese, 2011), e.g. using parts of the body (typically digits, but not only: see Saxe, 2014) or spatial arrangements in complex arithmetical calculations when number words are lacking.

Many researches have pointed out some typical steps in the way children progress in building up numbers intertwining language and gestures, e.g. using digits, and how they use them for counting and for adding. For example, Vergnaud uses an adaptation of the Piagetian notion of schème [he defines it in this way: "A scheme is the invariant organization of behaviour for a certain class of situations" (Vergnaud, 1997, p. 12)]. He talks about how, when a child uses a counting scheme, there may be a cognitive shift, related to gestures, that occurs:

> Another characteristic of schemes concerns the way cardinalisation is marked in speech: the last number pronounced represents the cardinality of the whole collection and not just the last object. This marking with speech comprises not only the repetition (1, 2, 3, 4, 5,.. 5) but also the accentuation (1, 2 3, 4…5). One can clearly see from this example that language is closely associated with the functioning of a scheme, and that it plays a role in producing perceptual-motor gestures whose organisation depends on both the nature and arrangement of the objects, and the problem to solve; associating an invariant number to a given collection. (Vergnaud, 1991, p. 80; my translation from French)

Basing more on neurological stances, a similar multi-step process is pointed out by Butterworth, Reeve and Reynolds for addition strategies:

> "Where two numbers or two disjoint sets, say 3 and 5, are to be added together, in the earliest stage the learner counts all members of the union of the two sets – that is, will count 1, 2, 3, and continue 4, 5, 6, 7, 8, keeping the number of the

> second set in mind. In a later stage, the learner will 'count-on' from the number of the first set, starting with 3 and counting just 4, 5, 6, 7, 8. At a still later stage, the child will count on from the larger of the two numbers, now starting at 5, and counting just 6, 7, 8. It is probably at this stage that addition facts are laid down in long term memory". (Butterworth, Reeve and Reynolds, 2011, p. 631)

Some recent researches both from the side of Ethno-mathematics and from that of Neurology seem to introduce a fresh and wider point of view about the issue of language and its role as a resource for arithmetic activities: for a survey from the point of view of neuroscience see Dehaene and Brannon (2011). A very intriguing example is given in the research of Butterworth, Reeve and Reynolds (2011), who point out how the word counting strategies are not the only ones that people can use for developing arithmetic competencies:

> "We tested speakers of Warlpiri and Anindilyakwa aged between 4 and 7 years old at two remote sites in the Northern Territory of Australia. These children used spatial strategies extensively, and were significantly more accurate when they did so. English-speaking children used spatial strategies very infrequently, but relied an enumeration strategy supported by counting words to do the addition task. The main spatial strategy exploited the known visual memory strengths of Indigenous Australians, and involved matching the spatial pattern of the augend set and the addend. These findings suggest that counting words, far from being necessary for exact arithmetic, offer one strategy among others. They also suggest that spatial models for number do not need to be one-dimensional vectors, as in a mental number line, but can be at least two dimensional." (*op. cit.,* p. 630)

Further researches from the side of neurology support such claims also for wider aspects of mathematics. For example a research of Varley et al. (2005) shows that:

> "once these resources [mathematical ones] are in place, mathematics can be sustained without the grammatical and lexical resources of the language faculty. As in the case of the relation between grammar and performance on "theory-of-mind" reasoning tasks (42), grammar may thus be seen as a co-opted system that can support the expression of mathematical reasoning, but the possession of grammar neither guarantees nor jeopardizes successful performance on calculation problems." (*op. cit.,* p. 470)

As well, Monti et al. point out that:

> "Our findings indicate that processing the syntax of language elicits the known substrate of linguistic competence, whereas algebraic operations recruit bilateral parietal brain regions previously implicated in the representation of magnitude. This double dissociation argues against the view that language provides the structure of thought across all cognitive domains." (Monti, Parsons and Osherson, 2012, p. 914)

Finally, some researches point out that the sense of numbers does not only base on a discrete approach, where it is crucial the one-one correspondence between external symbols and numerical representations, but also on an approximate number system, where for example estimation of numbers of two sets, when subitizing is not possible, are based on the ratio between their cardinality and not on their difference: see Gallistel and Gelman (2000). According to such researches, this continuous, analogic system during our evolution was encoded in our brains previously to the discrete one, and it is still active in us.

All such findings introduce a fresh perspective to the issue of tradition and of language, and of their role as a resource for arithmetic activities: an intriguing issue to be discussed in the panel. In particular some major questions are:

- How teachers can concretely base their task design for arithmetic on the linguistic and cultural roots of numbers?

- Is the embodied traditional approach to arithmetic, to be modified/extended by the findings of neurological researches on numbers?

Representing numbers in artefacts

Within the stream of semiotic representations of numbers, a specific analysis concerns the calculation tools, typically (but not only) abaci, which incorporate both specific representations of numbers and practices for doing arithmetical operations with them (for a survey see: Ifrah, 2001). They deeply intertwine with language and can be part of interesting didactical designs in primary school. Many teachers sometimes accompany them with modern technology introducing in the classroom both the concrete artefact and its simulation in a virtual technological environment. For a first example, N. Sinclair and her collaborators are integrating such embodied and traditional representations within tablets (Sinclair and Metzuyanim, 2014). They base on the hypothesis that the touch-screen devices enable an intuitive, embodied interface for arithmetic, which is suitable for young learners, allowing them to use their fingers and gestures to explore mathematics ideas and express mathematical understandings. For another example, Maschietto and Soury-Lavergne (this study, 2015) are using the old Pascal machine both in a concrete and in a virtual way to approach arithmetic in primary school.

These types of researches pose interesting questions for our panel:

- How the traditional instances are embodied in nowadays technology?
- Does the possible integration of cultural roots within the technological environments allow to bridge the gap between the “old fashioned” tradition and the NUC?

The panel consists of four scholars representing different parts of the world, complemented with one discussant. All of them have rich experience with mathematics teacher education in their countries/regions.

- Nadia Azrou is a mathematics university teacher at the university of Yahia Fares in Medea, Algeria and a PhD student. Her thesis subject is about proof at the university level. She is interested in teaching and learning mathematics at the undergraduate level, but also at the primary level in a multicultural context like the case of Algeria. As a YERME Network Group member, she is at the service of young researchers in mathematics education for providing information.

- Maria G. Bartolini Bussi is a Full Professor in Mathematics Education at the University of Modena and Reggio Emilia (Italy). She is the director of the University program for pre-primary and primary teacher education. She was a member of the ICMI Exectutive Committee (2007/2012). She is now the co-chair of the ICMI Study 23.
- Sarah Gonzalez is a Full Professor in Mathematics Education at Pontificia Universidad Católica Madre y Maestra (Dominican Republic). She is Vice Rector for Research and Innovation at the same university and has coordinated Professional Development Programs in Mathematics for Teachers, has worked in the development on the National Curriculum in Mathematics for Elementary and Middle school in Dominican Republic and is the Caribbean representative in the Interamerican Committee for Mathematics Education (IACME).
- Xuhua Sun is an Assistant Professor in Education at University of Macau, China, specialising in mathematics education from early childhood to secondary level. She conducts a range of research projects focused on children's mathematical development, curriculum and teacher professional development, with a special interest in Chinese history, culture and tradition in mathematics education. She is now the co-chair of the ICMI Study 23.
- The discussant is Man Keung Siu (The University of Hong Kong).

Abstracts

Spoken and written numbers in a post – colonial country: the case of Algeria (Nadia Azrou)

The aim of the presentation is to present some initial steps of a long-term study aimed at intervening in teacher education in a situation of encounter of different cultural influences in a post - colonial country: Algeria. Some preliminary analyses will be reported on how natural numbers are orally represented (spoken numbers) in different ways according to different languages. The long term perspective is to take profit from the existing differences to develop competencies concerning written numbers, and at the same time to enhance students' awareness about the roots of those differences, thus contributing to promote their cultural identities.

The number line: a “Western” teaching aid (Maria G. Bartolini Bussi)

This presentation aims at discussing a very popular teaching aid, the so-called number line, where whole numbers are introduced as labels on unit marks by means of a measuring process and where additions and subtractions can be realised, as operators, with jumps forwards and backwards. Traces of this early approach can be found in the teaching practices of most Western countries, but, surprisingly, not in the most popular Chinese textbooks. In the presentation, some Western literature is reviewed to sketch out the analysis of the number line as a teaching aid, from the historic-epistemological, cognitive and didactical perspectives.

Native American cultures tradition to whole number arithmetic (Sarah Gonzáles and Juana Caraballo)

In Latin America there are more than 23,000,000 natives that even today speak their own language and many are marginalised because they do not speak the Spanish language. They have their own conceptualisation of whole numbers. Many studies have been conducted on the Mathematics of these cultures. It is highly important for teachers to be able to understand their Mathematical approach of whole number arithmetic (WNA) to be able to teach these children. In this paper, a summary of some of the WNA of Incas are presented, and how an Ethnomathematics approach, as the theoretical base to teach Mathematics in this context, is used in order to diminish the exclusion in the mathematics education of native children of these cultures.

Chinese core tradition to whole number arithmetic (Xuhua Sun)

This presentation aims to discuss the ancient Chinese tradition to whole number arithmetic (WNA) and its influence on the current curriculum practices. From the linguistic and historic-epistemological perspectives, I reviewed some of the previous studies in literature to examine ancient traditions. Based on the Chinese linguistic habit, the early Chinese invented the most advanced number name and the most advanced calculation tools (counting rod and Suanpan or Chinese abacus), in which place value is the most overarching principle as the spirit of WNA. Traces of this influence can be found in current curriculum practices. Knowing number and calculation of addition/subtraction are closely connected. Place value is the most overarching principle. The composition and decomposition of a number and problem variation are the central approaches; their implications are discussed.

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PANEL ON TEACHER EDUCATION

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Abstract

The goal of the panel is to explore and discuss teacher education in different parts of the world and to emphasise the commonalities and differences not only in the panellists' countries but in a broad perspective. Different cultures have their own advantages and disadvantages, but rather than seeing these differences in opposition, we may view them as complementary and interrelated. Therefore, by looking at differences in the parts and processes of our different educational systems, we can learn from each other and develop a more integrated perspective on teacher education.

Two key issues frame these questions in our panellists' presentations. Firstly, there are shared imperatives in many countries and contexts for WNA primary-level teacher education to provide access to deepening teachers' mathematical understandings as well as developing tools that support their mathematics teaching. Panellists present different ways in which this might occur. Secondly, the panel presentations note structures, formats and content within primary maths teacher education programmes and the ways in which they feature within different contexts.

Key words: mathematics teacher education, teacher knowledge

Introduction

In the last decades, there are no doubts about the importance of teacher education for all domains of mathematics education (Adler et al., 2005). And there are no doubts about the great influence of social, political as well as family traditions, history, economic situation etc. on the organisation of teacher education in individual countries or regions. The main focus of the panel on teacher education is therefore teachers' professional education in different parts of the world.

As noted by Even and Ball (2009): "all countries face challenges in preparing and maintaining a high-quality teaching force of professionals who can teach mathematics effectively and who can help prepare young people for successful adult lives and for participation in the development and progress of society" (p. 1). But these challenges are not the same in all countries and/or regions. The necessity of cross-cultural comparison is one of the most urgent issues in mathematics education. Its importance is reflected in many projects, publications, events etc. While models and programmes of teacher education are researched and discussed, attention is paid also to their methods, contents, differences based on pupils' age levels and the influence of all of these on the mathematical education in schools. When speaking about teacher education we do not restrict our considerations to teachers at schools. In Novotná, Margolinas and Sarrazy's (2013) work, several categories of mathematics educators are characterised and their roles in mathematics education and professional development are discussed.

Further evidence of the worldwide recognition of the importance of primary mathematics teacher education is seen in international events. Examples of conferences paying significate attention to primary mathematics teaching and corresponding teacher training include the Symposia on Elementary Maths Teaching (www.semt.cz) focus on the teaching of mathematics to children within the age-range 5–12 years and the corresponding teacher education. At ICME congresses (http://www.mathunion.org/icmi/conferences/icme-international-congress-on-mathematical-education/introduction/), there are always Topic Study Groups focusing on primary mathematics. Substantial attention to primary mathematics is paid at CERME (http://www.mathematik.uni-dortmund.de/~erme/index.php?slab=conferences) conferences. The whole third volume of CERME 1 Proceedings (Krainer, Gofree and Berger, 1999) was devoted to teacher education. In the scientific programme of each of the CERME conferences, at least one Thematic Working Group (TWG) focusing on teacher education has been included.

There are several international projects focusing on primary mathematics teaching and teacher education. Two large studies where primary level pays a substantial role are: The IEA Teacher Education and Development Study in Mathematics (TEDS-M) that examined how different countries have prepared their teachers to teach mathematics in primary and lower-secondary school (http://www.iea.nl/teds-m.html); the FIRSTMATH study explores the connections between what teachers bring with them when they enter teaching and what is learned on the job as it concerns knowledge, beliefs, skills, and curricular content (http://firstmath.educ.msu.edu/).

Most research studies in the field of primary mathematics teacher education at the international level focus on curricula within teacher education and on the knowledge a primary teacher needs for teaching well. The most studied issues cover the structure of teacher training, admission of students into teacher education and their prospective career in the field, curricula for pre-service mathematics teachers, conditions of novice teachers, preparation of teachers for overcoming obstacles they will come across in their practice, history and development of systems of education in various countries, and international comparative studies of teacher education.

The areas in the spotlight of the panel are: How do traditions, familial and cultural differences, and historical development of primary mathematics education influence teacher education in different parts of the world? What are the similarities and differences in teacher education in different parts of the world? What are the strength and weaknesses of different systems of teacher education? What are the main concerns and questions in the field of primary teacher education? What are the differences from secondary teacher education? Do we address the difference?

One crucial question for teacher education is: Under which conditions can teachers' experiences from their mathematical education contribute to an increase in their didactical knowledge? It frames most studies about teacher education. Different teacher education systems deal with it in different ways, with more or less success. In their paper published in Theme 4, Barry, Novotná and Sarrazy look for reasons for the differences in pupils' approaches to applying the taught knowledge in new contexts. They argue that the type of education, general length of practice and school level have limited influence; the most influential variable is the teacher's pedagogical beliefs about didactical knowledge. The open question is how different teacher education models used in different historical, geographical areas with different educational and familial traditions cope with the request for increasing "teacher variability" (Sarrazy, 2002), which has been shown to crucially influence teachers' behaviour.

In the individual contributions to the panel, examples from teacher education models from different parts of the world will be presented by panellists and discussants coming from different parts of the world. The main goal of the panel is to explore and discuss teacher education in different parts of the world and to emphasize the commonalities and differences not only in the panellists' countries, but in a broad perspective.

Panellists and discussants

The panel consists of six scholars representing different parts of the world, complemented with two discussants. All of them have rich experience with mathematics teacher education in their countries/regions. The panellists, members of the IPC of the study:

- Maria G. Bartolini Bussi is a Full Professor in Mathematics Education at the University of Modena and Reggio Emilia (Italy). She is the director of the University program for pre-primary and primary teacher education. She was a member of the ICMI Executive Committee (2007/2012). She is now the co-chair of the ICMI Study 23.
- Sybilla Beckmann is Josiah Meigs Distinguished Teaching Professor of Mathematics at the University of Georgia, USA. She has developed and teaches mathematics courses for future elementary and middle grades teachers and has written a textbook for such courses. Her current research is on pre-service teachers' reasoning about multiplication, division, fractions, ratio, and proportional relationships.
- Maitree Inprasitha is a Head of Doctoral Program in Mathematics Education, Faculty of Education, Khon Kaen University and the President of the Society of Mathematics Education (TSMEd). He was a member of International Program Committee of EARCOME 4, EARCOME 5 and chair of an International Program Committee of EARCOME 6 to be held in Thailand in 2013. He also attended the CNAP project in Cambodia in 2013. He has been a Project Overseer of APEC- Lesson Study Project of APEC HRDWG since 2006 to present.

- Berinderjeet Kaur is a Full Professor in Mathematics Education at the National Institute for Education in Singapore. She was the mathematics consultant of TIMSS 2011 and is a member of the mathematics expert group for PISA 2015. Her primary research interests are in the area of classroom pedagogy of mathematics teachers and comparative studies in mathematics education. She is also actively involved in the professional development of mathematics teachers in Singapore and is the founding chairperson of conferences for mathematics teachers that started in 2005 and the founding editor of the Association of Mathematics Educators (AME) Yearbook series that started in 2009.
- Xuhua Sun is an Assistant Professor in Education at University of Macau, China, specializing in mathematics education from early childhood to secondary level. She conducts a range of research projects focused on children's mathematical development, curriculum and teacher professional development, with a special interest in applying Chinese open-class model to teacher education context in Macao. She is now the co-chair of the ICMI Study 23.
- Hamsa Venkat is a Full Professor and holds the SA Numeracy Chair at Wits focused on a 5-year research and development project in primary mathematics. The Numeracy Chair work involves the trialling and development of research-based interventions across ten government primary schools in one South African school district. Prior to this, Hamsa worked as a high school mathematics teacher and lecturer in London. She was awarded the 2005 British Educational Research Association dissertation award for making the most significant doctoral contribution to research in education in 2004.

The discussants are

- Deborah Loewenberg Ball (University of Michigan, USA) and Mike Askew (Wits School of Education, University of the Witwatersrand, Johannesburg, South Africa).

Abstracts

Constructing mathematical arguments using definitions with precision in middle-grades teacher education in the USA (Sybilla Beckmann)

One concern in the mathematical education of middle-grades teachers is that, in the limited time available, teachers should have adequate opportunities to learn mathematical forms of argumentation while also studying middle grades mathematics in depth, from the perspective of a teacher. We often look to geometry and to mathematics that is more advanced than teachers will teach to provide experiences in the careful use of definitions and in constructing mathematical arguments. Using evidence from a study of future grades 4 – 8 mathematics teachers, Beckmann will discuss how the multiplicative conceptual field, which encompasses multiplication, division, fraction, ratio, and proportional relationships, is a rich area for mathematical argumentation, including using definitions with precision. The multiplicative conceptual field is also a foundation for critical topics in secondary mathematics, including linear functions, rates of change, and slope, which makes it a prime candidate for middle-grades and secondary teachers to study in depth.

Traditional and contemporary approaches to teaching primary mathematics in Thailand (Maitree Inprasitha)

This contribution will focus on the traditional Thai approach to teaching primary mathematics and the unsatisfactory results that have thus far resulted from this traditional approach. It will include an analysis of the differences between Thai and Japanese mathematics textbook, which the latter is featured as a kind of ‘problem-solving based textbook.’ It will also discuss an idea on how to adapt the Japanese ‘lesson study and structured problem-solving teaching approach’, as an innovative for teaching primary mathematics. Finally, it provides an exemplar illustrated how 1st grade students learned to gain meaningful understanding of whole number arithmetic via mathematics activities in the actual class taught by fifth year intern students during the mid-semester of 2013 and 2014 academic years at two lesson study project schools.

Primary school curriculum in Singapore – Model Method (Berinderjeet Kaur)

The primary school mathematics curriculum in Singapore places emphasis on quantitative relationships when students learn the concepts of number and the four operations. The Model Method, an innovation in the teaching and learning of primary school mathematics, was developed by the primary school mathematics project team at the Curriculum Development Institute of Singapore in the 1980s. The method, a tool for representing and visualizing relationships, is a key heuristic students’ use for solving whole number arithmetic word problems. When students make representations, using the Part-Whole and Comparison models, the problem structure emerges and students are able to visualise the relationship between the known and unknown and determine what operation to use and solve the problem. The model method has proved to be effective for making number sense and solving arithmetic word problems in Singapore schools. Prospective primary school mathematics teachers are introduced to the method as part of their curriculum studies (Mathematics) during their pre-service teacher education at the National Institute of Education in Singapore.

Exploring relationships between mathematical and pedagogical content knowledge of primary teachers in South Africa (Hamsa Venkat)

The importance of representational flexibility and efficiency in mathematical working and learning is widely accepted. Within some formulations of professional knowledge though, the need to work with representations has tended to be located within the pedagogic content knowledge domain rather than within disciplinary learning per se. In South Africa, broad evidence continues to point to significant gaps among primary teachers in relation to mathematical content knowledge at, or close to, the level of teaching. This situation suggests the need to simultaneously direct attention to representations as object and tool, in order to support teachers' mathematical learning and their mathematics teaching within teacher education, beginning in the context of whole number arithmetic. This simultaneous orientation to representations was at the forefront of an in-service primary mathematics for teaching course where connecting between representations formed a central pillar of the work alongside the need to 'explain' choices and steps. In this presentation, I present a small slice of data illustrating shifts that suggest that emphasis on representation and explanation can potentially be simultaneously productive for both mathematical learning and teaching.

The goal, roles, and transposition: Chinese open-class approach and transposition to Macau and Italy (Xuhua Sun and Maria G. (Mariolina) Bartolini Bussi)

The open-class approach was established in the early 1950s by the Chinese Ministry of Education with the primary purpose of organising teacher study groups in schools. It is a more flexible professional development tool than the Japanese lesson study in terms of organization, budgeting, and timetabling. How this underrepresented tool plays critical and different roles in teacher recruitment, professional assessment, and professional research and development in primary teacher education to meet the goals of specialists (not generalists) and how this tool has been transposed and applied in Macao and Italy will be presented in the panel.

The discussants will react to the presentations, comment on them, bring additional information, and pose questions.

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PANEL ON SPECIAL NEEDS

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Abstract

Although the last decades have witnessed a serious growth in research onto the diagnosis, remediation and prevention of MLD, much work remains to be done. Longitudinal research is needed to identify developmental precursors and to delineate developmental trajectories of MLD. The neural basis of these difficulties and their association with classroom performance certainly need to be further explored. Understanding the different characteristics of MLD at different levels, the behavioural, the cognitive and the neurobiological, will inform appropriate educational interventions. The design and evaluation of these remedial interventions needs to be a priority on the agenda for future research. The goal of the panel is to explore and discuss the above issues and challenges, with an emphasis on the last issue, namely instructional goals and interventions for children with MLD.

Key words: dyscalculia, mathematical learning disabilities, special needs, whole number arithmetic

Introduction

Many children have difficulties or problems with learning mathematics. While these difficulties or problems may occur at any stage in learners' mathematical development, by far most attention of researchers and practitioners goes to the domain of early and elementary mathematics, and, more specifically, to the domain of whole number arithmetic (WNA). Even though the issues of diagnosis of and instruction to children with special mathematical learning needs is getting increasing research attention, research in this area is still lagging behind compared to other academic subjects such as reading. Hereafter we list some major open questions for research and practice.

First, there is the terminological issue. Defining mathematical learning difficulties, problems or disabilities (hereafter abbreviated as MLD) is not an easy task (Berch and Mazzocco, 2007). Despite the solid knowledge base that has been achieved in this field, more substantial progress in understanding and addressing MLD would be facilitated by establishing agreement on consistently used terminology and use of standardised criteria concerning the nature and seriousness of the disability. While certain definitions explicitly refer to a biologically based disorder, others emphasise the discrepancy between the child's mathematical achievement and his/her general intelligence as the main criteria, and still others capitalise on the response to intervention. But the field of MLD also lacks coherence and consensus about what constitutes "mathematics" in MLD. Within MLD research there is a history of predominance to focus on memorisation of arithmetic facts and automatisation of arithmetic procedures. A less (neuro)psychologically dominated and more interdisciplinary approach might bring a broader, more coherent and balanced perspective that takes into account both the views about mathematics learning as arithmetic and other equally important perspectives such as spatial and geometrical reasoning, mathematical relations and patterns, and other forms of mathematical thinking with more potential towards abstraction and generalisation (Hord and Xin, 2014; Mulligan, 2011). Evidently, besides children with MLD there are also other children requiring special mathematics educational support, but they are not diagnosed as MLD , such as children with intellectual disabilities, children with auditory, visual or motoric impairments, children with serious emotional and/or behavioural problems, or, finally, children with longstanding inappropriate instruction or environmental deprivation (De Smedt et al., 2012).

A second major concern of researchers in the field is to characterize the various cognitive mechanisms that are implicated in the development of MLD. Several cognitive explanations for the presence of MLD have been put forward. Most of the available research on MLD has dealt with domain-general cognitive factors, such as poor working memory and difficulties with the retrieval of phonological information of long-term memory. More recently (and against the background of findings from neuroimaging research), it has been proposed that MLD arises as a consequence of domain-specific impairments in number sense or the ability to represent and manipulate numerical magnitudes (Landerl et al., 2004). For example, children with MLD have particular difficulties in comparing two numerical magnitudes and in putting numbers on a number line, both of which are thought to measure one's understanding of numerical magnitude. Although various cognitive candidates have been put forward to explain the MLD, the existing body of data is still in its infancy. According to Karagiannakis et al. (2014), although the field has witnessed the development of many classifications, no single framework or model can be used for a comprehensive and fine interpretation of students' mathematical difficulties, not only for research purposes but also for informing mathematics educators. Starting from a multi-deficit neurocognitive approach and building on the available literature, these authors have recently proposed a classification model for MLD describing four cognitive domains within which specific deficits may reside.

Third, initial accounts of MLD in the 1970s suggested that MLD was due to brain abnormalities. With the advent of modern neuroimaging techniques, researchers have begun to address this issue. There is converging evidence for the existence of a frontoparietal network that is active during number processing and arithmetic (Ansari, 2008). Studies that examine this network in children with MLD are currently slowly but steadily emerging. These few studies consistently indicate that children with MLD have both structural and functional alterations in the abovementioned frontoparietal network, particularly in the intraparietal sulcus, which is the brain circuitry that supports the processing of numerical magnitudes, and (pre)frontal cortex, which is assumed to have an auxiliary role in the maintenance of intermediate mental operations in working memory. Furthermore, it has been suggested that these brain abnormalities in children with MLD are probably of a genetic origin, yet the genetic basis of MLD remains largely unknown and no genes responsible for mathematics (dis)abilities have been identified. Studies in the field of medical genetics have revealed that some disorders of a known genetic origin, such as Turner Syndrome and 22q11 Deletion Syndrome, show a consistent pattern of MLD. Furthermore, there is some early evidence of links to autism spectrum disorder and Asperger.

A fourth and final issue relates to the question: What are appropriate educational interventions for children with MLD? Originally, general perceptual-motor training was the dominant way of remediating learning disorders, but the effects of this type of training have been discounted. Interventions that target those specific components of mathematics with which a child with MLD has difficulty appear to be the most effective (Dowker, 2008). Such intervention involves the assessment of a child's strengths and weaknesses in mathematics and this profile is taken as an input to remediate specific components of mathematical skill. However, there remain a large number of major questions, such as: What is the appropriate moment to diagnose MLD and to start specific interventions? Do MLD children profit more individualized interventions organized out of the regular mathematics class or do they profit more from being integral part of the regular mathematics class? Do these children need a special kind of intervention or do they profit most from the same kind of instruction as children without MLD? More specifically, are conceptually-based and constructivist-oriented mathematics instruction also suitable for children with learning disabilities (Xin & Hord, 2013; Xin, Liu, Jones, Tzur, and Si, in press). Another issue is whether we do not have a blind spot when making assumptions about what children with MLD can do, rather than what they cannot do (Peltenburg et al., 2012). Finally, does the remedial instruction of children with MLD pay enough attention to other aspects of mathematics than whole number sense, such as on conceptual relationships that may develop from spatial reasoning. Clearly, it may not be productive to try to answer these major educational questions for all categories of children who have serious trouble with learning mathematics.

So, although the last decades have witnessed a serious growth in research onto the diagnosis, remediation and prevention of MLD, much work remains to be done. Longitudinal research is needed to identify developmental precursors and to delineate developmental trajectories of MLD. The neural basis of these difficulties and their association with classroom performance certainly need to be further explored. Understanding the different characteristics of MLD at different levels, the behavioural, the cognitive and the neurobiological, will inform appropriate educational interventions. The design and evaluation of these remedial interventions needs to be a priority on the agenda for future research. These interventions may not only treat the difficulties, but also prevent them.

The goal of the panel is to explore and discuss the above issues and challenges, with an emphasis on the last issue, namely instructional goals and interventions for children with MLD.

The panel consists of four scholars with complementary specializations in the domain of children with MLD and other special needs in the curricular domain of whole number arithmetic, complemented with one of the key-note speakers of the ICMI 23 conference, Professor Brian Butterworth (University College, London, UK), who is a world-leading scholar is the domain of the (neuro)cognitive roots of dyscalculia and its treatment, and who will act as a discussant in this panel.

- Anna Baccaglini-Frank is working as a post-doc at the University of Modena and Reggio Emilia (Italy). She has worked on several projects on low achievement and dyscalulia trying to construct a bridge between cognitive psychology and mathematics education for students with special needs, and was involved recently in the construction of a classification of MLD subtypes.
- Joanne Mulligan is an Associate Professor in Education at Macquarie University, Australia, specializing in mathematics education from early childhood to secondary level. She conducts a range of research projects focused on children's mathematical development, curriculum and assessment and teacher professional development, with a special interest in MLD.
- Marja Van den Heuvel-Panhuizen is professor in mathematics education at Utrecht University in The Netherlands. As a representative of the Freudenthal-oriented approach to teaching and learning mathematics, one of her research interest is investigating the mathematical potential of students with MLD.
- Yan Ping Xin is professor in special education at Purdue University, West Lafayette, Indiana, U.S.A. Dr. Xin's research includes effective instructional strategies in mathematics problem solving with students with learning disabilities or problems, computer-assisted differentiated instructional systems, and cross-cultural comparisons of mathematical learning practices and outcomes.

Abstracts

Does "dyscalculia" depend on initial primary school instruction? (Anna Baccaglini-Frank)

In Italy, "dyscalculic" students – and students with any other learning disabilities or handicaps – are typically included in the regular classrooms and receive instruction from the same teacher as the other students. During a 3-year project we designed didactical materials that provide all students (in first and second grade) with "hands on" (kinaesthetic-tactile) experiences involving manipulation of physical artefacts to develop mathematical meanings (including procedures) from these experiences and from consequent mathematical discussions. Persistent use of these materials shows to significantly reduce the number of children who can be classified as "dyscalculic" by third grade. In particular the children in the experimental classes develop a variety of strategies for addressing different mathematical situations. When assessed on calculation and compared to children in control classes, the children in the experimental classes show slower (by a few months) automatization of numerical facts, while their accuracy and variety of strategies used is consistently greater.

If the percentage of “dyscalculics” significantly depends (among other factors) on students’ initial mathematical experiences in school, does it make sense to keep on searching for who these children are, instead of investigating why some children fail to overcome difficulties in mathematical learning that others overcome?

Are MLD linked to a lack of underlying awareness of mathematical patterns and relationships that are more linked to spatial ability than development of number? (Joanne Mulligan)

Mathematics Learning Difficulties (MLD) may be traced to a lack of Awareness of Mathematical Pattern and Structure (AMPS) that is considered critical to the development of generalisation and relational thinking. Given the increasing influence of cognitive and neurocognitive sciences this perspective provides a much broader approach to both the research of MLD and the ways in which intervention programs can be developed. One of the key questions arising from the focus on AMPS is the study of conceptual connectivity within and between domains of knowledge (or disciplines). This may require mathematics to be reconceptualised as a coherent subject domain that develops from human interaction and that is reliant on conceptual relationships that develop from spatial sense and spatial reasoning. A lens on conceptual connectivity of spatial concepts, such as Awareness of Pattern and Structure, therefore, may offer a more complete picture of the learning that underpins WNA within mathematics.

It is time to reveal what children with MLD can do, rather than what they cannot (Marja Van den Heuvel)

Good teaching starts with getting to know what students know. Although this applies for all students, it is particularly true for weak learners. The problem with these learners is that they have low scores at mathematics tests, which may automatically lead to conclusions about their inability to solve demanding mathematics problems and coming up with their own solution methods. Unmasking such preconceived ideas is of vital importance for these students, and may open up new chances for their learning of mathematics. But how can we reveal what they know?

Conceptual Model-based Problem Solving: An integration of constructivist mathematics pedagogy and explicit strategy instruction (Yan Ping Xin)

This presentation will introduce a Conceptual Model-based Problem Solving (COMPS) approach that integrates constructivist mathematics pedagogy and explicit strategy instruction to promote concept development and mathematics problem-solving ability of students with learning disabilities or difficulties. Through nurturing fundamental mathematical ideas such as the concept of the composite unit, the COMPS program makes explicit the reasoning behind mathematics and therefore, students were able to make sense of abstract mathematical models. The COMPS program may be especially helpful for students with learning disabilities or difficulties who are likely to experience disadvantages in working memory and information organization.

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The Twenty-third ICMI Study: Primary Mathematics Study on Whole Numbers DISCUSSION DOCUMENT

1. Introduction and Rationale for ICMI Study 23

This document announces a new Study to be conducted by the International Commission on Mathematical Instruction. This Study, the twentythird led by ICMI, addresses for the first time mathematics teaching and learning in the primary school (and pre-school), taking into account international perspectives, socio-cultural diversity and institutional constraints. One of the challenges of designing the first ICMI primary school Study of this kind is the complex nature of mathematics at the early level. For this reason a focus area has been chosen, as central to the discussion, with a number of questions connected to it. The broad area of **Whole Number Arithmetic (WNA)** including operations and relations and arithmetic word problems form the core content of all primary mathematics curricula. The Study of this core content area is often regarded as foundational for later mathematics learning. However, the principles and main goals of instruction in the foundational concepts and skills in WNA are far from universally agreed upon, and practice varies substantially from country to country. An ICMI Study that provides a meta-level analysis and synthesis of what is known about WNA would provide a useful base from which to gauge gaps and silences and an opportunity to learn from the practice of different countries and contexts.

Whole numbers are part of everyday language in most cultures, but there are different views on the most appropriate age at which to introduce whole numbers in the school context. Whole numbers, in some countries, are introduced in the pre-school, where the majority of children attend before the age of 6 years. In some countries, primary schooling includes Grades 1-6; in others it includes Grades 1-5. Thus the entrance age of students for primary school may vary from country to country. For these reasons, this Study addresses teaching and learning WNA from the early grades, i.e., the periods in which WNA is systematically approached in the formal school, and in some contexts this includes the pre-school.

In January 2014, the International Programme Committee (IPC) for ICMI Study 23 met at the International Mathematical Union Secretariat, Berlin, and agreed upon four principles.

First it was decided that **cultural diversity** and how this diversity impinges on the early introduction of whole numbers would be one major focus. The IPC agreed that the Study will seek contributions from authors representative of as many countries as possible, especially those where cultural characteristics are less known but where these influence what is taught and learned. In order to foster an understanding of the different contexts where potential authors have developed their studies, each applicant for the Conference will be required to provide background information about this context.

Second, it was decided to find better ways to involve **policy makers** who have the responsibility to offer to every child the opportunity to go to school and to learn WNA. In connection with this aim, the IPC will also solicit contributions in the form of annotated video-clips about practical examples of WNA with potentially strong impact.

Third, it was decided to collect examples of experiences about **inclusive** teaching and learning, including students with special needs, considering that in some countries the education system provides special schools, classrooms and teachers whilst in others students are enrolled in mainstream classes.

Fourth, it was decided to focus on **teacher education and professional development**, considering that in order to teach elementary mathematics effectively there is a need for sound professional knowledge, both in mathematics and in pedagogy.

In order to meet this complex set of principles, the IPC delineated a set of **themes** to serve as the organizing framework for the Study Conference.

This discussion document presents the background of the Study, together with its challenges and aims and provides a description of the five organising themes. Because the Study Conference will be organized around discussion within each theme (with some overarching sessions) each proposed contribution to the Study should address the theme that it is most aligned with, and identify a second theme that it may also be related to. Finally the discussion document outlines the organisation, timing and location of the Study Conference and the timetable of the milestones leading up to the Study Conference and to ICMI publication.

1.1. Background of the Study

Countries differ enormously in providing pre-school programs (UNESCO, 2010) which are especially important for children from disadvantaged families: actually the OECD (2011) has reported that, in general, participation in pre-school produces better learning outcomes in later years such as for fifteen-year-old students. Primary schooling is compulsory in most countries (in all Western countries), although there is considerable variation in the facilities, resources and opportunities for students. This is the uneven context where mathematics teaching and learning takes place.

Mathematics is a central feature of early education and the content, quality and delivery of the curriculum is of critical importance in view of the kinds of citizens each country seeks to produce.

In the international literature there are many contributions about primary school mathematics. In many cases, especially in the West, early processes of mathematical thinking, often observed in early childhood (i.e., 3-8 year-old children), are also investigated by cognitive and developmental psychologists. They sometimes study the emergence of these processes in clinical settings, where children are stimulated by suitable models so as to observe the emergence of aspects such as one-to-one correspondences, counting, measuring and other processes). In several countries, Piaget's theory has been very influential despite criticism. Neuroscientists have also been studying for some years the emergence of "number sense". However, recent perspectives highlight that what is still needed is serious and deep interdisciplinary work with experts in mathematics education (UNESCO, 2013).

1.2. Key Challenges for ICMI Study 23

A recent document prepared by ICMI's Past President Michéle Artigue and commissioned by UNESCO (2012) discusses, from a political perspective, the main challenges in fundamental mathematics education. It reads:

> *We live in a world profoundly shaped by science and technology. Scientific and technological development has never been faster, has never had an impact as important and as immediate on our societies, whatever their level of development. The major challenges that the world has to face today, health, environment, energy, development, are both scientific and human challenges. In order to take up these challenges, the world needs scientists able to imagine futures that we barely see and able to make these possible, but it also needs that the understanding of these challenges, the debate on the proposed changes, are not reserved for a necessarily limited scientific elite, but are very widely shared. Nobody can now doubt that positive, sustainable and equitable evolutions cannot be achieved without the support and contribution of the great majority of the population. Nobody should thus doubt that the gamble of shared intelligence, that of quality education for all, and especially science education for all, including mathematics and technology education, are the only gambles we can take. This is even more the case in the current context of crisis. Without such an education, it is futile to speak of debate and citizens' participation.*

Drawing on these ideas, ICMI has acknowledged that it is timely to launch, for the first time in its history, an international Study that especially focuses on early mathematics education, which is both basic and fundamental mathematically. Primary school mathematics education has been present in other ICMI studies, but, in most cases, secondary school mathematics education was predominant. When foundational processes are concerned, a strong epistemological basis is needed. This is where the involvement of ICMI adds value with respect to analyses carried out in other fields. Such epistemological analysis was part of classical works of professional mathematicians (e.g., Klein, Smith, and Freudenthal) who played a major role in the history of ICMI and considered mathematics teaching as a whole (ICMI, 2008). It is pertinent to mention here a short quote by Felix Klein, the first President of ICMI, used as an epigraph in the website on the history of ICMI (ICMI, 2008).

> *I believe that the whole sector of Mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organised as an organic whole. It grew ever clearer to me that, without this general perspective, even the purest scientific research would suffer, inasmuch as, by alienating itself from the various and lively cultural developments going on, it would be condemned to the dryness which afflicts a plant shut up in a cellar without sunlight* (Felix Klein, 1923).

One cannot study school mathematics teaching without focusing also on the teacher's role and responsibility. The attention on mathematics teacher education and professional development has been a constant preoccupation of ICMI. The primary school and (more generally) early education deserve a special attention. The complex nature of arithmetic and its foundational importance for mathematics are well known by mathematicians and mathematics educators. However, primary school teachers work within systems which may or may not support a rigorous professional environment in which they are knowledgable and respected experts on both the mathematics and the pedagogy of what they teach. In some systems, teaching WNA may be treated as something that virtually any educated adult can do with little specific training; WNA may be viewed by some as straightforward and intuitive, and involving no more than showing children how to cope with everyday life and to carry out algorithms.

There are systems where primary mathematics teachers are specialists and other where they are generalists. It is not within the aims of this Study to enter deeply into the pedagogical debate about specialist vs generalist teachers in early education, as both models show advantages and disadvantages. What is important to highlight is that much is already known from research about productive ways to teach WNA, yet this knowledge cannot be enacted in systems in which teachers are not proficient in elementary mathematics and particular pedagogical approaches. Effective teacher education may require the development of a culture in which teachers are expected to be highly educated professionals.

2. Aims of the ICMI Study 23

This Study aims to produce and share knowledge about sustainable ways of promoting effective pedagogy in teaching and learning WNA for all, taking account of the large body of theory and research already existent, socio-cultural diversity and institutional constraints. In particular the following specific aims were acknowledged by the IPC, for the early teaching and learning of WNA:

- *bring together communities of international scholars representative of ICMI's diverse membership across regions and nationalities in addressing the theme of WNA for the production of a Study volume;*
- *provide a state-of-the-art expert reference group on the theme of WNA;*
- *contribute to knowledge, better understanding and resolution of the challenges that teaching and learning WNA faces in diverse contexts;*

- *collectively represent the wide variety of concerns in the field of WNA and reflect upon it;*
- *facilitate multi- and interdisciplinary approaches (including cooperation with other bodies and scientific communities) to advance research and development in WNA;*
- *disseminate scholarship in mathematics education — research; methodologies, theories, findings and results, practices, and curricula — in the theme of WNA;*
- *pave the way to the future by identifying and anticipating new research and development of WNA;*
- *act as a resource for researchers, teacher educators, policy and curriculum developers and analysts and the broad range of practitioners in mathematics and education;*
- *promote and assist discussion and action at the international, regional or institutional level.*

3. The Themes of the ICMI Study 23

The ICMI Study will be organised around five themes that provide complementary perspectives on approaches to early WNA in mathematics teaching and learning. Contributions to the separate themes will be distinguished by the theme's specific foci and questions, although it is expected that interconnections between themes will emerge and warrant attention.

The five themes are:

1. *The why and what of WNA*
2. *Whole number thinking, learning, and development*
3. *Aspects that affect whole number learning*
4. *How to teach and assess WNA*
5. *Whole numbers and connections with other parts of mathematics.*

Themes 1 and 2 address foundational aspects from the cultural-historic-epistemological perspective and from the (neuro) cognitive perspective. What is especially needed are reports about the impact of foundational aspects on practices (both at the micro-level of students and classrooms and at the macro-level of curricular choices).

Themes 3 and 4 address learning and teaching respectively, although it is quite difficult, sometimes, to separate the two aspects, because for example in some languages and cultures (eg. Chinese, Japanese, Russian) the two words collapse into only one.

Theme 5 addresses the usefulness (or the need) to consider WNA in connection with (or as the basis for) the transition to other kinds of numbers (e.g., rational numbers) or with other areas of mathematics, traditionally separated from arithmetic (e.g., algebra, geometry, modelling).

Each theme is outlined briefly and followed by exemplary questions that could be addressed in the submitted contributions. An overarching question which cuts across all the themes concerns teacher education and development:

- *How can each of the themes be effectively addressed in teacher education and professional development?*

3.1. The why and what of WNA

This theme will address cultural-historic-epistemological issues in WNA and their relation to traditional, present and possible future practices.

A sense of number is constructed through everyday experience, where culture and language play a major role, hence ethnomathematics has paid attention to the different grammatical constructions used in everyday talk (e.g., Maori numbers as actions; Aboriginal Australians' spatial approach to numbers). Ways of representing whole numbers and making simple calculations (e.g., with fingers or other body parts; with words; with tools, including mechanical and electronic calculators; with written algorithms) have enriched the meaning of whole numbers through the ages.

The base ten system is critical for our current sophisticated understanding of WNA. The long and difficult development of place value systems is well documented in the history of mathematics (the introduction of place value in China and India; the migration to Europe through the Arabic culture; the invention of zero; the strategies for mental calculation) and indicates the need to study place value and the base ten system deeply for understanding.

The above issues (and others) have been considered in different ways by different cultures throughout history. Besides the use of numbers in practical activities, there is evidence (in the history and in educational research) that the exploration of the properties of whole numbers, relations and operations paves the way towards the introduction, with young students too, of typical mathematical processes, such as generalizing, defining, arguing and proving.

Some references may be found in the ICMI Studies 10, 13, 16, 19.

The following possible questions will help to illuminate this theme further:

- *What goals underlie the teaching and learning of WNA?*
- *Taking a mathematical perspective (as practised by the current community of mathematicians) combined with an educational perspective, what are core mathematical ideas in developing pathways to WNA?*
- *What are distinctive features concerning whole number representation and arithmetic in your culture? What is the grammar of number? In what ways does language or ways of representing and using numbers influence approaches to calculation or problem solving? How do these features interact with the decimal place value system?*

- *What is the role of mathematical practices and habits of mind in teaching and learning WNA? How can teaching and learning WNA support the development of mathematical practices and habits of mind?*
- *How much is the base ten place value emphasized in your curriculum?*
- *How much computational facility is important for later mathematics learning, and learning in other areas? What about mental calculation? What about speed of calculation?*
- *How do policies and the educational environment and system support or not support a culture in which teaching WNA is seen as requiring detailed, specific professional knowledge?*
- *What were the main historic features and their origins of WNA in (ancient) west / east? What were some factors that led to such historic features? What were the effects on the development of mathematics curriculum?*
- *How does your curriculum develop understanding of the structural features of WNA and its extensions?*

3.2. Whole number thinking, learning, and development

This theme will address the relationships between cognitive and neurocognitive issues and traditional, present and possible future practices in the early teaching and learning of WNA.

The idea of number sense was in use for decades in the mathematics education literature before entering into the cognitive and neurocognitive literature, with some similarities and differences. (Neuro)cognitive scientists have investigated children's spontaneous tendency to focus on numerosity in their environment; the development of rapid and accurate perception of small numerosities (subitizing) in connection with visualization and structuring processes; the ability to compare numerical magnitudes; and the ability to locate numbers on a (mental) number line. There are models for children's informal knowledge of counting principles and informal counting strategies and their development into more formal and abstract arithmetic notions and procedures.

A recent focus concerns developmental dyscalculia, as a difficulty in mathematical performance resulting from impairment to those parts of the brain that are involved in arithmetical processing, without a concurrent impairment in general mental function.

Recent debates concern the embodied cognition thesis resulting in the evidence, shared by many researchers, that, although mathematics may be socially constructed, this construction is rooted in, and shaped by, the body and bodily experiences.

Some references may be found in OECD (2010), UNESCO (2013).

The following possible questions will help to illuminate this theme further:

- *To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children's WNA development?*
- *What can we learn from the (neuro-)cognitive studies in WNA? Do their findings essentially confirm insights that are present (and were already present for a long time) in the mathematics education community or do they point to truly new insights and recommendations about the kind of tasks and instructional approaches children need? How can we integrate different perspectives about the foundations and development of WNA concepts and skills?*
- *What are specific effects of the structure of the individual finger counting system on mental and linguistic quantity representation and arithmetic abilities in children, and even in older learners and adults?*
- *How can an embodiment framework be used to analyse and/or design educational approaches based on suitable representations, (e.g., through the number line) or on-line manipulatives and modern technological devices (touchscreens)?*
- *What are appropriate ways of analyzing the multimodal nature of mathematical thinking (e.g., the role of bodily motion and gesture)?*
- *What is the relationship between the embodied cognitive approach and traditional approaches, for example Montessori, Piaget, which had a strong influence of elementary school mathematics worldwide?*
- *How can the tools of the embodiment framework/analysis be integrated/combined with socio-cultural perspectives to compare/contrast approaches where embodiment is exploited or hindered?*
- *How can teachers be educated in order to exploit the (neuro-)cognitive foundations for WNA?*

3.3. Aspects that affect whole number learning

This theme will address some aspects affecting learning of WNA in both positive and negative ways.

Socio-cultural aspects influence enumeration practices, algorithms and representations as well as metaphors or models (e.g., the number line). Hence students' language and culture may help or hinder the construction of WNA not only in schools but also in informal settings. On the one hand, the recourse to tools from the history of mathematics (e.g. counting sticks; different kind of abaci; reproduction of ancient mechanical calculators) may be effective to foster learning of WNA with explicit reference to the local culture. On the other hand, intentionally designed tools may address the effective learning processes evidenced in the literature (e.g., technological tools including the multitouch ones).

Low achievement in WNA is a major focus in debates at all levels, from school practice to international studies. Literature shows that it may depend on very different aspects: context variables (e.g., marginalized students; migrant and refugee students; education in fragile democracies), institutional variables (e.g. different languages in school and out of school context), learning disabilities (dyscalculia; sensual impairment for deaf and blind students); affect factors (e.g., self-beliefs, anxiety, motivation, gender issues); didactical obstacles (e.g., a too limited approach as in the case of teaching addition separate from subtraction or multiplication as a repeated addition only); or epistemological obstacles (related to the historical process of constructing WNA by mankind).

Some references may be found in the ICMI Studies 17, 22 and, for general issues concerning the contexts, UNESCO (2010).

The following possible questions will help to illuminate this theme further:

- *What are the features of your language related to whole numbers, operations and word problems that could affect learning in a positive or negative way? How these features are mirrored in formal or informal settings?*
- *What main challenges for learning WNA are faced by marginalized students or, in general, in difficult contexts?*
- *What main challenges are faced for learning WNA by students with sensual impairments (blind and deaf)?*
- *What main challenges are faced for learning WNA by dyscalculic students ?*
- *In your country are students with special needs enrolled in mainstream classes (inclusive systems) or in special education classes? To what extent may the strategies for learning WNA especially developed for students with special needs be useful for all students?*
- *In your country is there evidence that the literature on either didactical or epistemological obstacles has impact on classroom practice?*
- *Which tools (from the ancient or new technologies) are useful to enrich classroom activity for all or to help low achievers in WNA? Is there evidence on effective use of traditional manipulatives (including the ones rooted in local cultures), virtual manipulatives, technologies (including the recently developed multi touch technologies)? Are there classroom studies on the comparison of different kinds of tools?*
- *What strategies may be implemented by teachers in relation to the above issues?*

3.4. How to teach and assess WNA

This theme will address general and specific approaches to the teaching, assessing and learning of WNA. WNA appears in standards documents for mathematics of every country (see http://www.mathunion.org/icmi/other-activities/database-project/introduction/), and in specific international studies (e.g. the Learner's perspective Study, with sixteen country teams). In some countries independent research communities have also developed projects on teaching and assessing WNA, which, in some cases, are internationally acknowledged (e.g. Realistic Mathematics Education in the Netherlands; NCTM Curriculum and Evaluation Standards in US; Davydov's math curriculum in Russia; the Theory of Didactical Situations in France). In the ethnomathematics trend, projects sensitive to the local cultures and traditions have been developed (e.g. in Australia, Latin America, USA and Canada). A specialized Symposium on Elementary Mathematics Teaching (SEMT) has been held every second year in Prague since 1991.

Some other focus issues may be the following: the role of textbooks and future teaching aids (e.g., multimedia; e-books) for WNA; tools to approach specific elements of WNA (e.g., manipulatives, technologies); specific strategies for some fields (e.g., for word problems, the Chinese tradition of problems with variation; Singapore's model method; the extended literature on word problems and relations with real life situations); examples of practices rooted in local culture; metacognitive aspects in national curricula (e.g. early approaches to mathematical thinking processes).

In recent years the assessment debate at the local and school level has been very much biased by the results of international studies (e.g. OECD PISA, TIMSS), which are likely to produce assessment driven curricula. An ICMI Study on assessment was produced in the early 90s (ICMI Study 6), but updating might be necessary to establish current relevance and the impact of the international studies.

Some references for this theme may be found in the proceedings of ICMI Congress and Regional Conferences http://www.mathunion.org/icmi/Conferences/introduction/.

The following possible questions will help to illuminate this theme further:

- *What are the consequences of policy decision making related to evidence-based WNA teaching in comparison with policy decision making based on opinion?*
- *How is the intended curriculum reflected in textbooks and other teaching aids?*
- *What are the changes (if any) that have resulted from the use of technology in the teaching of WNA?*
- *How complety is our understanding of the development of the place value system, and at what points in the/your curriculum are key features of place value explored in greater depth?*

- *How does the/your curriculum foster the transition from a counting or additive view of number to a ratio/multiplicative/measurement view of number?*
- *How do children acquire WNA concepts and procedures outside of school? How can teachers build upon the knowledge children acquire outside school?*
- *What are the approaches that have proven to be effective in your school setting to teach elements of WNA, for example number sense, cardinality, ordering, operations (subtraction with re-grouping, etc.), problem solving, estimation, representing, mental computation…?*
- *Problem-solving contexst: should they be realistic? Should they be authentic? Always? What is the place (if any) of traditional word problems? What is the role of (real world) context in WNA? Are they always necessary?*
- *How can we develop positive attitudes toward mathematics while teaching WNA?*
- *How can teachers promote the development of student's metacognitive strategies during the learning of WNA?*
- *What main challenges are faced by teachers when teaching and assessing WNA?*
- *What innovative assessment approaches are used to evaluate the learning outcomes of WNA? What are the changes (if any) in assessment of WNA that have resulted from the media appeal of international studies like PISA or TIMSS?*

3.5. Whole numbers and connections with other parts of mathematics

This theme will address WNA in terms of its interrelationships with the broader field of mathematics.

Some connections are of central concern: pre-algebra and algebraic thinking (e.g. looking for patterns; schemes for the solution of world problems); geometry or spatial thinking (e.g., triangular or square numbers and similar; number lines); rational numbers and measurement (e.g. Davydov's curriculum for arithmetic); statistical literacy (e.g. mean, median and mode, interval, scale, and graphical representation).

Evidence suggests that the earliest formation of WNA can support the learning of mathematics as a connected network of concepts and, viceversa, embedding WNA in the broad field of mathematics can foster a better understanding.

Some references for this theme may be found in the ICMI Studies 9,12,14,18.

The following possible questions will help to illuminate this theme further:

- *How can WNA teaching and learning contribute to understanding other interconnected mathematical ideas and build on one another to make students view mathematics as a coherent body of knowledge?*
- *In your country, to what extent are connections between WNA and other Mathematics topics pointed out in the curriculum syllabus and textbooks, and how are they approached? i.e WNA and measurement, WNA and elementary statistics? Pre-algebra patterns, WNA and algebra?*
- *In your system/country are symbolic and non-symbolic approaches to word problems compared? To what extent are connections made between base ten arithmetic and polynomial arithmetic? To what extent are the rules of arithmetic/properties of operations used as a guide in learning manipulation of algebraic expressions?*
- *In your country/system to what extent are connections between WNA and other Mathematics topics stressed in the teachers' education programs?*
- *In what ways does the connection between WNA and specific themes in other areas of Mathematics contribute to students' understanding of these themes?*
- *What learning conditions enable students to make connections between WNA and other mathematics topics?*
- *In which ways do the practice of connecting WNA to other areas of mathematics contribute to the development of mathematical thinking?*
- *How does the connection of WNA with other areas of mathematics improve communication of mathematical ideas?*
- *How can technology be used to make connections between WNA and other mathematics topics?*
- *How does the use of representations in WNA teaching and learning contribute to building connections with other mathematical areas? For example, to what extent is the number line used to exhibit the connections between WNA and arithmetic of fractions?*

4. The Study Conference

ICMI Study 23 is designed to enable teachers, teacher educators, researchers and policy makers around the world to share research, practices, projects and analyses. Although reports will form part of the program, substantial time will also be allocated for collective work on significant problems in the field, that will eventually form part of the Study volume. As in every ICMI Study, the ICMI Study 23 is built around an International Conference and directed towards the preparation of a published volume.

The Conference will be organized around working groups on the themes: these groups will meet in parallel during the time of the Conference. In each working group, the IPC will organise the discussion starting from the contributions, assuming that each participant has carefully reviewed the contributions of their working group. Some special sessions presenting video-clips of practice will be organized, to share meaningful examples of WNA. Thus, there will be plenty of time for discussion of submitted papers, as well as possible plans for future collaborative activity.

The Conference language is English. However, native speakers and more expert participants will do every effort to ensure that every participant may take active part in the discussion.

4.1. Location and dates

The Study Conference will take place in Macau, China and will be hosted by the University of Macau (**June 3-7, 2015**), with an opening on June 3 at 9AM and closing on June 7 at 2PM. Arrival day is June 2; departure may be scheduled as from the night of June 7.

Every effort will be made to assist participants with visa applications, if needed.

4.2. Participation

As is the usual practice for ICMI studies, participation in the Study Conference will be by invitation only for the authors of submitted contributions which are accepted. Proposed contributions will be reviewed and a selection will be made according to the quality of the work, the potential to contribute to the advancement of the Study, with explicit links to the themes contained in the Discussion Document and the need to ensure diversity among the perspectives. The number of invited participants will be limited to approximately 100 people.

Unfortunately, an invitation to participate in the Conference does not imply financial support from the organizers, and participants should finance their own attendance at the Conference. Funds are being sought to provide partial support to enable participants from non-affluent countries to attend the Conference, but it is unlikely that more than a few such grants will be available. Further information about the access to such grants will be available in the ICMI Study 23 website

http://www.umac.mo/fed/ICMI23/

4.3. ICMI Study 23 Products

The **first product** of the ICMI Study 23 is an electronic volume of Proceedings, to be made available first on the Conference website and later in the ICMI website: it will contain all the accepted papers as reviewed papers in a Conference Proceedings (with ISBN number).

The **second product** is a gallery of commented videoclips about practices in WNA, to be hosted in the Conference website and, possibly, later, in the ICMI website.

The **third product** is the ICMI Study volume. The volume will be informed by the papers, the videoclips and the discussions at the Study Conference as well as its outcomes, but it must be appreciated that there will be no guarantee that any of the papers accepted for the Study Conference will appear in the book. The Study book will be an edited volume published by Springer as part of the New ICMI Study Series. The editors and the editing process and content will be the subject of discussion among the IPC considering also the framework prepared for the Study Conference. It is expected that the organization of the volume will follow the organization and themes of this Discussion Document, although some changes might be introduced to exploit the impact of the discussion raised during the Conference. A report on the Study and its outcomes will be presented at the 13th International Congress on Mathematical Education, to be held in Hamburg, Germany (24-31 July 2016). It is hoped that the Study volume will also be published in 2016.

5. Call for Contribution to ICMI Study 23

The IPC for ICMI Study 23 invites submissions of contributions of several kinds: theoretical or cultural-historic-epistemological-essays (with deep connection with classroom practice, curricula or teacher education programs); position papers discussing policy and practice issues; discussion papers related to curriculum issues; reports on empirical studies; video-clips on explicit classroom or teacher professional education practice. The possibility of submitting short video-clips is a novelty of the ICMI Study 23. Video-clips show in a visual way examples of non-verbal communication, dynamic moments of significance or oddity, impressive performances or crucial incidents about teaching and learning WNA (including teacher professional education and development). Hence, in addition to usual reports, video-clips with accompanying short paper (see below) are welcome.

To ensure a rich and varied discussion, participation from countries with different economic capacity or with different cultural heritage and practices is encouraged.

The IPC encourages people who are not familiar with such Conferences to submit early (see the deadlines below) in order to receive assistance for finalizing their contribution (this assistance concerns the choice of the topic of the contribution and the structure of the paper, not the editing of English language). In this way the IPC inaugurates a new tradition of helping newcomers (including practitioners) to the international mathematics education community. This implies a process of supporting the writing of a contribution which the IPC judges as having the potential to contribute to the Study (see below).

An invitation to the Conference does not imply that a formal presentation of the submitted contribution will be made during the Conference or that the paper will appear in the Study volume published after the Conference.

5.1. Submissions

The ICMI Study 23 website is opened at the address:

http://www.umac.mo/fed/ICMI23/

The website will be regularly updated with information about the Study and the Study Conference and will be used for sharing the contributions of those invited to the Conference in the form of Conference pre-proceedings.

Two kinds of submissions are welcome:

Papers prepared in English (the language of the Conference) according to a template (max 8 pages).

Video-clips (5-8 minutes) with **English subtitles** together with an accompanying **paper** prepared according to a **template** (max 6 pages) together with the author's declaration of having collected **informed consent forms** signed by the participants. The English subtitles are required also in the videos with English speakers, in order to help the understanding of the interaction for non native speakers. Blurring faces of participants for privacy reasons, when needed, must be ensured by the applicants before sending the videos.

The files are to be saved with the name:

Familyname_name

Accepted file extensions are the following:

Papers: .doc; .docx; .odt together with a .pdf copy.

Videos: .mp4; 3gp.

In both cases, the indication of the working group - theme (1st and 2nd choice) where the paper or the video-clip is expected to be discussed must be included.

In both cases, also the **context form** has to be filled out by all the author(s) as completely as possible to help readers to understand the context of the contribution and interpret the contribution accordingly.

The template, the context form and the informed consent form will be available in the ICMI Study 23 website.

It is not allowed to submit two papers with the same first author.

Information about the technical way of submitting a paper or a video+paper will be available soon in the ICMI Study 23 website.

http://www.umac.mo/fed/ICMI23/

5.2. Deadlines

August 31, 2014: People who believe they need assistance for finalizing their contribution must submit a tentative copy with an appropriate form (assistance form) for requiring assistance no later than August 31, 2014. Their submissions will be examined immediately. The author will receive by September 30 the information of the decision (rejected, accepted pending revision, accepted in the present form). In the second case an IPC member will act as "tutor" to help the final preparation of the paper. Then the final paper will undergo the standard review process. The assistance form will be available in the ICMI Study 23 website.

October 15, 2014: Submissions by people who do not require assistance must be sent no later than October 15, 2014, but earlier if possible.

February 2015: Proposals will be reviewed, decisions made about invitations for the Conference (to be held in June 2015) and notification of these decisions sent by the end of February.

Information about visa, costs and details of accomodation will be available on the ICMI Study 23 website:

http://www.umac.mo/fed/ICMI23/

Further information may be asked at the following address:

icmiStudy23@gmail.com

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7. References

The references are limited to documents from ICMI or other international bodies.

The list of ICMI Studies is available at http://www.mathunion.org/icmi/Conferences/icmi-studies/introduction/.

The following references can be downloaded free from the websites (last visited on April 1, 2014).

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